CHAPTER V

# M-ideals in Banach algebras

## V.1 Preliminary results

This chapter deals with $M$-ideals in Banach algebras. Quite surprisingly, their purely geometric definition forces $M$-ideals to get into contact with the underlying algebraic structure. To motivate the main idea behind the following, let us briefly come back to one of the very first examples in this book:
Take $\mathfrak{A} = C(K)$, the Banach algebra of continuous functions on the compact space $K$, and recall (Example I.1.4(a)) that here the $M$-ideals correspond precisely to the (closed) ideals, i.e. $J \subset C(K)$ is an $M$-ideal if and only if for some closed set $D \subset K$,

$$J = J_D := \{f \in C(K) \mid f_{|D} = 0\}.$$

Let us follow $J_D$'s way through the higher duals of $C(K)$ just a little bit further: We have $C(K)^{**} = C(K'')$ for some suitable compact $K''$ and

$$C(K'') = J_D^{\perp\perp} \oplus_\infty J_2,$$

where the latter can only happen if for some clopen set $\widehat{D} \subset K''$

$$J_D^{\perp\perp} = J_{\widehat{D}} = \chi_{\widehat{D}} C(K'').$$

Hence, for each $M$-ideal $J$ in $C(K)$ there is an idempotent element $p \in C(K'')$ such that

$$J^{\perp\perp} = p\, C(K)^{**}.$$

It is surely rather tempting to try to prove that – at least in some special cases – the algebraic and the geometric structure of a Banach algebra are linked in a similar way.
And, as a matter of fact, there is such a connection: $M$-ideals are always subalgebras, in the case of the more "classical" Banach algebras, they are mostly generated by idempotents in the bidual or, at least, they are (algebraic) ideals.

Before we actually start to formulate and prove the central results of this chapter, we have to recall some definitions and a little more: To be able to pursue the above idea in the context of an arbitrary Banach algebra we need a product on the bidual $\mathfrak{A}^{**}$ of a Banach algebra. Fortunately, there is such a thing. What will make things a little bit more uncomfortable is the fact that one can define two products, both in quite a natural way, which are in general different from each other. For mnemonic reasons, we take the freedom of sometimes writing mappings on the right, when we now define these products.

**Definition 1.1** *Let $\mathfrak{A}$ be a Banach algebra. Let $a, b \in \mathfrak{A}$, $f, g \in \mathfrak{A}^*$ and $F, G \in \mathfrak{A}^{**}$. We then put*

$$\begin{array}{rclcrcl} fa(b) & := & f(ab) & \qquad & (b)af & := & (ba)f \\ Ff(a) & := & F(fa) & & (a)fF & := & (af)F \\ FG(f) & := & F(Gf) & & (f)F.G & := & (fF)G \end{array}$$

*The product $FG$ is called the first, $F.G$ the second Arens multiplication.*

When both products coincide, $\mathfrak{A}$ is called *Arens regular*. Examples of Arens regular Banach algebras are $C^*$-algebras as well as function algebras, whereas mere commutativity of $\mathfrak{A}$ generally does not suffice (see e.g. [134]). We also refer to the survey [176] for more information along these lines.

Let us agree upon some further notation:
Those operators on $\mathfrak{A}$ that multiply with a given element $a$ from the left resp. right are denoted by $L_a$ and $R_a$, respectively. When this distinction is not necessary, i.e. if $a$ commutes with all elements in $\mathfrak{A}$, then we will also write $M_a$. To emphasize which Arens product in $\mathfrak{A}^{**}$ is under consideration we shall use the notation $L_F^i$ resp. $R_F^i$ to denote left or right multiplication with respect to the $i$-th Arens multiplication.
In the following theorem we collect some results in connection with the Banach algebra $\mathfrak{A}^{**}$. In the proofs of all of them nothing but Definition 1.1 is involved. Nevertheless, some of them are somewhat cumbersome to show.

**Theorem 1.2** *Let $\mathfrak{A}$ be a Banach algebra.*

(a) *Whenever $\mathfrak{J}$ is a left (right) ideal in $\mathfrak{A}$, the same is true for $\mathfrak{J}^{\perp\perp}$ in $\mathfrak{A}^{**}$, independent of the Arens multiplication under consideration.*

(b) *If $H : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then so is $H^{**} : \mathfrak{A}^{**} \to \mathfrak{B}^{**}$ if $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$ are furnished with the same Arens product. Similarly, if $H : \mathfrak{A} \to \mathfrak{B}$ is an antihomomorphism (i.e. $H(ab) = H(b)H(a)$ for all $a, b \in \mathfrak{A}$), then $H^{**}$ has the same property whenever $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$ are provided with different Arens products.*

(c) *For every subalgebra $\mathfrak{B}$, both Arens products on $\mathfrak{B}^{**} = \mathfrak{B}^{\perp\perp}$ agree with the ones inherited from $\mathfrak{A}^{**}$. In particular, $\mathfrak{B}^{\perp\perp}$ is a subalgebra of $\mathfrak{A}^{**}$.*

(d) *For all $a \in \mathfrak{A}$ and $F \in \mathfrak{A}^{**}$ we have*

$$i_{\mathfrak{A}}(a)F = i_{\mathfrak{A}}(a).F = L_a^{**}(F)$$

*and*

$$Fi_{\mathfrak{A}}(a) = F.i_{\mathfrak{A}}(a) = R_a^{**}(F).$$

(e) *For fixed $F \in \mathfrak{A}^{**}$, the mappings $R_F^1$ and $L_F^2$ are weak$^*$ continuous.*

Note that according to (d), the mappings $R_a^1$, $R_a^2$, $L_a^1$ and $L_a^2$ are weak* continuous for each $a \in \mathfrak{A}$.
We will call a Banach algebra *unital* if it contains a two-sided unit which we usually denote by $e$. A *left $\lambda$-approximate unit* for $\mathfrak{A}$ is a net $(p_\alpha)$ such that $\|p_\alpha\| \leq \lambda$ and

$$\lim_\alpha p_\alpha a = a$$

for all $a \in \mathfrak{A}$. *Right approximate units* are defined in a similar manner. The connection between the concepts of unit and approximate unit is given by the next theorem.

**Theorem 1.3** *Let $\mathfrak{A}$ be a Banach algebra.*

(a) *Suppose that $\mathfrak{A}$ has a bounded right approximate unit $(q_\alpha)$. Then each weak* cluster point of $(q_\alpha)$ is a right unit for $\mathfrak{A}^{**}$ with respect to the first Arens product. The analogous result is valid for left approximate units, when we substitute the first by the second Arens product.*

(b) *If $\mathfrak{A}^{**}$ has a right unit $q$ with respect to either of the Arens multiplications then there exists a right approximate unit $(q_\alpha)$ in $\mathfrak{A}$ with*

$$\lim_\alpha \|q_\alpha\| = \|q\|.$$

Using Theorem 1.2(e) the proof of (a) is straightforward. It can also be found in [85], where a proof of (b) is given as well.
For reasons which will become clear later on, we would like to indicate an independent proof, which makes use of the following variant of the principle of local reflexivity. Compared to the more classical variants, it admits the additional degree of freedom (d).

**Theorem 1.4** *Let $X$ be a Banach space, $F \subset X^{**}$, $G \subset X^*$, $H \subset L(X)$ finite dimensional subspaces and put*

$$F_H := \operatorname{lin}\{h^{**}x^{**} \mid h \in H, x^{**} \in F\} + F.$$

*Then for each $\varepsilon > 0$ there is an operator $T : F_H \to X$ with*

(a) $T|_{F\cap X} = Id$

(b) *For all $g \in G$ and $f \in F$ we have $g(Tf) = f(g)$.*

(c) *For all $x \in F_H$,*
$$\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|.$$

(d) *For all $h \in H$,*
$$\|(hT - Th^{**})|_F\| < \varepsilon\|h\|.$$

Proof: [58] □

Let us show how this result can be used to prove Theorem 1.3(b): To this end order the set

$$\mathsf{A} = \{(F,G,H,\varepsilon) \mid q \in F \subset \mathfrak{A}^{**}, G \subset \mathfrak{A}^*, H \subset \mathfrak{A}, \varepsilon > 0\}$$

where $F$, $G$ and $H$ are finite dimensional subspaces, by

$$(F,G,H,\varepsilon) \prec (\widetilde{F},\widetilde{G},\widetilde{H},\widetilde{\varepsilon}) \qquad \Longleftrightarrow \qquad F \subset \widetilde{F},\ G \subset \widetilde{G},\ H \subset \widetilde{H},\ \varepsilon > \widetilde{\varepsilon}$$

and identify a space $H \subset \mathfrak{A}$ with the operator space $\{L_h \mid h \in H\}$. We next choose for every $\alpha \in \mathsf{A}$ an operator $T_\alpha$ that satisfies the conditions (a) – (d) listed in Theorem 1.4 for the subspaces and the number $\varepsilon$ determined by $\alpha$. Let

$$q_\alpha := T_\alpha(q).$$

We have by condition (d) and since $L_x^{**} = L_{i_{\mathfrak{A}}(x)}$ for either of the Arens multiplications (Theorem 1.2(d)) that for every $a \in \mathfrak{A}$

$$\lim_\alpha \|aq_\alpha - T_\alpha(aq)\| = \lim_\alpha \|L_a T_\alpha(q) - T_\alpha L_a^{**}(q)\| = 0$$

By property (a) of Theorem 1.4 and the above,

$$\lim_\alpha \|aq_\alpha - a\| = \lim_\alpha \|aq_\alpha - T_\alpha(aq)\| = 0$$

for all $a \in \mathfrak{A}$, and accordingly, $q_\alpha$ is a right approximate unit. Finally, property (c) of Theorem 1.4 gives

$$\lim_\alpha \|q_\alpha\| = \|q\|.$$

Note that by the above, a right (left) unit with respect to the second (first) Arens product is a right (left) unit for the first (second) one as well. (□)

We will use Theorem 1.4 in much the same way in Section V.3.

Indispensable for our purposes is the theory of hermitian elements in an arbitrary Banach algebra as developed in [84] and [86]. Let us recall, for the sake of easy reference, some definitions and fundamental results.
The first concept we will use several times is the *state space* of a unital Banach algebra $\mathfrak{A}$. It is defined by

$$\mathsf{S}_{\mathfrak{A}} = \{\varphi \in \mathfrak{A}^* \mid \varphi(e) = 1 = \|\varphi\|\}.$$

We denote by

$$v(a, \mathfrak{A}) = \{\varphi(a) \mid \varphi \in \mathsf{S}_{\mathfrak{A}}\}$$

the *numerical range* of an element $a \in \mathfrak{A}$. An element $a$ is called *hermitian* if and only if

$$v(a, \mathfrak{A}) \subset \mathbb{R}.$$

This is of course an interesting definition only when $\mathfrak{A}$ is a complex vector space. In general, we will omit the reference to the algebra in question and write $v(a)$ instead of $v(a, \mathfrak{A})$. Finally, we denote by $\mathbb{H}(\mathfrak{A})$ the (real) subspace of hermitian elements of $\mathfrak{A}$.

We close this introductory section with a small sample of the beautiful results which exist in connection with numerical ranges. They will turn out to be important cornerstones for our reasoning in the following section.

**Theorem 1.5** *Let $\mathfrak{A}$ be a unital Banach algebra. Then, for each element $\psi \in \mathfrak{A}^*$, there are $\lambda_1, \ldots, \lambda_4 \geq 0$ and $\psi_1, \ldots, \psi_4 \in \mathsf{S}_{\mathfrak{A}}$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq \sqrt{2}\mathrm{e}\|\psi\|$ such that*

$$\psi = \sum_{n=1}^{4} i^n \lambda_n \psi_n.$$

PROOF: [86, p. 100] □

**Theorem 1.6** *Let $h \in B_{\mathfrak{A}}$ be hermitian and suppose that $\varphi \in \mathsf{S}_{\mathfrak{A}}$ satisfies*

$$1 = \varphi(h) = \|h\|.$$

*Then $\varphi$ is multiplicative on the algebra generated by the unit $e$ and $h$.*

PROOF: [86, p. 77] □

**Proposition 1.7** *An element $h$ of a $C^*$-algebra is self-adjoint if and only if it is hermitian.*

PROOF: [84, p. 47] □

The following result, known as the Vidav-Palmer theorem, is an important means to distinguish $C^*$-algebras from general Banach algebras.

**Theorem 1.8** *A unital Banach algebra $\mathfrak{A}$ is a $C^*$-algebra if and only if it is generated by its hermitian elements, that is if $\mathfrak{A} = \mathbb{H}(\mathfrak{A}) + i\mathbb{H}(\mathfrak{A})$. In this case the involution is given by $(h_1, h_2 \in \mathbb{H}(\mathfrak{A}))$*

$$(h_1 + ih_2)^* = h_1 - ih_2.$$

PROOF: [84, §6] □

As an application of the Vidav-Palmer theorem we would like to prove that the bidual of a $C^*$-algebra is a $C^*$-algebra. This fact will be used in Section V.4. We first need a lemma.

**Lemma 1.9** *If $\mathfrak{A}$ is a unital Banach algebra, then for each $F \in \mathfrak{A}^{**}$ we have*

$$v(F, \mathfrak{A}^{**}) = \overline{\{F(\varphi) |\ \varphi \in \mathsf{S}_{\mathfrak{A}}\}}.$$

PROOF: Theorem 12.2 in [84]. For a different approach see Theorem 3.4 and Corollary 5.8 of [23]. □

**Theorem 1.10** *Let $\mathfrak{A}$ be a $C^*$-algebra.*

(a) *Then $\mathfrak{A}$ is Arens regular.*
(b) *The bidual algebra is a $C^*$-algebra as well.*
(c) *On $\mathfrak{A}^{**}$ multiplication from the left and the right and the involution are weak$^*$ continuous.*

PROOF: We put $f^*(x) := \overline{f(x^*)}$ and $F^*(f) := \overline{F(f^*)}$ for $f \in \mathfrak{A}^*$, $F \in \mathfrak{A}^{**}$. This defines an isometric linear involution in the sense of [85, p. 63] on $\mathfrak{A}^{**}$. Also, by definition of the Arens products we have

$$(FG)^* = G^*.F^* \qquad (*)$$

for $F, G \in \mathfrak{A}^{**}$.
Let us now suppose in addition that $\mathfrak{A}$, and hence $\mathfrak{A}^{**}$, is unital. We first claim:

$$F = F^* \qquad \Longrightarrow \qquad F \text{ hermitian}.$$

To show this, let $\varphi \in \mathsf{S}_{\mathfrak{A}}$ so that $\varphi = \varphi^*$ (here Proposition 1.7 enters). It follows that

$$F(\varphi) = F^*(\varphi) = \overline{F(\varphi^*)} = \overline{F(\varphi)},$$

whence $F(\varphi) \in \mathbb{R}$. An appeal to Lemma 1.8 finishes the proof of our claim.
Now we can deduce that each $F \in \mathfrak{A}^{**}$ has a representation

$$F = \frac{F + F^*}{2} + i\frac{F - F^*}{2i} \in \mathbb{H}(\mathfrak{A}^{**}) + i\,\mathbb{H}(\mathfrak{A}^{**}).$$

Consequently, $\mathfrak{A}^{**}$ is a $C^*$-algebra for the above involution by the Vidav-Palmer theorem, no matter which Arens product on $\mathfrak{A}^{**}$ is chosen. But with this information at hand we derive from $(*)$ and the identity

$$(FG)^* = G^*F^*,$$

which is of course valid in any $C^*$-algebra, that in fact the two Arens products coincide. Thus we have shown both (a) and (b) for unital $C^*$-algebras.
In the nonunital case it is a simple matter to adjoin an identity to $\mathfrak{A}$ (cf. [85, p. 67]) so that $\mathfrak{A}$ embeds isometrically as a self-adjoint subalgebra into a unital $C^*$-algebra $\mathfrak{B}$ ($= \mathfrak{A} \oplus \mathbb{C}$ with a suitable norm). By what we already know, $\mathfrak{A}^{**}$ embeds isometrically as a self-adjoint subalgebra into the unital $C^*$-algebra $\mathfrak{B}^{**}$, hence (a) and (b) in the general case, too.
(c) follows now from Theorem 1.2 and the definition of the involution. □

Using representation theory for $C^*$-algebras, one may identify the bidual with an algebra of operators on some Hilbert space, the so-called universally enveloping von Neumann algebra. We refer to [592, p. 122] for this view of Theorem 1.10(b).

## V.2 The general case of unital algebras

In this section we will follow the lane that we indicated at the beginning of the last section. As it should be expected, results won't be as strong in general as in the case of the algebra $C(K)$.
Let us retain for the moment the idea from the introduction to the previous section and suppose that a given $M$-ideal $\mathfrak{J}$ in a unital Banach algebra has the property that the associated $M$-projection $P$ in the bidual operates by multiplication with a certain idempotent element $p$. Then, of course, we could gain information on $p$ by looking at the element $P(e)$. In general, we would then have to decide whether multiplication with this element from an appropriate side and with the help of one of the Arens products gives rise to an $M$-projection. In this situation, Proposition I.1.2 will be helpful. We will hence start our investigation of $M$-ideals in Banach algebras with an examination of the properties of the mapping

$$\Delta_{\mathfrak{A}} : Z(\mathfrak{A}) \to \mathfrak{A}, \quad T \mapsto T(e).$$

(Here $Z(\mathfrak{A})$ denotes the centralizer of $\mathfrak{A}$ which was introduced in Section I.3.) We will occasionally omit the index and simply write $\Delta$, when this is unlikely to cause any confusion.

Although there are Banach algebras with a trivial centralizer, yet containing a nontrivial $M$-ideal ($L(H)$ is an example of this, see Corollary VI.1.13 and Example I.1.4(d)), we may, in view of a general $M$-ideal structure theory of Banach algebras $\mathfrak{A}$, restrict our attention to studying the centralizer. This is due to the fact that $M$-ideals in $\mathfrak{A}$ correspond to certain $M$-projections in $Z(\mathfrak{A}^{**})$. In fact, a satisfactory characterisation of this correspondence is the subject of part (c) in the following result. (For part (a) recall the definition of $Z_{\mathbb{R}}(\mathfrak{A})$ from I.3.7.)

**Theorem 2.1** *Let $\mathfrak{A}$ be a unital Banach algebra and denote by $\Delta_{\mathfrak{A}} : Z(\mathfrak{A}) \longrightarrow \mathfrak{A}$ the mapping $T \longmapsto T(e)$. Then the following assertions hold.*

(a) *For all $T \in Z(\mathfrak{A})$ the inclusion $v(\Delta(T)) \subset v(T)$ holds. It follows in particular that the range of $Z_{\mathbb{R}}(\mathfrak{A})$ under $\Delta$ is contained in the set of hermitian elements of $\mathfrak{A}$.*

(b) *$\Delta(P)$ is a projection for each $M$-projection $P \in Z(\mathfrak{A})$.*

(c) *An $M$-projection $P$ on $\mathfrak{A}^{**}$ is associated with an $M$-ideal $\mathfrak{J}$ in $\mathfrak{A}$ (i.e. $P(\mathfrak{A}^{**}) = \mathfrak{J}^{\perp\perp}$) if and only if $\Delta_{\mathfrak{A}^{**}}(P)|_{\mathsf{S}_{\mathfrak{A}}}$ is lower semicontinuous.*

(d) *$\Delta$ is multiplicative.*

(e) *$\Delta$ is an isometry.*

(f) *$\Delta$ preserves spectral radii and spectra, relative to $Z(\mathfrak{A})$ and $\Delta(Z(\mathfrak{A}))$.*

For the proof we need the following lemma. To understand what is going on, recall the definition of a split face from Example I.1.4(c).

**Lemma 2.2** *Suppose $\mathfrak{J}$ is an $M$-ideal in $\mathfrak{A}$, $\mathfrak{A}^* = \mathfrak{J}^{\perp} \oplus_1 \mathfrak{J}_1$. Then*

$$F_0 := \mathsf{S}_{\mathfrak{A}} \cap \mathfrak{J}^{\perp} \qquad \text{and} \qquad F_1 := \mathsf{S}_{\mathfrak{A}} \cap \mathfrak{J}_1$$

*are complementary split faces of $\mathsf{S}_{\mathfrak{A}}$. Furthermore,*

$$\operatorname{lin} F_0 = \mathfrak{J}^{\perp} \qquad \text{and} \qquad \operatorname{lin} F_1 = \mathfrak{J}_1.$$

Proof: Let $\varphi \in \mathsf{S}_{\mathfrak{A}}$ and denote by $Q$ the $L$-projection onto $\mathfrak{J}^{\perp}$. Suppose that both $\|Q\varphi\|$ and $\|(Id - Q)\varphi\|$ are strictly positive and write $\varphi = \|Q\varphi\|\varphi_0 + \|(Id - Q)\varphi\|\varphi_1$ where $\varphi_0 = \|Q\varphi\|^{-1}Q\varphi \in \mathfrak{J}^{\perp}$ and $\varphi_1 = \|(Id - Q)\varphi\|^{-1}(Id - Q)\varphi \in \mathfrak{J}_1$. Evaluation at $e$ then gives

$$1 = \|Q\varphi\|\varphi_0(e) + \|(Id - Q)\varphi\|\varphi_1(e)$$

which is only possible when $1 = \varphi_0(e) = \varphi_1(e)$. Hence, $\varphi_0, \varphi_1$ are elements of $\mathsf{S}_{\mathfrak{A}}$ and thus

$$\mathsf{S}_{\mathfrak{A}} = \operatorname{co}(F_0 \cup F_1).$$

Uniqueness of the above representation is clear since $\mathfrak{A}^* = \mathfrak{J}^{\perp} \oplus_1 \mathfrak{J}_1$. It remains to show that $F_0$ and $F_1$ are faces. By symmetry, we may restrict ourselves to $F_0$. Suppose that $\psi \in F_0$ is a convex combination in $\mathsf{S}_{\mathfrak{A}}$, $\psi = \lambda\psi_1 + (1 - \lambda)\psi_2$. Then

$$\psi = \lambda Q\psi_1 + (1 - \lambda)Q\psi_2,$$

and we find $\|Q\psi_1\| = \|Q\psi_2\| = 1$. This gives $\|(Id - Q)\psi_1\| = \|(Id - Q)\psi_2\| = 0$ and consequently, $\psi_1, \psi_2 \in F_0$, i.e. $F_0$ is a face. The last part of the lemma follows directly from Theorem 1.5 and the fact that $\mathsf{S}_{\mathfrak{A}} = \operatorname{co}(F_0 \cup F_1)$. □

PROOF OF THEOREM 2.1:

(a) This follows from the fact that for each $\psi \in \mathsf{S}_{\mathfrak{A}}$ the functional $T \mapsto \psi(T(e))$ belongs to $\mathsf{S}_{L(\mathfrak{A})}$. The statement about $T \in Z_{\mathbb{R}}(\mathfrak{A})$ is a consequence of this observation and the fact that all operators in $Z_{\mathbb{R}}(\mathfrak{A})$ are hermitian (Lemma I.3.8).

(b) We retain the notation of Lemma 2.2 and put furthermore $z = P(e)$. By Lemma 2.2, $z$ has norm one and is hermitian as a consequence of (a) (note that an $M$-projection belongs to $Z_{\mathbb{R}}(\mathfrak{A})$). Consequently, Theorem 1.6 applies, and each $\varphi \in F_1$ must be multiplicative on the algebra generated by $e$ and $z$ so that

$$\varphi(z^2) = \varphi(z)^2 = 1 = \varphi(z).$$

In a similar way, we find for $\psi \in F_0$

$$\psi\left((e-z)^2\right) = \psi(e-z)^2 = 1,$$

which implies $\psi(z^2) = \psi(z) = 0$. Now the above lemma applies and shows that $z$ and $z^2$ coincide on $\mathsf{S}_{\mathfrak{A}}$, a property that extends to all of $\mathfrak{A}^*$ by Theorem 1.5, and we are done.

(c) Observe first that by part (a)

$$z(\mathsf{S}_{\mathfrak{A}}) = v(z) \subset v(P) = [0,1].$$

Hence, the use of the expression "lower semicontinuous" makes sense. To prove the result, suppose that $P$ is associated with some $M$-ideal $\mathfrak{J}$ in $\mathfrak{A}$, and let $(s_\alpha)$ be a net contained in the set $(r \geq 0)$

$$E_r := \{s \in \mathsf{S}_{\mathfrak{A}} \mid z(s) \leq r\}$$

which we must prove to be closed. Suppose $(s_\alpha)$ has the weak* limit $s \in \mathsf{S}_{\mathfrak{A}}$. We let $F_0$ and $F_1$ as in Lemma 2.2 and write

$$s_\alpha = (1-t_\alpha) f_\alpha^{(0)} + t_\alpha f_\alpha^{(1)},$$

where $f_\alpha^{(i)} \in F_i$ and $t_\alpha \in [0,1]$. Making the assumptions that

$$f_\alpha^{(0)} \xrightarrow{w^*} f^{(0)} \in F_0, \qquad s_\alpha \xrightarrow{w^*} s \in \mathsf{S}_{\mathfrak{A}}, \qquad t_\alpha \longrightarrow t \in [0,1]$$

and that, furthermore,

$$f_\alpha^{(1)} \xrightarrow{w^*} f^{(1)} = \tau\varphi_0 + (1-\tau)\varphi_1,$$

where $\varphi_i \in F_i$ (note that everything is compact here), we obtain

$$s = (1-t) f^{(0)} + t\left(\tau\varphi_0 + (1-\tau)\varphi_1\right).$$

This gives

$$z(s) = t(1-\tau) \leq t = \lim_\alpha t_\alpha = \lim_\alpha z(s_\alpha) \leq r,$$

and the first half of the proof is done. Now for the other one: Let $P$ be an $M$-projection on $\mathfrak{A}^{**}$ such that $z := \Delta_{\mathfrak{A}^{**}}(P)$ is lower semicontinuous on $\mathsf{S}$. By Theorem I.1.9 there is an $L$-projection $P_*$ on $\mathfrak{A}^*$ such that $(P_*)^* = P$. We denote $F_0 = \mathsf{S}_{\mathfrak{A}} \cap \ker(P_*)$, and we have to show that $\ker(P_*)$ is weak* closed. To do so, we employ the Krein-Smulyan theorem and suppose that $(\psi_\alpha)$ is any bounded net in $\ker(P_*)$ with limit $\psi \in \mathfrak{A}^*$. Use Theorem 1.5 to write

$$\psi_\alpha = (Id - P_*)\psi_\alpha = \sum_{\nu=1}^{4} i^\nu \mu_\nu^{(\alpha)} \psi_\nu^{(\alpha)}, \qquad (*)$$

with $\psi_\nu^{(\alpha)} \in \mathsf{S}_{\mathfrak{A}}$ and $0 \le \mu_\nu^{(\alpha)} \le C$ for some constant $C$; $\nu = 1, \dots, 4$. Note that we can choose the $\psi_\nu^{(\alpha)}$ in $F_0$. Now observe that the set $z^{-1}(0) \cap \mathsf{S}_{\mathfrak{A}} = \{s \in \mathsf{S}_{\mathfrak{A}} \mid z(s) \le 0\}$ is weak* closed by lower semicontinuity. The next thing to note is that

$$z^{-1}(0) \cap \mathsf{S}_{\mathfrak{A}} = F_0,$$

which easily results from $z|_{\ker(P_*)} = 0$ and $z|_{\operatorname{ran}(P_*)} = 1$. Hence, $F_0$ is weak* compact. Consequently, we may again suppose convergence of the nets involved in $(*)$:

$$\mu_\nu^{(\alpha)} \longrightarrow \mu_\nu \in [0, C], \qquad \psi_\nu^{(\alpha)} \longrightarrow \psi_\nu \in F_0.$$

Hence,

$$\psi = \sum_{\nu=1}^{4} \mu_\nu \psi_\nu \in \operatorname{lin} F_0 \subset \ker(P_*),$$

and the proof of (c) is finished.

(d) Since by bitransposition $Z(\mathfrak{A})$ is a subalgebra of $Z(\mathfrak{A}^{**})$ (Corollary I.3.15) and since the map $\Delta_{\mathfrak{A}}$ is nothing else but $\Delta_{\mathfrak{A}^{**}}|_{Z(\mathfrak{A})}$, it is sufficient to prove the claim for dual Banach algebras. In this case, however, $Z(\mathfrak{A})$ is generated by elements of the form $\sum_{i=1}^n \alpha_i P_i$ where the $P_i$ denote $M$-projections with $P_i P_j = P_j P_i = 0$ (see Theorem I.3.14). Put $\Delta(P_i) = z_i$. Now, it is clearly enough to show that $P_1 P_2 = P_2 P_1 = 0$ implies $z_1 z_2 = z_2 z_1 = 0$. But by assumption, $P_1 + P_2$ is an $M$-projection and so, according to (b), we have $(z_1 + z_2)^2 = z_1 + z_2$. This leads to $z_1 z_2 + z_2 z_1 = 0$ and

$$(-z_1 z_2)^2 = z_1 z_2 z_1 z_2 = -z_1 z_1 z_2 z_2 = -z_1 z_2.$$

In the same way it follows that $-z_2 z_1$ is idempotent. But $-z_1 z_2 = z_2 z_1$ and hence,

$$-z_1 z_2 = (-z_1 z_2)^2 = (z_2 z_1)^2 = -z_2 z_1.$$

This implies $z_1 z_2 = z_2 z_1 = 0$, as desired.

(e) Using the same reduction procedure as above, it is clearly enough to show that

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\| = \max_{1 \le i \le n} |\alpha_i|,$$

where $z_i = \Delta(P_i)$ for some orthogonal projections $P_i \in Z(\mathfrak{A}^{**})$. But one inequality is clear since $\|\Delta\| = 1$ and the other follows, since for fixed $z_j$

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\| \geq \left\| z_j \sum_{i=1}^{n} \alpha_i z_i \right\| = |\alpha_j|.$$

(f) With the help of the spectral radius formula $\rho(a) = \lim_n \sqrt[n]{\|a^n\|}$ this follows from (d) and (e). □

We now come to the most far-reaching result on $M$-ideals in arbitrary unital Banach algebras. The reader should compare part (b) below to what we said at the beginning of this section. We will use this part of the result in Section V.4.

**Theorem 2.3** *Suppose that $\mathfrak{A}$ is a unital Banach algebra.*

(a) *Every $M$-ideal $\mathfrak{J}$ of $\mathfrak{A}$ is a subalgebra, but in general not an ideal.*

(b) *Let $\mathfrak{J}$ be an $M$-summand of a unital Banach algebra $\mathfrak{A}$, $P$ the $M$-projection with range $\mathfrak{J}$, and put $z = P(e)$. Then $z\mathfrak{A}z \subset \mathfrak{J}$.*

Proof: (a) Suppose first that $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are complementary $M$-summands, that the $M$-projection $P$ maps $\mathfrak{A}$ onto $\mathfrak{J}_1$, and put $z = P(e)$. Let $m_1, m_2 \in \mathfrak{J}_2$ have unit norm and suppose that

$$m_1 m_2 = j_1 + j_2$$

with $j_i \in \mathfrak{J}_i$ and $j_1 \neq 0$. For any $|\kappa| = 1$ we have

$$(z + m_1)(z + \kappa m_2) = z + m_1 z + \kappa z m_2 + \kappa j_1 + \kappa j_2.$$

By Lemma 2.2 we may select $\varphi \in \mathsf{S}_{\mathfrak{A}} \cap \mathfrak{J}_2^{\perp}$ and fix $\kappa_0$, $|\kappa_0| = 1$, such that $\kappa_0 \varphi(j_1) > 0$. $M$-summands are invariant under hermitian operators (Corollary I.1.25), and, whenever $h \in \mathbb{H}(\mathfrak{A})$, the operators $L_h$ and $R_h$ are hermitian [84, p. 47]. Hence, $m_1 z + \kappa_0 z m_2 + \kappa_0 j_2 \in \mathfrak{J}_2$, since $z$ is hermitian by Theorem 2.1(a), and we find

$$\varphi\left((z + m_1)(z + \kappa_0 m_2)\right) = \varphi(z) + \kappa_0 \varphi(j_1) > 1.$$

But this is impossible, because $\|(z + m_1)(z + \kappa_0 m_2)\| \leq 1$. Consequently, $m_1 m_2 \in \mathfrak{J}_2$ whenever $m_1, m_2 \in \mathfrak{J}_2$ which means that $M$-summands are subalgebras. Suppose now that $\mathfrak{J}$ is an $M$-ideal. By what we have just shown, $\mathfrak{J}^{\perp\perp}$ must be a subalgebra of $\mathfrak{A}^{**}$ (for any Arens product), a property that passes to $\mathfrak{J}$ due to the fact that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^{**}$. The fact that $\mathfrak{J}$ need not be an ideal will be shown on page 231.

(b) Denote by $\mathfrak{J}_2$ the $M$-summand complementary to $\mathfrak{J}_1 := \mathfrak{J}$. By the above, $z\mathfrak{J}_i z \subset \mathfrak{J}_i$ for $i = 1, 2$, and the proof will be complete once it is shown that $z\mathfrak{J}_2 z = 0$. Suppose that there is $m \in \mathfrak{J}_2$ with $\|zmz\| = 1$. Since $z$ is in $\mathfrak{J}_1$,

$$\|(\pm zmz + z)^2\| \leq \|\pm zmz + z\|^2 = 1.$$

This leads to $\|(zmz)^2 \pm 2zmz + z\| \leq 1$ and

$$\begin{aligned} 4 &= \left\| 2zmz + \left(z + (zmz)^2\right) + 2zmz - \left(z + (zmz)^2\right) \right\| \\ &\leq \|(z + zmz)^2\| + \|(z - zmz)^2\| \ \leq\ 1 + 1, \end{aligned}$$

which is absurd. □

## V.3 Inner $M$-ideals

In almost all the known cases, there is only one practicable way to obtain more information on the structure of an $M$-ideal in a given unital Banach algebra than the one delivered by Theorem 2.3: It consists in showing that the evaluation of the corresponding $M$-projection on the bidual at the unit element $e$ yields an element which, to some extent, is responsible for the $M$-projection itself.
The concept which underlies this idea has been alluded to at the beginning of the previous section and will be made precise right now. We will study this type of $M$-ideal here in a rather abstract setting. The more concrete examples will be provided in future sections of the present and the following chapter.

Let $\mathfrak{A}$ be a Banach algebra and recall that $\mathrm{Mult}(\mathfrak{A})$ denotes the multiplier algebra of $\mathfrak{A}$ (Definition I.3.1). An element $T \in \mathrm{Mult}(\mathfrak{A})$ is called left (right) inner if there is an element $t \in \mathfrak{A}$ such that, for all $a \in \mathfrak{A}$,

$$T(a) = L_t(a) = ta$$

or, respectively,

$$T(a) = R_t(a) = at$$

for all $a \in \mathfrak{A}$. The subalgebra of $\mathfrak{A}$ consisting of all those elements of $\mathfrak{A}$ that give rise to a left (right) inner element in $\mathrm{Mult}(\mathfrak{A})$ is denoted by

$$\mathrm{Mult}^l_{inn}(\mathfrak{A}) \qquad (\mathrm{Mult}^r_{inn}(\mathfrak{A})).$$

Analogously we define

$$Z^l_{inn}(\mathfrak{A}) \qquad (Z^r_{inn}(\mathfrak{A})).$$

Note that $\mathrm{Mult}^l_{inn}(\mathfrak{A})$ and $\mathrm{Mult}^r_{inn}(\mathfrak{A})$ are subalgebras of $\mathfrak{A}$, whereas $\mathrm{Mult}(\mathfrak{A})$ is a subalgebra of $L(\mathfrak{A})$. The fact that there are two different reasonable multiplications on $\mathfrak{A}^{**}$ makes the corresponding definition for $\mathfrak{A}^{**}$ somewhat unpleasant: We denote by

$$\mathrm{Mult}^{l,i}_{inn}(\mathfrak{A}^{**}) \qquad (\mathrm{Mult}^{r,i}_{inn}(\mathfrak{A}^{**})) \qquad Z^{l,i}_{inn}(\mathfrak{A}^{**}) \qquad (Z^{r,i}_{inn}(\mathfrak{A}^{**}))$$

the respective subalgebra of $\mathfrak{A}^{**}$ furnished with the $i$-th Arens product.

**Definition 3.1** *Let $\mathfrak{A}$ be a Banach algebra.*

(a) *An $M$-ideal $\mathfrak{J} \subset \mathfrak{A}$ is called* left (right) inner, *if the $M$-projection $P : \mathfrak{A}^{**} \to \mathfrak{J}^{\perp\perp}$ is right (left) inner with respect to the first (second) Arens multiplication on $\mathfrak{A}^{**}$.*

(b) *A subspace $\mathfrak{J}$ is called a* two-sided inner $M$-ideal *if and only if there is $p \in \mathfrak{A}^{**}$ such that $L^2_p$ as well as $R^1_p$ define $M$-projections $\mathfrak{A}^{**} \to \mathfrak{J}^{\perp\perp}$.*

(c) *We call an $M$-ideal* inner *if it is either left or right inner.*

We think that some remarks are in order:

(a) The necessity for using different multiplications in the definition of an inner $M$-ideal is due to the fact that $M$-projections in dual spaces are always weak$^*$ continuous (Theorem I.1.9), a property that is shared only by one of the respective Arens multiplications (Theorem 1.2). In fact, when e.g. also $a^{**} \mapsto a^{**}.p$ defines an $M$-projection on $\mathfrak{A}^{**}$, then by its weak$^*$ continuity, Theorem 1.2(d) and Theorem 1.2(e) we must have for all $a^{**} \in B_{\mathfrak{A}^{**}}$, being approximated in the weak$^*$ topology by $(x_\alpha)$ in $B_{\mathfrak{A}}$,

$$a^{**}.p = w^*\text{-}\lim_\alpha x_\alpha.p = w^*\text{-}\lim_\alpha x_\alpha p = a^{**}p.$$

Also, the reader might find it strange that we defined left inner $M$-ideals in terms of right inner $M$-projections. However, we decided to accept this asymmetry in favour of a symmetric characterisation of left inner $M$-ideals in terms of left ideals (Theorem 3.2).

(b) With the same reasoning as above, we find using Theorem I.3.14(c) that

$$Z_{inn}^{l,2}(\mathfrak{A}^{**}) \supset Z_{inn}^{l,1}(\mathfrak{A}^{**})$$

as well as

$$Z_{inn}^{r,1}(\mathfrak{A}^{**}) \supset Z_{inn}^{r,2}(\mathfrak{A}^{**}).$$

It is, for the time being, not clear whether a similar relation holds for the algebras $\mathrm{Mult}_{inn}^{l,i}(\mathfrak{A}^{**})$ and $\mathrm{Mult}_{inn}^{r,i}(\mathfrak{A}^{**})$.

Nevertheless, a glimpse at the proof of Lemma 3.4 below should convince the reader that $\mathrm{Mult}_{inn}^{l,2}(\mathfrak{A}^{**})$ and $\mathrm{Mult}_{inn}^{r,1}(\mathfrak{A}^{**})$ are the more pleasant objects to work with.

(c) Let $\mathfrak{J}$ be a two-sided inner $M$-ideal of $\mathfrak{A}$ and $p$ be the element in $\mathfrak{A}^{**}$ that gives rise to the $M$-projections $R_p^1$ and $L_p^2$ onto $\mathfrak{J}^{\perp\perp}$. It follows by Proposition I.1.2 that

$$p.a^{**} = a^{**}p$$

for all $a^{**} \in \mathfrak{A}^{**}$ which is an "almost central" behaviour.

(d) Note further that, at least when $\mathfrak{A}^{**}$ is unital, every $M$-ideal $\mathfrak{J}$, which is a left as well as a right inner $M$-ideal is a two-sided inner $M$-ideal. This follows, since according to Proposition I.1.2, for every pair of $M$-projections $L_p^2$ and $R_{\widetilde{p}}^1$ with $\mathfrak{J}^{\perp\perp}$ as a common range we have

$$p = L_p^2(e) = R_{\widetilde{p}}^1(e) = \widetilde{p}.$$

(e) Let us mention some examples which show that (algebraic) ideals of $\mathfrak{A}$, which are $M$-ideals at the same time, are not necessarily inner:

For the first one we make a Banach algebra out of any Banach space $X$ containing an $M$-ideal $J$ by defining $xy = 0$ for any $x, y \in X$. A more serious example is this: Let $J$ be an $M$-ideal of a Banach space $X$, the former of which does not enjoy the MAP. (To see that this is possible, fix e.g. a space $Y$ without the MAP and note that $J := Y \oplus_\infty c_0$ is an $M$-ideal in the space $X := Y \oplus_\infty \ell^\infty$.) Then $A(X, J)$, the space of approximable operators from $X$ to $J$, is a right ideal as well as, by Proposition VI.3.1, an $M$-ideal of $A(X) \cong X^* \widehat{\otimes}_\varepsilon X$. However, were $A(X, J)$ inner, there would be an approximate identity for this ideal by Theorem 3.2 below, which is impossible by the assumptions made on $Y$.

We will see in Sections V.4 and V.6 that this type of $M$-ideal is present in $C^*$-algebras, commutative Banach algebras and in the algebra $L(X)$ for a number of Banach spaces. In fact, in each of these cases, all $M$-ideals will turn out to be inner. We will prove in Proposition VI.4.10 that $K(X)$ is automatically an inner $M$-ideal once it is an $M$-ideal in $L(X)$.

The next theorem gives a characterisation of inner $M$-ideals which will be used in what follows.

**Theorem 3.2** *Let $\mathfrak{A}$ be a unital Banach algebra and $\mathfrak{J} \subset \mathfrak{A}$ a closed subspace. Then the following are equivalent:*

(i) *$\mathfrak{J}$ is a left inner $M$-ideal.*

(ii) *$\mathfrak{J}$ is a left ideal as well as an $M$-ideal and contains a right 1-approximate unit for $\mathfrak{J}$.*

(iii) *$\mathfrak{J}$ is a left ideal and contains a right approximate unit $(p_\alpha)$ that satisfies*

$$\limsup_\alpha \|sp_\alpha + t(e-p_\alpha)\| \le 1 \qquad \forall s,t \in B_{\mathfrak{A}}.$$

*This equivalence remains true after exchanging "left" and "right" accordingly.*
*If, in addition, $\mathfrak{J}$ is a two-sided inner $M$-ideal, then the net $(p_\alpha)$ in* (iii) *can be chosen as a two-sided approximate unit and to satisfy*

$$\limsup_\alpha \|sp_\alpha + t(e-p_\alpha)\| \le 1 \qquad \forall s,t \in B_{\mathfrak{A}}$$

*and*

$$\limsup_\alpha \|p_\alpha s + (e-p_\alpha)t\| \le 1 \qquad \forall s,t \in B_{\mathfrak{A}}$$

*simultaneously.*

Proof: (iii) $\Rightarrow$ (ii): The condition imposed on $(p_\alpha)$ implies $\limsup_\alpha \|p_\alpha\| \le 1$ and so, by Theorem 1.3, we may suppose $\|p_\alpha\| \le 1$ for all $\alpha$. We have to check that $\mathfrak{J}$ is indeed an $M$-ideal.
We will apply Theorem I.2.2. With this goal in mind, suppose that $j_1, j_2, j_3 \in B_{\mathfrak{J}}$, $x \in B_X$ and $\varepsilon > 0$ are arbitrarily given. Select an index $\alpha$ such that $\|j_i p_\alpha - j_i\| < \frac{\varepsilon}{2}$ for $i = 1,2,3$ as well as $\|j_i p_\alpha + x(e-p_\alpha)\| < 1 + \frac{\varepsilon}{2}$ for $i = 1,2,3$ and put $j = xp_\alpha$. Then $j \in \mathfrak{J}$,

$$\|j_i + x - j\| \ \le \ \|j_i p_\alpha - j_i\| + \|j_i p_\alpha + x(e-p_\alpha)\| \ < \ 1+\varepsilon,$$

and $\mathfrak{J}$ must be an $M$-ideal.

(ii) $\Rightarrow$ (i): By Theorem 1.3, $\mathfrak{J}^{\perp\perp}$ contains a right unit with respect to the first Arens multiplication which we denote by $p$. Since $\mathfrak{J}^{\perp\perp}$ is a left ideal (Theorem 1.2(a)), the mapping $R_p^1$ is a projection onto this space. But $M$-projections are unique among norm one projections with the same range (Proposition I.1.2), and so $\mathfrak{J}$ is a left inner $M$-ideal.

(i) $\Rightarrow$ (iii): Suppose that $R_p^1 : \mathfrak{A}^{**} \to \mathfrak{J}^{\perp\perp}$ is an $M$-projection for some $p \in \mathfrak{A}^{**}$. Then $\mathfrak{J}^{\perp\perp}$ is a left ideal, and hence so is $\mathfrak{J}$. Let now $F \subset \mathfrak{A}^{**}$, $G \subset \mathfrak{A}^*$ and $H \subset \mathfrak{A}$ run through the respective sets of finite dimensional subspaces and order the set

$$B = \{(F,G,H,\varepsilon) \mid F \subset \mathfrak{A}^{**},\ G \subset \mathfrak{A}^*,\ H \subset \mathfrak{A},\ \varepsilon > 0\}$$

by

$$(F,G,H,\varepsilon) \prec (\widetilde{F},\widetilde{G},\widetilde{H},\widetilde{\varepsilon}) \qquad \Longleftrightarrow \qquad F \subset \widetilde{F},\ G \subset \widetilde{G},\ H \subset \widetilde{H},\ \varepsilon > \widetilde{\varepsilon}.$$

Identify the space $H \subset \mathfrak{A}$ with the operator space $\{L_h \mid h \in H\}$ and choose for every $\beta \in B$ an operator $T_\beta$ that satisfies the conditions (a) – (d) listed in Theorem 1.4 for the subspaces and the number $\varepsilon$ determined by $\beta$. Let $p_\beta^0 := T_\beta(p)$. (Note that $T_\beta(a^{**})$

is eventually defined for all $a^{**} \in \mathfrak{A}^{**}$.) Then, as in the proof of Theorem 1.3, we may conclude that $(p^0_\beta)$ is a right approximate unit. As was also shown there,

$$\lim_\beta \|ap^0_\beta - T_\beta(ap)\| = \lim_\beta \|L_a T_\beta(p) - T_\beta L^{**}_a(p)\| = 0$$

for all $a \in \mathfrak{A}$, and so, for $s, t \in B_{\mathfrak{A}}$, we have by property (c) of Theorem 1.4

$$\begin{aligned} \limsup_\beta \|sp^0_\beta + t(e - p^0_\beta)\| &= \limsup_\beta \|T_\beta(sp) + T_\beta(t) - T_\beta(tp)\| \\ &= \limsup_\beta \|T_\beta\left(sp + t(e-p)\right)\| \\ &\leq 1. \end{aligned}$$

To finish the proof, we have to force the $p^0_\beta$ into $\mathfrak{J}$. To this end, we take a net $(q_\beta)$ from $B_{\mathfrak{J}}$ with $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$-$\lim_\beta q_\beta = p$. Note that we may use the same index set. Since by (b) of Theorem 1.4, $w^*$-$\lim_\beta p^0_\beta = p$, and since $p^0_\beta - q_\beta \in \mathfrak{A}$, we must have $w$-$\lim_\beta (p^0_\beta - q_\beta) = 0$. In passing to an appropriate $\|\,.\,\|$-convergent net of convex combinations we obtain

$$\|\,.\,\|\text{-}\lim_\alpha \sum_{i=1}^{N_\alpha} t_{i,\alpha}\left(p^0_{\beta_{i,\alpha}} - q_{\beta_{i,\alpha}}\right) = 0,$$

and

$$\lim_\alpha \max_{i \leq N_\alpha} \|xp^0_{\beta_{i,\alpha}} - x\| = 0.$$

We now put

$$p_\alpha = \sum_{i=1}^{N_\alpha} t_{i,\alpha} q_{\beta_{i,\alpha}}.$$

Hence we obtain

$$\begin{aligned} \lim_\alpha \|xp_\alpha - x\| &\leq \lim_\alpha \left( \left\| xp_\alpha - \sum_{i=1}^{N_\alpha} t_{i,\alpha} xp^0_{\beta_{i,\alpha}} \right\| + \left\| \sum_{i=1}^{N_\alpha} t_{i,\alpha} xp^0_{\beta_{i,\alpha}} - x \right\| \right) \\ &\leq \lim_\alpha \max_{i \leq N_\alpha} \|xp^0_{\beta_{i,\alpha}} - x\| \\ &= 0. \end{aligned}$$

In a similar fashion, for all $s, t \in B_{\mathfrak{A}}$,

$$\lim_\alpha \|sp_\alpha + t(e - p_\alpha)\| \leq 1,$$

and (iii) follows.

We are thus left with the case of a two-sided inner $M$-ideal $\mathfrak{J}$. Recall (Remark (d) after Definition 3.1) that there is $p \in \mathrm{Mult}^{l,2}_{inn}(\mathfrak{A}^{**}) \cap \mathrm{Mult}^{r,1}_{inn}(\mathfrak{A}^{**})$ with

$$L^2_p \mathfrak{A}^{**} = R^1_p \mathfrak{A}^{**} = \mathfrak{J}^{\perp\perp}.$$

To refine the properties of $(p_\alpha)$ in this situation, we use the index set

$$B^* = \{(F, G, H_L, H_R, \varepsilon) \mid p \in F \subset \mathfrak{A}^{**},\ G \subset \mathfrak{A}^*,\ H_L, H_R \subset \mathfrak{A},\ \varepsilon > 0\}$$

ordered in the now obvious way, and identify the spaces $H_L \subset \mathfrak{A}$ and $H_R \subset \mathfrak{A}$ with the operator spaces $\{L_h \mid h \in H_L\}$ and $\{R_h \mid h \in H_R\}$. Keeping these minor changes in mind, we may proceed as above. □

We continue our investigation of inner $M$-ideals with a proposition that deals with the behaviour of such $M$-ideals with respect to subalgebras.
As the case of the disk algebra $A(\mathbb{D})$, which we will now regard as a subalgebra of $C(\overline{\mathbb{D}})$, already shows, it is not to be expected that for subalgebras $\mathfrak{B}$ and inner $M$-ideals $\mathfrak{J}$ the space $\mathfrak{J} \cap \mathfrak{B}$ again is an $M$-ideal, much less an inner one. (To be more concrete, let $J = \{f \in A(\mathbb{D}) \mid f(0) = 0\}$. This cannot be an $M$-ideal in $A(\mathbb{D})$ by Example I.1.4(b), though it is of the above form.)

**Proposition 3.3** *Suppose that $\mathfrak{A}$ is a unital Banach algebra and let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ with $e \in \mathfrak{B}$. If $p \in \mathfrak{B}^{\perp\perp} \cap \operatorname{Mult}^{r,1}_{inn}(\mathfrak{A}^{**})$ is idempotent, it gives rise to a right inner $M$-ideal $\mathfrak{J}$ in $\mathfrak{A}$ by virtue of*

$$\mathfrak{A}^{**}p = \mathfrak{J}^{\perp\perp}$$

*if and only if*

$$\mathfrak{B}^{**}p = \mathfrak{J}_0^{\perp\perp}$$

*for some $M$-ideal $\mathfrak{J}_0$ in $\mathfrak{B}$. If one of these conditions is satisfied then $\mathfrak{J}_0 = \mathfrak{J} \cap \mathfrak{B}$, and $\mathfrak{J}$ is nontrivial if and only if $\mathfrak{J} \cap \mathfrak{B}$ is. In this case, one can find a left approximate unit $(p_\alpha)$ for $\mathfrak{J}$ in $B_{\mathfrak{J}\cap\mathfrak{B}}$ with the property that, for all $s, t \in B_{\mathfrak{A}}$,*

$$\limsup_\alpha \|p_\alpha s + (e - p_\alpha)t\| \le 1.$$

*A similar statement holds for left and two-sided inner $M$-ideals.*

Proof: Plainly, since $p \in \mathfrak{B}^{\perp\perp}$, we have for all $\psi \in \mathsf{S}_{\mathfrak{A}}$ and any $\rho \in \mathbb{R}$

$$p(\psi) \le \rho \qquad \Longleftrightarrow \qquad p(\psi_{|\mathfrak{B}}) \le \rho$$

which leads to

$$\{\psi \in \mathsf{S}_{\mathfrak{A}} \mid p(\psi) \le \rho\} = r^{-1}(\{\varphi \in \mathsf{S}_{\mathfrak{B}} \mid p(\varphi) \le \rho\})$$

where $r : \mathsf{S}_{\mathfrak{A}} \to \mathsf{S}_{\mathfrak{B}}$ denotes the restriction map. An examination of this equation quickly reveals that $\{\psi \in \mathsf{S}_{\mathfrak{A}} \mid p(\psi) \le \rho\}$ is weak* closed if and only if the set $\{\varphi \in \mathsf{S}_{\mathfrak{B}} \mid p(\varphi) \le \rho\}$ has the same property. This in turn implies that $p$ as a function on $\mathsf{S}_{\mathfrak{A}}$ is lower semicontinuous if and only if $p$ shows the same behaviour when considered as a function on $\mathsf{S}_{\mathfrak{B}}$. Hence, according to Theorem 2.1(c), $\mathfrak{A}^{**}p = \mathfrak{J}^{\perp\perp}$ for some $M$-ideal $\mathfrak{J}$ of $\mathfrak{A}$ is equivalent to $\mathfrak{B}^{\perp\perp}p = \mathfrak{J}_0^{\perp\perp}$ for some $M$-ideal $\mathfrak{J}_0$ of $\mathfrak{B}$. But then $\mathfrak{J}$ and $\mathfrak{J}_0$ are connected by

$$\mathfrak{J}_0 = \{b \in \mathfrak{B} \mid bp = b\} = \mathfrak{J} \cap \mathfrak{B}.$$

Next, since $e \in \mathfrak{B}$, we conclude that $\mathfrak{J} \ne \mathfrak{A}$ (for an ideal $\mathfrak{J}$ of $\mathfrak{A}$) is equivalent to $\mathfrak{J} \cap \mathfrak{B} \ne \mathfrak{B}$. Since $\mathfrak{J} = \{0\}$ if and only if $p = 0$ we have $\mathfrak{J} \cap \mathfrak{B} = \{0\}$ if and only if $\mathfrak{J} = \{0\}$. This proves that $\mathfrak{J}$ is not trivial if and only if $\mathfrak{J}_0$ is not either.

Let us finally construct the required net $(p_\alpha)$: To this aim, start with $(p_\alpha^0)$ in $B_{\mathfrak{J}}$ as delivered by Theorem 3.2, and select furthermore $q_\alpha \in B_{\mathfrak{B}\cap\mathfrak{J}}$ converging also to $p$ in the $\sigma(\mathfrak{A}^{**},\mathfrak{A}^*)$-topology. The claim then follows by an application of the same blocking technique as used in the proof of Theorem 3.2. ◻

As mentioned before, part of what follows will be devoted to the study of particular examples of inner $M$-ideals. We start this enterprise in this section with the description of a rather general situation in which these $M$-ideals appear quite naturally. We first need a lemma.

**Lemma 3.4** *For every Banach algebra $\mathfrak{A}$,*

$$\mathrm{Mult}^l_{inn}(\mathfrak{A})^{\perp\perp} \subset \mathrm{Mult}^{l,2}_{inn}(\mathfrak{A}^{**})$$

*This statement persists when "left" is changed to "right" and "2" to "1".*

PROOF: We will apply Theorem I.3.6. To this end, choose $f^{**} \in \mathrm{Mult}^l_{inn}(\mathfrak{A})^{\perp\perp}$ and let $a^{**}, b^{**} \in \mathfrak{A}^{**}$ with

$$\|a^{**} + \lambda b^{**}\| \le r$$

for all $\lambda$ with $|\lambda| \le \|f^{**}\|$. Use the principle of local reflexivity (Theorem 1.4) to find nets $(a_\alpha)$ and $(b_\alpha)$ with

$$\|a_\alpha + \lambda b_\alpha\| \le r + \varepsilon_\alpha$$

for all $|\lambda| \le \|f^{**}\|$, $(\varepsilon_\alpha)$ tending to zero, $w^*\text{-}\lim_\alpha a_\alpha = a^{**}$ and $w^*\text{-}\lim_\alpha b_\alpha = b^{**}$. If $(f_\beta)$ denotes a net in $\mathrm{Mult}^l_{inn}(\mathfrak{A})$ with the property that $\|f_\beta\| \le \|f^{**}\|$ and $w^*\text{-}\lim_\beta f_\beta = f^{**}$, then, by the weak* continuity of the mapping $R_{b_\alpha}$ (Theorem 1.2), we have that

$$\|a_\alpha + f^{**}b_\alpha\| \le \limsup_\beta \|a_\alpha + f_\beta b_\alpha\| \le r + \varepsilon_\alpha$$

for all $\alpha$. By the weak* continuity of the map $L^2_{f^{**}}$ (Theorem 1.2),

$$\|a^{**} + f^{**}.b^{**}\| \le \limsup_\alpha \|a_\alpha + f^{**}.b_\alpha\| \le r,$$

and we are done. The proof of the "r,1"-case is similar. ◻

**Proposition 3.5** *Let $\mathfrak{A}$ be a unital Banach algebra and $\mathfrak{B}$ be a subalgebra of* $\mathrm{Mult}^l_{inn}(\mathfrak{A})$ *such that $e \in \mathfrak{B}$. Suppose further that $\mathfrak{J}$ is an $M$-ideal in $\mathfrak{B}$.*

- (a) *The space $\mathfrak{A}_{\mathfrak{J}} := \overline{\mathrm{lin}}\,\{ja_0 \mid j \in \mathfrak{J}, a_0 \in \mathfrak{A}\}$ is a right inner $M$-ideal, and there is a left approximate unit $(p_\alpha)$ in $B_{\mathfrak{J}}$ for $\mathfrak{A}_{\mathfrak{J}}$ with $\limsup_\alpha \|p_\alpha s + (e - p_\alpha)t\| \le 1$ for all $s, t \in B_{\mathfrak{A}}$.*
- (b) *We have $\mathfrak{A}_{\mathfrak{J}} = \{a \in \mathfrak{A} \mid \lim_\alpha p_\alpha a = a\}$ as well as $\mathfrak{J} = \mathfrak{B} \cap \mathfrak{A}_{\mathfrak{J}}$. Moreover, $\mathfrak{A}_{\mathfrak{J}}$ is trivial if and only if $\mathfrak{J}$ has this property.*

Also here we are in a position where the cases for "left" and, respectively, "right" can be treated in a completely analogous way.

PROOF: (a) Since $\mathfrak{B}$ is a function algebra, Theorem 4.1 below shows that

$$\mathfrak{J}^{\perp\perp} = p.\mathfrak{B}^{\perp\perp}$$

for some $p \in \mathfrak{B}^{\perp\perp}$. We have $p \in \mathrm{Mult}^l_{inn}(\mathfrak{A})^{\perp\perp}$ and so, by the above lemma, $p \in \mathrm{Mult}^{l,2}_{inn}(\mathfrak{A}^{**})$. Since $\mathfrak{B}$ is unital and $\mathfrak{J}^{\perp\perp} \subset \mathfrak{A}_{\mathfrak{J}}^{\perp\perp}$ we have $p \in \mathfrak{A}_{\mathfrak{J}}^{\perp\perp}$. Furthermore, $pja_0 = ja_0$ for all $j \in \mathfrak{J}$ and each $a_0 \in \mathfrak{A}$ and hence, $pa = a$ for all $a \in \mathfrak{A}_{\mathfrak{J}}$. By weak* continuity of the second Arens multiplication, we have for each $F \in \mathfrak{A}_{\mathfrak{J}}^{\perp\perp}$ with $F = w^*\text{-}\lim_\alpha f_\alpha$ for some net $f_\alpha \in \mathfrak{A}_{\mathfrak{J}}$

$$p.F = w^*\text{-}\lim_\alpha pf_\alpha = F.$$

This shows that $p$ is a left unit for $\mathfrak{A}_{\mathfrak{J}}^{\perp\perp}$ with respect to the second Arens product. Since $\mathfrak{A}_{\mathfrak{J}}^{\perp\perp}$ is a right ideal, it follows that $\mathfrak{A}_{\mathfrak{J}}$ must be a right inner $M$-ideal. The net $(p_\alpha)$ is obtained just as in the proof of Theorem 3.2 with the aid of a net $(q_\alpha)$ in $B_{\mathfrak{J}}$ converging to $p$ in the $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$-topology.

(b) The first equation is immediate from (a), and the second is clear by virtue of

$$\mathfrak{J} = \{b \in \mathfrak{B} \mid \lim_\alpha p_\alpha b = b\} = \mathfrak{A}_{\mathfrak{J}} \cap \mathfrak{B}.$$

Finally, $\mathfrak{J} = \mathfrak{B}$ if and only if $p = e$ which in turn is equivalent to $\mathfrak{A}_{\mathfrak{J}} = \mathfrak{A}$. Similarly, $\mathfrak{J} = \{0\}$ is the same as $p = 0$ and $\mathfrak{A}_{\mathfrak{J}} = \{0\}$. □

**Corollary 3.6** *Let $X$ be a Banach space and $\mathfrak{J}$ an $M$-ideal of a subalgebra $\mathfrak{B}$ of* $\mathrm{Mult}(X)$ *with $Id \in \mathfrak{B}$. Then*

$$X_{\mathfrak{J}} := \overline{\mathrm{lin}}\,\{j(x) \mid j \in \mathfrak{J},\ x \in X\}$$

*is an $M$-ideal in $X$.*

Proof: Put $\mathfrak{A} = L(X)$ in the above proposition. Then, by Lemma VI.1.1 below, $\mathfrak{B} \subset \mathrm{Mult}^l_{inn}(\mathfrak{A})$, and, by the above, we find a net $(P_\alpha)$ in $B_{\mathfrak{J}}$ with

$$\limsup_\alpha \|P_\alpha T_1 + (Id - P_\alpha)T_2\| \le 1$$

for all $T_1, T_2 \in B_{L(X)}$. This easily implies that

$$\limsup_\alpha \|P_\alpha x_1 + (Id - P_\alpha)x_2\| \le 1$$

for all $x_1, x_2 \in B_X$. Moreover, for all $x \in X$, $(P_\alpha x)$ converges to $x$ in norm, and since the range of each $P_\alpha$ is contained in $X_{\mathfrak{J}}$, one can proceed exactly as in the proof of Theorem 3.2 to settle our claim. □

It remains an open problem whether a nontrivial $M$-ideal in $\mathrm{Mult}(X)$ generates a nontrivial $M$-ideal in $X$ in Corollary 3.6. What can be shown is that $X^{**}$ always contains a nontrivial $M$-ideal as soon as $\mathrm{Mult}(X)$ is known to be nontrivial [638, Korollar 4.15].
Let us conclude this section with an example which shows that there are $M$-ideals in Banach algebras which are neither inner nor ideals: Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital Banach algebras such that both

$$\mathrm{Mult}^l_{inn}(\mathfrak{A}_1) \setminus \mathrm{centre}(\mathfrak{A}_1)$$

and

$$\mathrm{Mult}^r_{inn}(\mathfrak{A}_2) \setminus \mathrm{centre}(\mathfrak{A}_2)$$

contain idempotent elements $l$ and $r$, respectively. Put $\mathfrak{A} := \mathfrak{A}_1 \oplus_\infty \mathfrak{A}_2$. Under pointwise multiplication, $\mathfrak{A}$ becomes a Banach algebra. By uniqueness of $M$-projections with a prescribed range (Proposition I.1.2) and by the assumptions made on $l$ and $r$,

$$L_{(l,0)} \neq R_{(l^*,0)} \qquad \text{and} \qquad R_{(0,r)} \neq L_{(0,r^*)}$$

for all idempotent elements $l^*$ and $r^*$ of $\mathfrak{A}_1$ resp. $\mathfrak{A}_2$. Then $(a,b) \longmapsto (la, br)$ defines an $M$-projection on $\mathfrak{A}$ which belongs neither to $\mathrm{Mult}^l_{inn}(\mathfrak{A})$ nor to $\mathrm{Mult}^r_{inn}(\mathfrak{A})$, and, consequently, the $M$-summand

$$\{(la, br) \mid a \in \mathfrak{A}_1, b \in \mathfrak{A}_2\}$$

is not inner, since it is not an ideal.
To be more specific, let $\mathfrak{A}_1 = L(\ell^\infty(2))$, $\mathfrak{A}_2 = L(\ell^1(2))$, and consider

$$l = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = r.$$

Then $T \mapsto lT$ is an $M$-projection on $\mathfrak{A}_1$, and $T \mapsto Tr$ is an $M$-projection on $\mathfrak{A}_2$ so that $l$ and $r$ meet the above requirements. This choice even yields a semisimple Banach algebra so that there is no hope to extend the results on $M$-ideals in commutative or $C^*$-algebras to be proved in the next section to the class of semisimple Banach algebras.

# V.4 Commutative and $C^*$-algebras

We investigate in this section the structure of the $M$-ideals in two rather common types of Banach algebras.
Whereas in the case of a $C^*$-algebra there is a neat correspondence between $M$-ideals and closed two-sided ideals, the situation for commutative Banach algebras is, due to the fact that commutativity generally doesn't have too deep an influence on the geometry of these spaces, less clear. The following result contains what can be said for this class of Banach algebras.

**Theorem 4.1** *In a unital commutative Banach algebra $\mathfrak{A}$ all $M$-ideals of $\mathfrak{A}$ are inner. In particular, $\mathfrak{J} \subset \mathfrak{A}$ is an $M$-ideal if and only if it is a closed ideal and there is an approximate unit $(p_\alpha)$ in $B_{\mathfrak{J}}$ with*

$$\limsup_\alpha \|p_\alpha s + (e - p_\alpha)t\| \leq 1 \qquad \forall s, t \in B_{\mathfrak{A}}.$$

Proof: We demonstrate that all $M$-ideals of $\mathfrak{A}$ are inner. The rest of the conclusion then follows from Theorem 3.2.
Now, by Theorem 2.3(b) we have $(e-z)\mathfrak{J}^{\perp\perp}(e-z) = 0$ – recall that $z$ denotes the image of the unit under the associated $M$-projection in the bidual of $\mathfrak{A}$ – which implies that $zj = jz = j$ for all $j \in \mathfrak{J}$. Consequently, $z$ is a unit for $\mathfrak{J}^{\perp\perp}$, whenever an Arens product is chosen that fits the direction from which $z$ multiplies (remember that even for commutative Banach algebras, $\mathfrak{A}$ need not be Arens regular). Since $z \in \mathfrak{J}^{\perp\perp}$, we must

have that $L_z^{(2)} = R_z^{(1)}$ is a projection onto $\mathfrak{J}^{\perp\perp}$ whose norm does not exceed one. But there is only one such thing (Proposition I.1.2), and the result follows. $\square$

The example of an ideal in the convolution algebra $L^1(G)$, where $G$ denotes a locally compact abelian group, shows in light of Theorem I.1.8 that no substantially better result is to be expected here. However in the special case that $\mathfrak{A}$ is a function algebra, more can be done.

**Theorem 4.2** *Let $\mathfrak{A}$ be a function algebra on a compact space $K$ and $\mathfrak{J}$ a subspace of $\mathfrak{A}$. Then the following conditions are equivalent:*

(i) *$\mathfrak{J}$ is the annihilator of a $p$-set of $\mathfrak{A}$.*

(ii) *$\mathfrak{J}$ is an ideal of $\mathfrak{A}$ containing a bounded approximate unit.*

(iii) *$\mathfrak{J}$ is an $M$-ideal of $\mathfrak{A}$.*

We recall that a $p$-set of a function algebra $\mathfrak{A}$ is an intersection of a family of peak sets where this last expression means a set of the form $D = \{a = 1\} = a^{-1}(\{1\})$ for some $a \in \mathfrak{A}$ with $\|a\| = 1$. What we need to know about these sets is contained in the following result.

**Lemma 4.3** *Let $\mathfrak{A}$ be a function algebra on a compact space $K$. Then a closed subset $D$ of $K$ is a $p$-set for $\mathfrak{A}$ if and only if for all $\varepsilon > 0$ and for every open set $U$ containing $D$ there exists $a \in \mathfrak{A}$ with*

$$\|e - a\| \le 1 + \varepsilon, \qquad a|_D = 0 \qquad \textit{and} \qquad \left|(e - a)|_{K\setminus U}\right| < \varepsilon.$$

Proof: The sufficiency follows from Bishop's "$\frac{1}{4}$-$\frac{3}{4}$-condition" [585, p. 47], and the necessity is a consequence of [239, Lemma II.12.2]. $\square$

Proof of Theorem 4.2:

(i) $\Rightarrow$ (ii): This conclusion follows easily from the above result.

(ii) $\Rightarrow$ (iii): Since $\mathfrak{A}$ embeds isometrically as well as algebraically into a $C(K)$-space, it follows that $\mathfrak{A}^{**}$ furnished with one of the Arens products (in fact, in this case both products coincide) again is a function algebra.
For this reason, the unit, which $\mathfrak{J}^{\perp\perp}$ must contain by assumption, is necessarily a characteristic function. It follows that $\mathfrak{J}$ is an $M$-ideal.

(iii) $\Rightarrow$ (i): Let $\mathfrak{J}$ be an $M$-ideal of $\mathfrak{A}$. The equation $\mathrm{Mult}^l_{inn}(C(K)) = C(K)$ permits us to apply Proposition 3.5, and there must be a closed set $D$ in $K$ with $J_D = C(K)_{\mathfrak{J}}$ and $\mathfrak{J} = J_D \cap \mathfrak{A}$. By the same result, we obtain an approximate unit $(p_\alpha)$ for the $M$-ideal $J_D$ of $C(K)$ contained in $B_{\mathfrak{J}} = B_{J_D} \cap \mathfrak{A}$ such that $\|e - p_\alpha\| \to 1$. It is easy to check that $(p_\alpha)$ can be characterised by the condition

$$\forall \varepsilon > 0 \; \forall U \supset D,\; U \,\text{open}\; \exists \alpha_0 \; \forall \alpha > \alpha_0 \quad \left|(e - p_\alpha)|_{K\setminus U}\right| < \varepsilon,$$

which is in light of Lemma 4.3 all we have to know in order to show that $D$ is a $p$-set for $\mathfrak{A}$. $\square$

Let us remark that the implication (i) $\Rightarrow$ (iii) can also be deduced from Proposition I.1.20; see the remarks preceding that result. Also, we note that one even gets a 1-approximative unit in (ii).

Another close relative of the class of Banach algebras of type $C(K)$ is the class of $C^*$-algebras, which will be treated next.

**Theorem 4.4** *Every $M$-ideal in a $C^*$-algebra $\mathfrak{A}$ is inner. There is, in addition, a one-to-one correspondence between the $M$-ideals and the closed two-sided ideals of $\mathfrak{A}$.*

PROOF: Let $\mathfrak{J}$ be an $M$-ideal of $\mathfrak{A}$. Since in this case an element is hermitian if and only if it is self-adjoint (Proposition 1.7), we find with the aid of Corollary I.3.15 that $h\mathfrak{J} \subset \mathfrak{J}$ and $\mathfrak{J}h \subset \mathfrak{J}$ for all $h \in \mathbb{H}(\mathfrak{A})$. Since $\mathfrak{A} = \mathbb{H}(\mathfrak{A}) + i\mathbb{H}(\mathfrak{A})$, the space $\mathfrak{J}$ must be a two-sided ideal. Since a two-sided closed ideal $\mathfrak{J}$ of a $C^*$-algebra is supported by a central idempotent $z$ (which has to be of norm one) in $\mathfrak{A}^{**}$, i.e. $\mathfrak{J}^{\perp\perp} = z\mathfrak{A}^{**} = \mathfrak{A}^{**}z$ (see Proposition 4.5(b) below), $\mathfrak{J}$ is inner. For the converse, suppose that $\mathfrak{J}$ is a two-sided ideal, and again, denote by $z$ the supporting idempotent. We have ($\mathfrak{A}^{**}$ always has a unit, see below)

$$\begin{aligned} \|a\|^2 &= \|(az + a(e-z))^*(az + a(e-z))\| \\ &= \|a^*az + a^*a(e-z)\| \\ &= \max\{\|a^*az\|, \|a^*a(e-z)\|\}, \end{aligned}$$

since the last equation takes place in the commutative $C^*$-algebra generated by $z$ and $a^*a$. Because $z$ is self-adjoint, central and idempotent, the last expression becomes

$$\max\{\|az\|^2, \|a(e-z)\|^2\},$$

and hence, multiplication with $z$ yields an $M$-projection on $\mathfrak{A}^{**}$. It follows that $\mathfrak{J}^{\perp\perp}$ and $\mathfrak{A}^{**}(e-z)$ are complementary $M$-summands in $\mathfrak{A}^{**}$ and so, $\mathfrak{J}^{\perp}$ is an $L$-summand (see Theorem I.1.9) in $\mathfrak{A}^*$: $\mathfrak{J}$ must be an inner $M$-ideal. ◻

We still owe the following proposition. A $C^*$-algebra $\mathfrak{A}$ is called a $W^*$-algebra if, as a Banach space, it is a dual space. Equivalently (e.g. [592, p. 133]), a $W^*$-algebra is a unital $C^*$-subalgebra of $L(H)$ which is closed for the weak operator topology; usually the latter type of algebra is called a von Neumann algebra. It is known (e.g. [592, p. 135]) that a $W^*$-algebra has a uniquely determined predual so that we can unambiguously make reference to its weak$^*$ topology. In the next proposition we call a $W^*$-algebra a *$D^*$-algebra* if multiplication from the left and the right and the $*$-operation are weak$^*$ continuous. As a matter of fact, *every* $W^*$-algebra is a $D^*$-algebra [554, p. 18]. However, in order to keep the proof of Proposition 4.5 self-contained we did not want to rely on that result. Instead, what we will use here is the fact that, for a $C^*$-algebra $\mathfrak{A}$, the algebra $\mathfrak{A}^{**}$ is a $D^*$-algebra, which we proved in Theorem 1.10.

**Proposition 4.5**

(a) *A $D^*$-algebra has a unit. In particular, the bidual of any $C^*$-algebra is unital.*

(b) *For every weak$^*$ closed two-sided ideal $\mathfrak{J}$ of a $D^*$-algebra $\mathfrak{A}$ there is a central idempotent $p$ in $\mathfrak{A}$ such that*

$$\mathfrak{J} = p\mathfrak{A}.$$

*It follows in particular that for any two-sided ideal $\mathfrak{J}$ in an arbitrary $C^*$-algebra $\mathfrak{A}$ there is a central idempotent $p$ in $\mathfrak{A}^{**}$ such that $\mathfrak{J}^{\perp\perp} = p\mathfrak{A}^{**}$.*

PROOF: Since (b) is an easy consequence of (a) (let $p =$ the unit of $\mathfrak{J}$), we content ourselves with a proof of (a).

Fix a maximal abelian $C^*$-subalgebra $\mathfrak{M}$ of $\mathfrak{A}$, which is necessarily weak$^*$ closed by the $D^*$-condition. It follows that $\mathfrak{M}$ is a dual space, and, consequently, $B_{\mathfrak{M}}$ must contain an extreme point. Therefore, $\mathfrak{M} \cong C(K)$ for some compact (not only locally compact) $K$, so $\mathfrak{M}$ has a unit, $e$. Let $a \in \mathfrak{A}$ and put $h = (a - ae)^*(a - ae)$. Since $e$ is a unit for $\mathfrak{M}$, we clearly have $h\mathfrak{M} = 0$, and likewise $\mathfrak{M}h = 0$. Hence, because $\mathfrak{M}$ is maximal abelian, $h \in \mathfrak{M}$. It follows that $h = 0$ and thus $ae = a$. Similarly $ea = a$, so we have found a unit for $\mathfrak{A}$. ◻

Theorem 4.4 above enables us to give an example which was promised in Section I.1, see Remark (c) following Theorem I.1.9.

**Proposition 4.6** *There is a Banach space $X$ without nontrivial $M$-ideals such that $X^*$ contains a nontrivial $L$-summand. In fact, there is a Banach space $X$ without nontrivial $M$-ideals such that $X^*$ contains uncountably many nontrivial pairwise not isometrically isomorphic $L$-summands.*

PROOF: The first-mentioned example has already been produced in Example IV.1.8.

For the harder example we rely on the deep results of R. T. Powers [508]. Let $X$ be the UHF-algebra of type $(2^n)$, that is the norm closure of all operators on $\ell^2$ which have, for some $n \in \mathbb{N}$, a block diagonal matrix representation

$$\begin{pmatrix} A & 0 & 0 & \dots \\ 0 & A & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

with some $2^n \times 2^n$-matrix $A$. It is known that $X$ is a $C^*$-algebra without nontrivial two-sided ideals [361, Prop. 10.4.18], yet $X^{**}$ has uncountably many weak$^*$ closed two-sided ideals. This is an easy corollary of the work of Powers who proved in [508, Th. 4.8] (see also Theorem 12.3.14 in [361]) that there is a family $\{\rho_\lambda |\ 0 \le \lambda \le \frac{1}{2}\}$ of states on $X$ with corresponding representations $\pi_\lambda : X \to L(H_\lambda)$ such that the von Neumann algebra $M_\lambda :=$ the bicommutant of $\pi_\lambda(X)$ in $L(H_\lambda)$, i.e. $M_\lambda = \overline{\pi_\lambda(X)}^{w*}$, is $*$-isomorphic to $M_\mu$ if and only if $\lambda = \mu$. Now $\pi_\lambda$ has a unique weak$^*$ continuous extension to a $*$-homomorphism $\widehat{\pi}_\lambda$ from $X^{**}$ onto $M_\lambda$ (see e.g. [592, p. 121]) so that $\mathfrak{J}_\lambda := \ker(\widehat{\pi}_\lambda)$ is a weak$^*$ closed two-sided ideal in $X^{**}$. Powers's result entails $\mathfrak{J}_\lambda \neq \mathfrak{J}_\mu$ if $\lambda \neq \mu$ since otherwise ($\overset{*}{\cong}$ meaning "$*$-isomorphic to")

$$M_\lambda \overset{*}{\cong} X^{**}/\mathfrak{J}_\lambda = X^{**}/\mathfrak{J}_\mu \overset{*}{\cong} M_\mu.$$

Hence $X^{**}$ contains uncountably many weak$^*$ closed $M$-ideals which have to be $M$-summands by Corollary II.3.6(b). Theorem I.1.9 now shows that there are uncountably many $L$-summands in $X^*$. No two of these $L$-summands are isometrically isomorphic, since otherwise two of the $M_\lambda$ would be isometrically isomorphic and hence by a result due to Kadison [356] even $*$-isomorphic. □

Our next aim is to show that also the centralizer admits a purely algebraic interpretation in the case of $C^*$-algebras. To this end recall that the centroid of an (arbitrary) algebra $\mathfrak{A}$ is defined as the set of all those linear mappings defined on $\mathfrak{A}$ which satisfy the relation

$$T(ab) = aT(b) = T(a)b$$

for all $a, b \in \mathfrak{A}$. Whenever $\mathfrak{A}$ is a $C^*$-algebra or more generally a Banach algebra where the multiplication separates points (meaning $\forall a \neq 0\ \exists b \neq 0 :\ ab \neq 0$), such a mapping is continuous, as a standard application of the closed graph theorem shows. Note further that the centroid can be identified with the centre of $\mathfrak{A}$ whenever $\mathfrak{A}$ has a unit.

**Theorem 4.7** *Let $\mathfrak{A}$ be a $C^*$-algebra and $T \in L(\mathfrak{A})$. Then the following are equivalent:*

(i) *$T$ is in $Z(\mathfrak{A})$.*
(ii) *$T$ belongs to the centroid of $\mathfrak{A}$.*
(iii) *There is an element $a^{**}$ in the centre of $\mathfrak{A}^{**}$ such that $a^{**}\mathfrak{A} \subset \mathfrak{A}$ and $T = M_{a^{**}}$.*

Proof: (i) ⇒ (ii): Since the operators in $Z(\mathfrak{A})$ commute with $L_h$ for any hermitian element $h$ (Corollary I.3.15) we have $T(hb) = hT(b)$ for all $h \in \mathbb{H}(\mathfrak{A})$ and every $b \in \mathfrak{A}$. Since $\mathfrak{A} = \mathbb{H}(\mathfrak{A}) + i\mathbb{H}(\mathfrak{A})$ we must have $T(ab) = aT(b)$ for all $a \in \mathfrak{A}$. Similarly, $T(ab) = T(a)b$ for all $a, b \in \mathfrak{A}$, and this gives (ii).

(ii) ⇒ (iii): Let $T$ be an element in the centroid. Since $\mathfrak{A}$ is Arens regular (Theorem 1.10(a)), we have

$$T^{**}(FG) = T^{**}(F)G = FT^{**}(G)$$

for all $F, G \in \mathfrak{A}^{**}$, and because $\mathfrak{A}^{**}$ always has a unit (Proposition 4.5(a)), it follows

$$T^{**}(F) = T^{**}(e)F = FT^{**}(e).$$

Hence, putting $a^{**} = T^{**}(e)$, we clearly have $a^{**}\mathfrak{A} \subset \mathfrak{A}$, and so, $a^{**}$ is what we were searching for.

(iii) ⇒ (i): Since the centre of $\mathfrak{A}^{**}$ is weak$^*$ closed (this follows e.g. from the fact that $\mathfrak{A}$ is Arens regular (Theorem 1.10(a))), it is algebraically isomorphic to a dual $C(K)$-space. As a consequence, the elements of the form $\sum_{i=1}^n \lambda_i p_i$, where the $\lambda_i$ are complex numbers and the $p_i$ are idempotent, are dense. We are thus left with showing that each central idempotent $p \in \mathfrak{A}^{**}$ gives rise to an $M$-projection. But this has been done in the proof of Theorem 4.4. □

We next wish to give an application to the homological theory of Banach algebras. To this end, we have to recall some definitions. Unfortunately, to keep things within reasonable bounds, we must refer the reader to the literature for almost all details.

A *cochain complex* $C$ is a sequence of vector spaces $C^n$, $n \geq 0$, together with linear maps $\partial^n$, $n \geq -1$,

$$0 \xrightarrow{\partial^{-1}} C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \xrightarrow{\partial^2} \cdots$$

such that $\partial^n\partial^{n-1} = 0$ for all $n \in \mathbb{N}$. We put

$$\begin{aligned} Z^n(C) &= \ker \partial^n \\ B^n(C) &= \operatorname{ran} \partial^{n-1} \\ H^n(C) &= Z^n(C)/B^n(C) \end{aligned}$$

so that $H^n(C)$ measures failure of exactness of the cochain complex at stage $n$. In particular, $H^n(C) = 0$ if and only if $C$ is exact at $C^n$, i.e. $\ker \partial^n = \operatorname{ran} \partial^{n-1}$. A *cochain homomorphism* $\psi$ between cochain complexes $C_1$ and $C_2$ is a sequence $(\psi^n)_{n\geq 0}$ of linear mappings $\psi^n : C_1^n \to C_2^n$ such that $\partial_2^{n-1}\psi^{n-1} = \psi^n\partial_1^{n-1}$ for all $n \in \mathbb{N}$. A diagram of cochain complexes

$$\cdots \xrightarrow{\psi} C \xrightarrow{\varphi} \cdots$$

is said to be exact at $C$ if $\operatorname{ran}\psi^n = \ker\varphi^n$ for all $n$. Let

$$0 \longrightarrow C_1 \xrightarrow{\psi} C_2 \xrightarrow{\varphi} C_3 \longrightarrow 0$$

be an exact sequence of cochain complexes. (Typically, $C_1$ is a complex of subspaces of the members of $C_2$, whereas $C_3$ is a complex of quotients of the $C_2^n$. Knowledge of $H(C_1)$ and $H(C_3)$ together with the procedure described below then facilitates the computation of $H(C_2)$.) Then there is always a *long exact cohomology sequence*

$$0 \longrightarrow H^0(C_1) \xrightarrow{H^0(\psi)} H^0(C_2) \xrightarrow{H^0(\varphi)} H^0(C_3) \xrightarrow{d^0} H^1(C_1) \xrightarrow{H^1(\psi)} \cdots .$$

The maps $H^n(\psi)$ and $H^n(\varphi)$ are (well-) defined by

$$\begin{aligned} H^n(\psi)([z_1^n]) &= [\psi^n(z_1^n)] \qquad z_1^n \in Z^n(C_1), \\ H^n(\varphi)([z_2^n]) &= [\varphi^n(z_2^n)] \qquad z_2^n \in Z^n(C_2), \end{aligned}$$

where we used brackets to indicate cohomology classes. For the definition of $d^n$ note that, by exactness, each $\varphi^n$ is surjective, and this, together with a bit of chasing diagrams yields that

$$d^n([z_3^n]) = [(\psi^{n+1})^{-1}\partial^{n+1}(\varphi^n)^{-1}(z_3^n)] \qquad z_3^n \in Z^n(C_3)$$

is in fact well-defined. For more details, see [310, IV.2].

We now come to a more specific example. Let $\mathfrak{A}$ be a Banach algebra and denote by $\mathfrak{X}$ a two-sided Banach $\mathfrak{A}$-module. (By this we mean that $\mathfrak{A}$ operates uniformly continuously from the left and the right on the Banach space $\mathfrak{X}$ such that, for all $a, b \in \mathfrak{A}$ and any $x \in \mathfrak{X}$, the associative laws $(ab)x = a(bx)$ and $x(ab) = (xa)b$ hold.) We write $C^n(\mathfrak{A}, \mathfrak{X})$ for the space of $n$-linear maps from $\mathfrak{A}^n$ into $\mathfrak{X}$ whenever $n > 0$. We furthermore put $C^0(\mathfrak{A}, \mathfrak{X}) = \mathfrak{X}$. Define mappings $\delta^n : C^{n-1}(\mathfrak{A}, \mathfrak{X}) \longrightarrow C^n(\mathfrak{A}, \mathfrak{X})$ for every $n \in \mathbb{N}$ as follows: Whenever $T \in C^{n-1}(\mathfrak{A}, \mathfrak{X})$ then

$$\begin{aligned} \delta^n T(a_1, \ldots, a_n) &= a_1 T(a_2, \ldots, a_n) - T(a_1 a_2, \ldots, a_n) \\ &\quad + T(a_1, a_2 a_3, \ldots, a_n) - \ldots + (-1)^{n-1} T(a_1, \ldots, a_{n-1}a_n) \\ &\quad + (-1)^n T(a_1, \ldots, a_{n-1}) a_n. \end{aligned}$$

Note that $\delta^{n+1}\delta^n = 0$, and so $C(\mathfrak{A},\mathfrak{X}) := (C^n(\mathfrak{A},\mathfrak{X}))_{n\geq 0}$ becomes a cochain complex. Note also that each map $\delta^n$ is bounded. The resulting spaces

$$H^n(\mathfrak{A},\mathfrak{X}) := H^n(C(\mathfrak{A},\mathfrak{X}))$$

are usually called *n-th (Hochschild) cohomology groups*. For a nice explanation of the ideas underlying this construction, see for example [348] or [522]. Of course, in each of these groups some information on the $\mathfrak{A}$-bimodule $\mathfrak{X}$ and hence on $\mathfrak{A}$ itself is encoded. To illustrate this point by means of an example, we recall that, by a result of B. E. Johnson, a locally compact group $G$ is amenable if and only if $H^1(L^1(G),\mathfrak{X}^*) = 0$ for every dual Banach $L^1(G)$-bimodule. (The latter means that the operation of $\mathfrak{A}$ on $\mathfrak{X}^*$ is in addition weak$^*$ continuous.) For more on this topic the reader should consult e.g. [347] or, for some more recent developments, [301] and [369].
Suppose that $\mathfrak{Y}$ is an $\mathfrak{A}$-submodule of $\mathfrak{X}$. Then there is an exact sequence

$$0 \longrightarrow C(\mathfrak{A},\mathfrak{Y}) \stackrel{i}{\longrightarrow} C(\mathfrak{A},\mathfrak{X}) \stackrel{q}{\longrightarrow} C(\mathfrak{A},\mathfrak{X}/\mathfrak{Y}) \longrightarrow 0,$$

where $i^n : C^n(\mathfrak{A},\mathfrak{Y}) \to C^n(\mathfrak{A},\mathfrak{X})$ simply enlarges the range of a multilinear mapping from $\mathfrak{A}$ to $\mathfrak{Y}$, and $q^n : C^n(\mathfrak{A},\mathfrak{X}) \to C^n(\mathfrak{A},\mathfrak{X}/\mathfrak{Y})$ compresses a map $T \in C^n(\mathfrak{A},\mathfrak{X})$ with the aid of the quotient map $\pi_{\mathfrak{Y}} : \mathfrak{X} \to \mathfrak{X}/\mathfrak{Y}$ to $\pi_{\mathfrak{Y}}T$. There are, in general, two points which deserve some additional attention in constructing connecting homomorphisms $d^n$ as above: surjectivity of the mappings $H^n(\varphi)$ and continuity of the resulting homomorphisms. That these conditions can be satisfied is, in general, by no means clear. One situation in which these problems can be remedied is that in which $M$-ideals enter the scene.

**Proposition 4.8** *Suppose that $\mathfrak{A}$ is a separable Banach algebra, that $\mathfrak{X}$ is a Banach $\mathfrak{A}$-bimodule and that $\mathfrak{Y}$ is a submodule which is an $M$-ideal in $\mathfrak{X}$ at the same time. Assume further that either*
(a) *$\mathfrak{A}$ has the $\lambda$-approximation property or*
(b) *$\mathfrak{Y}^*$ is an $L^1$-space.*
*Then one has a long exact sequence*

$$0 \longrightarrow H^0(\mathfrak{A},\mathfrak{Y}) \stackrel{H^0(i)}{\longrightarrow} H^0(\mathfrak{A},\mathfrak{X}) \stackrel{H^0(q)}{\longrightarrow} H^0(\mathfrak{A},\mathfrak{X}/\mathfrak{Y}) \stackrel{d^0}{\longrightarrow} H^1(\mathfrak{A},\mathfrak{Y}) \stackrel{H^1(i)}{\longrightarrow} \cdots$$

*such that all the connecting homomorphisms $d^0, d^1, \dots$ are bounded.*

Proof: Let $T = \delta^n_{\mathfrak{X}/\mathfrak{Y}} T_0 \in Z^n(\mathfrak{A},\mathfrak{X}/\mathfrak{Y})$ be given. We shall tacitly make use of the identification $C^n(\mathfrak{A},\mathfrak{X}) = L(\mathfrak{A}\widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \mathfrak{A},\mathfrak{X})$. By hypothesis and the fact that separability as well as the bounded approximation property – with a different constant, though – pass to finite projective tensor products, we find, with the aid of Theorem II.2.1, a lifting $\widetilde{T} \in C^n(\mathfrak{A},\mathfrak{X})$ for $T$ with $\|\widetilde{T}\| \leq \lambda^n \|T\|$ when (a) is satisfied and with $\|\widetilde{T}\| = \|T\|$ whenever (b) holds. It follows that

$$q^{n+1}\delta^{n+1}_{\mathfrak{X}}\widetilde{T} = \delta^{n+1}_{\mathfrak{X}/\mathfrak{Y}} q^n \widetilde{T} = \delta^{n+1}_{\mathfrak{X}/\mathfrak{Y}} \delta^n_{\mathfrak{X}/\mathfrak{Y}} T_0 = 0,$$

so $\delta_{\mathfrak{X}}^{n+1}\widetilde{T}$ takes its values in $\mathfrak{Y}$, and, formally, there is $T_1 \in C^{n+1}(\mathfrak{A}, \mathfrak{Y})$ with $i^{n+1}T_1 = \delta^{n+1}\widetilde{T}$. Put $d^n(T) = [T_1]$, the equivalence class of $T_1$, and convince yourself (if you didn't before) that $d^n$ is well-defined. Since

$$\|d^n(T)\| = \|T_1\| = \|i^{n+1}T_1\| = \|\widetilde{T}\| \leq \lambda^n \|T\|$$

if (a) is valid and a similar estimate holds in case (b), $d^n$ is bounded, and we are done. □

Using Theorem 4.4 and the above proposition one obtains:

**Corollary 4.9** *Suppose that $\mathfrak{A}$ is a separable $C^*$-subalgebra of the $C^*$-algebra $\mathfrak{B}$ and that $\mathfrak{J}$ is a two-sided closed ideal of $\mathfrak{B}$. If either $\mathfrak{A}$ has the bounded approximation property or else, if $\mathfrak{J}$ is commutative, then one has a long exact sequence*

$$0 \longrightarrow H^0(\mathfrak{A}, \mathfrak{J}) \xrightarrow{H^0(i)} H^0(\mathfrak{A}, \mathfrak{B}) \xrightarrow{H^0(q)} H^0(\mathfrak{A}, \mathfrak{B}/\mathfrak{J}) \xrightarrow{d^0} H^1(\mathfrak{A}, \mathfrak{J}) \xrightarrow{H^1(i)} \cdots$$

*with bounded connecting homomorphisms $d^n$.*

We end our short digression to the homological theory of $C^*$-algebras with an illustration of how long exact sequences help one with calculations. (The proof below is, however, far from being self-contained.) Recall that a von Neumann algebra is called approximately finite dimensional if it is the (von Neumann) inductive limit of finite dimensional von Neumann algebras.

**Corollary 4.10** *Suppose $\mathfrak{A} \subset L(H)$ is a separable $C^*$-algebra enjoying the bounded approximation property such that the closure of $\mathfrak{A}$ in the weak operator topology is an approximately finite dimensional von Neumann algebra. Then*

$$H^n(\mathfrak{A}, L(H)/K(H)) \cong H^{n+1}(\mathfrak{A}, K(H))$$

*for all $n \in \mathbb{N}$.*

Proof: Write $\mathfrak{A}^-$ for the closure of $\mathfrak{A}$ in the weak operator topology. One of the main results in [350] (Theorem 6.1) shows that we have in the present situation

$$H^n(\mathfrak{A}, L(H)) = H^n(\mathfrak{A}^-, L(H)).$$

By a result of Kadison and Ringrose (see e.g. [522, Theorem 7.4]), $H^n(\mathfrak{A}^-, L(H)) = 0$. By Corollary 4.9 we then have an exact sequence

$$0 = H^n(\mathfrak{A}, L(H)) \to H^n(\mathfrak{A}, L(H)/K(H)) \to H^{n+1}(\mathfrak{A}, K(H)) \to H^{n+1}(\mathfrak{A}, L(H)) = 0.$$

But this clearly implies that $H^n(\mathfrak{A}, L(H)/K(H)) \cong H^{n+1}(\mathfrak{A}, K(H))$, which is what we had to show. □

It should be remarked that there is one important class of $C^*$-algebras which satisfies the hypothesis of the above result: It is known that a nuclear $C^*$-algebra $\mathfrak{A}$ always has the metric approximation property (see [123] or [375] for these matters) and that $\mathfrak{A}^{**}$ is approximately finite dimensional in this case ([122] and [200]). For more on this topic see the Notes and Remarks section.

## V.5 Some further approximation theorems

The chief concern of this section is to present some results similar to Theorem 3.2. We preferred to collect them at this place, mainly because they all exhibit more or less the same idea: Whenever $P : \mathfrak{A}^{**} \to \mathfrak{J}^{\perp\perp}$ is the $M$-projection corresponding to an $M$-ideal $\mathfrak{J}$, then all of these results aim at finding a net $(z_\alpha)$ converging to $z = P(e)$ in the weak$^*$ sense subject to some constraints that have shown to be useful in special situations.
Our first result in this chain will find its application in the following section, when it will be shown that the $M$-ideals of $L(X)$ are inner whenever $X$ is uniformly smooth or uniformly convex.

**Proposition 5.1** *Suppose $\mathfrak{A}$ is a unital Banach algebra, and let $\mathfrak{J}$ be an $M$-ideal in $\mathfrak{A}$. Denote by $P$ be the associated $M$-projection on $\mathfrak{A}^{**}$ and put $z = P(e)$. Then $\mathfrak{J}$ is left inner if and only if there is a net $(t_\alpha) \subset \mathfrak{J}$ approximating $z$ in the weak$^*$ topology such that for all $a \in \mathfrak{A}$*

$$\lim_{\lambda \to 0} \limsup_{\alpha} \frac{\|t_\alpha + \lambda a(e - t_\alpha)\| - 1}{\lambda} = 0$$

*and*

$$\lim_{\lambda \to 0} \limsup_{\alpha} \frac{\|\lambda a t_\alpha + e - t_\alpha\| - 1}{\lambda} = 0.$$

*An analogous result holds in the case of right inner $M$-ideals.*

Proof: That left inner $M$-ideals always admit nets of the above type is an immediate consequence of Theorem 3.2: If $(t_\alpha)$ is a net the existence of which was shown by this result, then

$$\lim_{\lambda \to 0} \limsup_{\alpha} \frac{\|t_\alpha + \lambda a(e - t_\alpha)\| - 1}{\lambda} \le \lim_{\lambda \to 0} \frac{\max\{\lambda\|a\|, 1\} - 1}{\lambda} = 0,$$

and one similarly obtains the other equation. For the converse, suppose $\mathfrak{A}^* = \mathfrak{J}^\perp \oplus_1 \mathfrak{J}^*$, and let us first prove that

$$\mathfrak{A}^{**}(e - z) \subset (\mathfrak{J}^*)^\perp.$$

By weak$^*$ continuity of the first Arens product, and since $(\mathfrak{J}^*)^\perp$ is weak$^*$ closed, we may reduce our efforts and show only that

$$\mathfrak{A}(e - z) \subset (\mathfrak{J}^*)^\perp.$$

This holds, in light of Lemma 2.2, once it has been demonstrated that $\varphi(a(e - z)) = 0$ for all $a \in \mathfrak{A}$ and $\varphi \in F_1$. (Here, $F_1$ stands for the set $\mathsf{S}_{\mathfrak{A}} \cap \mathfrak{J}^*$.) Let us suppose to the contrary that for some $a \in \mathfrak{A}$ and an appropriate $\varphi \in F_1$

$$\varphi(a(e - z)) = \lambda_0,$$

with $0 < \lambda_0 < 1$. Put

$$a_n = z + \lambda_0^n a(e - z).$$

Then, by assumption on $z$,

$$\begin{aligned}\lambda_0 &= \frac{\varphi(a_n)-1}{\lambda_0^n} \\ &= \lim_\alpha \frac{\varphi(t_\alpha+\lambda_0^n a(e-t_\alpha))-1}{\lambda_0^n} \\ &\leq \limsup_\alpha \frac{\|t_\alpha+\lambda_0^n a(e-t_\alpha)\|-1}{\lambda_0^n}\end{aligned}$$

and hence

$$\lambda_0 \leq \lim_{n\to\infty} \limsup_\alpha \frac{\|t_\alpha+\lambda_0^n a(e-t_\alpha)\|-1}{\lambda_0^n} = 0,$$

which contradicts our assumption on $\lambda_0$. Repeating the same argument with the expression

$$\lambda_0 a t_\alpha + e - t_\alpha$$

will give $\mathfrak{A}^{**}z \subset \mathfrak{J}^{\perp\perp}$. But this permits us to infer that $R_z^{(1)}$ is actually onto, and the result once again follows by an application of Proposition I.1.2. □

We wish to present a second approximation theorem of this type (Theorem 5.4 below). This will be prepared by a result deserving independent interest (Theorem 5.3), which in turn requires the following lemma.

**Lemma 5.2** *Let $X$ be a Banach space and suppose $S \subset X^*$ is convex and weak$^*$ compact. If $\Omega$ is a compact convex subset of the complex plane, then*

$$\overline{\mathrm{co}}^{w*}\, \Omega S = \overline{\mathrm{co}}^{\|\ \|}\, \Omega S.$$

PROOF: Fix $\delta > 0$ and pick $z_1, \ldots, z_n \in \mathbb{C}$ with

$$\Omega \subset \Omega_\delta := \mathrm{co}\,\{z_1, \ldots, z_n\} \subset (1+\delta)\Omega.$$

Since $\mathrm{co}\,\Omega_\delta S = \mathrm{co}\,(\bigcup_{j=1}^n z_j S)$ is weak$^*$ compact and thus norm closed, we have

$$\overline{\mathrm{co}}^{w*}\, \Omega S \subset \mathrm{co}\,\Omega_\delta S \subset (1+\delta)\overline{\mathrm{co}}^{\|\ \|}\, \Omega S.$$

Letting $\delta \to 0$ gives $\overline{\mathrm{co}}^{w*}\, \Omega S \subset \overline{\mathrm{co}}^{\|\ \|}\, \Omega S$, which is the less trivial inclusion. □

**Theorem 5.3** *Let $\mathfrak{A}$ be a unital Banach algebra and suppose $F \in \mathfrak{A}^{**}$. Then, for some net $(\varepsilon_\alpha)$ of positive numbers tending to zero, there is a net $(f_\alpha) \subset \mathfrak{A}$ converging to $F$ in the weak$^*$ topology with* $\limsup_\alpha \|f_\alpha\| \leq \|F\|$ *and*

$$v(f_\alpha) \subset v(F) + B(0, \varepsilon_\alpha).$$

PROOF: Let $M$ and $M^*$ be subsets of a Banach space $X$ and its dual, respectively. In the following, we denote the *real* polar of $M$ by

$$M^\pi := \{x^* \in X^* \mid \mathrm{Re}\, x^*(x) \leq 1\ \forall x \in M\}$$

and write $M^*_\pi$ for the set

$$M^*_\pi := \{x \in X \mid \operatorname{Re} x^*(x) \le 1 \ \forall x^* \in M^*\}.$$

In addition, $v_*(F)$ denotes the *lower* numerical range of $F \in \mathfrak{A}^{**}$, which is defined by

$$v_*(F) = \{F(\psi) \mid \psi \in \mathsf{S}_\mathfrak{A}\}.$$

Suppose that $\Omega$ is a compact convex subset of the plane containing an interior point. Our claim is then – except for the additional norm condition – equivalent to showing that, for any such set,

$$\overline{\{a \in \mathfrak{A} \mid v(a) \subset \Omega\}}^{w*} = \{F \in \mathfrak{A}^{**} \mid v(F) \subset \Omega\},$$

where the first set is identified with its canonical image in $\mathfrak{A}^{**}$. Since $v(\lambda e + a) = \lambda + v(a)$ for all $\lambda \in \mathbb{C}$ and every $a \in \mathfrak{A}$, we may suppose that $0 \in \operatorname{int}\Omega$ without spoiling the argument. Then $0 \in \{a \in \mathfrak{A} \mid v(a) \subset \Omega\}$ and hence, by the bipolar theorem, the above reduces to

$$\{a \in \mathfrak{A} \mid v(a) \subset \Omega\}^{\pi\pi} = \{F \in \mathfrak{A}^{**} \mid v(F) \subset \Omega\}.$$

Now, $\{a \in \mathfrak{A} \mid v(a) \subset \Omega\} = (\Omega^\pi \mathsf{S}_\mathfrak{A})_\pi$ and similarly

$$(\Omega^\pi \mathsf{S}_\mathfrak{A})^\pi = \{F \in \mathfrak{A}^{**} \mid v_*(F) \subset \Omega\} = \{F \in \mathfrak{A}^{**} \mid v(F) \subset \Omega\},$$

where the last equation holds by virtue of $\overline{v_*(F)} = v(F)$ (Lemma 1.9). By assumption, $0 \in \operatorname{int}\Omega$, so $\Omega^\pi$ is compact and convex, which together with Lemma 5.2 gives

$$\overline{\text{co}}^{\,w*}\, \Omega^\pi \mathsf{S}_\mathfrak{A} = \overline{\text{co}}^{\,\|\,\|}\, \Omega^\pi \mathsf{S}_\mathfrak{A}$$

Putting all the pieces together, we obtain

$$\begin{aligned}\{a \in \mathfrak{A} \mid v(a) \subset \Omega\}^{\pi\pi} &= (\overline{\text{co}}^{\,w*}\, \Omega^\pi \mathsf{S}_\mathfrak{A})^\pi \\ &= (\Omega^\pi \mathsf{S}_\mathfrak{A})^\pi \\ &= \{F \in \mathfrak{A}^{**} \mid v(F) \subset \Omega\},\end{aligned}$$

which gives the claim. We now adjust the net thus obtained by a convex combinations argument so as to satisfy the condition $\limsup_\alpha \|f_\alpha\| \le \|F\|$. To do so pick any net $(g_\alpha)$ in $\mathfrak{A}$ such that $\|g_\alpha\| \le \|F\|$ and $g_\alpha \xrightarrow{w*} F$. (One may employ the same index set.) Then $f_\alpha - g_\alpha \xrightarrow{w} 0$ so that there are $\overline{f}_\alpha \in \operatorname{co}\{f_\beta \mid \beta \ge \alpha\}$ and $\overline{g}_\alpha \in \operatorname{co}\{g_\beta \mid \beta \ge \alpha\}$ such that $\|\overline{f}_\alpha - \overline{g}_\alpha\| \to 0$. Hence $(\overline{f}_\alpha)$ still has the numerical range property, but also inherits the desired norm property from the net $(\overline{g}_\alpha)$. ◻

Since the $\overline{f}_\alpha$ are "convex blocks" of the $f_\alpha$, we shall refer to the above convex combinations argument (which we shall use several times later and have first encountered in the proof of Theorem 3.2) also as the "blocking technique".

**Theorem 5.4** *Let $\mathfrak{A}$ be a unital Banach algebra, and suppose that $\mathfrak{J}$ is an $M$-summand in $\mathfrak{A}^{**}$. Denote by $P$ the corresponding $M$-projection. Then there is a net $(z_\alpha) \subset \mathfrak{A}$ with weak* limit $z = P(e)$ such that for a suitable net of positive numbers $(\varepsilon_\alpha)$ tending to zero*

$$v(z_\alpha) \subset [0,1] + B(0, \varepsilon_\alpha)$$

*and furthermore*

$$\lim_\alpha \|\mu(e - z_\alpha) + \lambda z_\alpha\| = \max\{|\lambda|, |\mu|\}$$

*for all complex numbers $\lambda, \mu$. If $\mathfrak{J}$ is the bipolar of an $M$-ideal $\mathfrak{J}_0 \subset \mathfrak{A}$, then we may suppose that $z_\alpha \in \mathfrak{J}_0$ for all $\alpha$.*

PROOF: We first show how to produce a net for which the second relation holds. To do this, one has to observe that for an $M$-projection $P$ the equation

$$\|\mu(Id - P) + \lambda P\| = \max\{|\mu|, |\lambda|\}$$

holds. Since $\Delta$ is isometric on $\mathfrak{A}^{**}$ (Theorem 2.1(e)), it follows that

$$\max\{|\mu|, |\lambda|\} = \|\mu(Id - P) + \lambda P\| = \|\mu(Id - P)(e) + \lambda P(e)\|.$$

Let $(T_\alpha)$ be a net of operators gained with the aid of the local reflexivity principle similarly to the proof of Theorem 3.2. Then it follows for $z^0_\alpha := T_\alpha(P(e))$

$$\begin{aligned} \lim_\alpha \|\mu(e - z^0_\alpha) + \lambda z^0_\alpha\| &= \lim_\alpha \|\mu(e - T_\alpha(P(e))) + \lambda T_\alpha(P(e))\| \\ &= \lim_\alpha \|T_\alpha(\mu(e - P(e)) + \lambda P(e))\| \\ &= \max\{|\mu|, |\lambda|\}. \end{aligned}$$

A blocking technique similar to the one used in the proof of Theorem 3.2 allows us to pass to the required net $(z_\alpha)$ in $B_{\mathfrak{J}}$.
A net which fulfills the requirements concerning the numerical range is provided by Theorem 5.3 (note $v(P) = [0, 1]$); and by again using our blocking technique we obtain a net sharing both properties simultaneously. □

## V.6 Inner $M$-ideals in $L(X)$

In this last section of the present chapter we will investigate the $M$-ideal structure of the Banach algebra of operators $L(X)$. Spaces of bounded linear operators will also be the subject of the next chapter. Whereas in the final chapter we will put emphasis on Banach *space* methods and also consider spaces of operators which act between different Banach spaces, we will employ here Banach *algebra* techniques and apply the methods developed so far. Our knowledge on the $M$-ideal structure of $L(X)$ is rather incomplete. All that is known at this moment is contained in the following theorem.

**Theorem 6.1** *Let $J$ be an $M$-ideal of $L(X)$. Then*

(a) *$J$ is left (right) inner if $\mathbb{K} = \mathbb{C}$ and $X$ is uniformly convex (uniformly smooth),*
(b) *$J$ is left (right) inner if $\mathbb{K} = \mathbb{C}$ and $X = L^1(\mu)$ or a predual of a function algebra ($X$ is an $L^1$-predual or a function algebra on some compact Hausdorff space $K$),*
(c) *$J$ is left (right) inner for arbitrary scalar fields if $J$ is an $M$-summand and the centralizer $Z(X)$ (the Cunningham algebra $\mathrm{Cun}(X)$) is trivial,*
(d) *if $K(X)$, the space of compact operators on $X$, is an $M$-ideal in $L(X)$, then it is an inner $M$-ideal. Here, again, no restriction is imposed on the scalar field.*

The proof of part (d) will be presented in the next chapter (Proposition VI.4.10). Also, the proof of (c) will be postponed; (c) is a special case of Theorem VI.1.2. Finally, (a) and (b) can be attacked with tools that were provided in the foregoing sections. We mention in addition that one can read the following result between the lines of the proof of Theorem VI.2.3:

*An $M$-ideal in $L(X)$ is right inner for the real space $X = C(K)$.*

We now start preparing the proof of part (a). First, let us recall the definition of the modulus of convexity of a Banach space $X$: For $\varepsilon \in (0, 2]$ this function is defined by

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \frac{\|x+y\|}{2} \,\middle|\, x, y \in S_X, \|x-y\| = \varepsilon \right\}.$$

A Banach space is called *uniformly convex* if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The proof of (a) in the following lemma can be found in [422, Section 1.e], whereas (c) is an immediate consequence of (a) and the definition involved. For part (b) we refer to [274, Lemma 5.1].

**Lemma 6.2** *Let $X$ be a Banach space.*

(a) *The modulus of convexity of $X$ can equivalently be defined by*

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \,\middle|\, x, y \in B_X, \|x-y\| \geq \varepsilon \right\}.$$

*Furthermore, the function $\varepsilon \mapsto \delta_X(\varepsilon)/\varepsilon$ is nondecreasing on $(0, 2]$.*

(b) *The function $\delta_X$ is continuous on $(0, 2)$.*

(c) *If $X$ is uniformly convex, then $\delta_X$ is strictly increasing as is its inverse function $\delta_X^{-1}$, and*

$$\lim_{\varepsilon \to 0} \delta_X(\varepsilon) = \lim_{\varepsilon \to 0} \delta_X^{-1}(\varepsilon) = 0.$$

The modulus of smoothness is defined for $\tau > 0$ by

$$\rho_X(\tau) = \inf \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 \,\middle|\, \|x\| = 1, \|y\| = \tau \right\},$$

and $X$ is said to be *uniformly smooth* if and only if $\lim_{\tau \to 0} \rho_X(\tau)\tau^{-1} = 0$. We will make use of the following theorem. Again, the reader should consult [422] for proofs, examples and further literature.

**Theorem 6.3**

(a) *A Banach space $X$ is uniformly convex if and only if $X^*$ is uniformly smooth.*

(b) *Every uniformly convex (and thus also every uniformly smooth) Banach space is reflexive.*

Our first step towards the proof of Theorem 6.1(a) is the following lemma.

**Lemma 6.4** *Let $X$ be uniformly convex and suppose that there are operators $A, T \in L(X)$ with $\|A\| \leq 1$, $\|T\| \leq 1 + \varepsilon$, and $\|Id - \lambda T\| \leq 1 + \varepsilon$ whenever $0 \leq \lambda \leq 2$. Then*

$$\|T + \lambda A(Id - T)\| \leq 1 + \varepsilon + \lambda(1 + \varepsilon)\delta_X^{-1}(\varepsilon + \lambda).$$

Proof: Suppose that $T$ is as above and fix $\lambda > 0$ as well as $y \in S_X$. If $\|Ty\| \le 1-\lambda(1+\varepsilon)$ then, by the triangle inequality,

$$\| (T + \lambda A(Id - T))\, y\| \le 1,$$

which permits us to assume that $\|Ty\| > 1 - \lambda(1+\varepsilon)$. Set

$$u = \frac{Ty}{1+\varepsilon} \qquad \text{and} \qquad v = \frac{y - Ty}{1+\varepsilon}.$$

It follows by assumption that $\|u \pm v\| \le 1$, and since $u = \frac{1}{2}[(u+v)+(u-v)]$, we have $\delta_X(\|2v\|) \le 1 - \|u\|$, hence $\|u\| \le 1 - \delta_X(\|2v\|)$. By assumption on $y$ and definition of $u$, $[1-\lambda(1+\varepsilon)](1+\varepsilon)^{-1} \le \|u\|$ and so

$$\frac{1-\lambda(1+\varepsilon)}{1+\varepsilon} \le 1 - \delta_X(2\|v\|).$$

Regrouping terms and using the fact that $\delta_X^{-1}$ is increasing, we arrive at

$$\|v\| \le 2\|v\| \le \delta_X^{-1}\left(1 - \frac{1}{1+\varepsilon} + \lambda\right) \le \delta_X^{-1}(\varepsilon + \lambda).$$

This gives

$$\| (T + \lambda A(Id - T))\, y\| \le 1 + \varepsilon + \lambda\|(Id - T)y\| \le 1 + \varepsilon + \lambda(1+\varepsilon)\delta_X^{-1}(\varepsilon + \lambda),$$

and we are done. □

We now come to the

Proof of Theorem 6.1(a):
We first consider a uniformly convex Banach space $X$. Let $J \subset L(X)$ be an $M$-ideal and, as before, denote by $z \in J^{\perp\perp}$ the hermitian idempotent that corresponds to $J$. By Theorem 5.4, there is a net $(T_\alpha) \subset J$ of operators converging with respect to the weak$^*$ topology of $L(X)^{**}$ to $z$ such that

$$\begin{aligned} \|Id - \lambda T_\alpha\| &= \|(Id - T_\alpha) + (1-\lambda)T_\alpha\| \\ &\le (1+\varepsilon_\alpha)\max\{1, |1-\lambda|\} \\ &= 1 + \varepsilon_\alpha \end{aligned}$$

for $0 \le \lambda \le 2$ and $\varepsilon_\alpha \to 0$. Using Lemma 6.2(b) and (c) and Lemma 6.4 we find for all $A \in B_{L(X)}$

$$\begin{aligned} \lim_{\lambda\to 0} \limsup_{\alpha} \frac{\|T_\alpha + \lambda A(Id - T_\alpha)\| - 1}{\lambda} &\le \lim_{\lambda\to 0} \limsup_{\alpha} \frac{\varepsilon_\alpha + \lambda(1+\varepsilon_\alpha)\delta_X^{-1}(\varepsilon_\alpha + \lambda)}{\lambda} \\ &= \lim_{\lambda\to 0} \delta_X^{-1}(\lambda) \\ &= 0. \end{aligned}$$

Repeating the above argument with $z$ replaced by $Id - z$ we get that for any $A \in B_{L(X)}$

$$\lim_{\lambda \to 0} \limsup_{\alpha} \frac{\|\lambda A T_\alpha + Id - T_\alpha\| - 1}{\lambda} = 0,$$

and the theorem follows by an appeal to Proposition 5.1.
The case of uniformly smooth spaces follows from this by duality (Theorem 6.3). □

**Corollary 6.5** *Let $X$ be a complex Banach space and $J$ an $M$-ideal of $L(X)$. If $X$ is uniformly convex then $J$ is a left ideal. Is $X$ uniformly smooth, then $J$ must be a right ideal, and if $X$ is both uniformly smooth and uniformly convex, then $J$ is a two-sided ideal.*

PROOF: Use the fact that, by reflexivity of $X$, the mapping $T \mapsto T^*$ is an algebraic anti-isomorphism, and Theorem 3.2. □

**Corollary 6.6** *Let $1 < p < \infty$. Then $K(\ell^p)$ is the only nontrivial $M$-ideal of $L(\ell^p)$.*

PROOF: That $K(\ell^p)$ is an $M$-ideal in $L(\ell^p)$ was mentioned already several times (Example I.1.4(d)) and will eventually be proved in Example VI.4.1. To prove uniqueness note that by a result from [276] (see also [307]), $K(\ell^p)$ is the only two-sided closed ideal in $L(\ell^p)$. Since $\ell^p$ is a well-known example of a uniformly smooth and uniformly convex Banach space we may now invoke Corollary 6.5. □

Whereas in the above the norm condition on the approximating net of Theorem 5.4 was used, the proof of Theorem 6.1(b) requires the restrictions put on the numerical range of the elements of the involved net.
Before we go into the details let us make some remarks on how the theory of numerical ranges for elements in Banach algebras transfers to the special case of the Banach algebra $L(X)$.

In the case of the algebra $L(X)$, the set $\mathsf{S}_{L(X)}$ as a whole is, of course, highly nonaccessible in general. However, concerning $v(T, L(X))$ one may restrict one's attention to the subset $\Pi(X)$ of $\mathsf{S}_{L(X)}$ which is defined by

$$\Pi(X) := \{(x^*, x) \in S_{X^*} \times S_X \mid x^*(x) = 1\}$$

and define the *spatial numerical range* of an operator $T$ to be

$$V(T) := \{x^*(Tx) \mid (x^*, x) \in \Pi(X)\}.$$

In this way, one has access to the set $v(T, L(X))$ by virtue of

$$v(T, L(X)) = \overline{\text{co}}\, V(T).$$

(See [84, Chapter 3] for further details.) In particular, the notion of a *hermitian operator* may also be defined by

$$T \text{ hermitian } \iff V(T) \subset \mathbb{R}.$$

Note that the above notion has already been used in Section I.1; with the previous remarks it will be possible to combine the results obtained in that section with those that were presented in the last one. More precisely, we will profit from the fact that sometimes operators with the property that $V(T)$ is distributed along a small strip around the real axis are perturbations of elements in $\mathrm{Mult}(X)$ or $\mathrm{Cun}(X)$:

**Lemma 6.7** *Let $\mathfrak{A}$ be a function algebra on a compact space $K$ and put $R(\varepsilon) = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \le \varepsilon\}$. If $T \in L(\mathfrak{A})$ has the property that, for some $\varepsilon > 0$, $V(T) \subset R(\varepsilon)$, then there is an $a \in \mathfrak{A}$ such that $\|T - M_a\| \le 2\varepsilon$. Similarly, if $X = L^1(\mu)$ and an operator $T$ on $X$ has the above property, then there is an operator $S \in \mathrm{Cun}(X)$ such that $\|T - S\| \le 2\varepsilon$.*

For the proof we have to recall some details from the theory of function algebras. Those points $k \in K$ with $\delta_k \in \mathrm{ex}\, B_{\mathfrak{A}^*}$ are said to belong to the *Choquet boundary* of $\mathfrak{A}$, denoted by $\mathrm{ch}\,\mathfrak{A}$. It is known [585, Theorem 7.18] that $k$ is in the Choquet boundary of $\mathfrak{A}$ if and only if $\{k\}$ is a $p$-set, and, consequently, for each such point, $\mathfrak{J}_k := \ker \delta_k$ is an $M$-ideal by Theorem 4.2.

PROOF: Put $a := T(e)$ and let $k \in \mathrm{ch}\,\mathfrak{A}$. Since $\mathfrak{J}_k$ is an $M$-ideal, and since for any $x \in \mathfrak{A}$ $\xi := \delta_k(x)e - x$ is in $\mathfrak{J}_k$, we have by Proposition I.1.24

$$\varepsilon\|\xi\| \ge d(T\xi, \mathfrak{J}_k) = |\delta_k(T\xi)| = |\delta_k(ax - Tx)|.$$

Since $\|\xi\| \le 2\|x\|$, this yields

$$\|ax - Tx\| = \sup_{k \in \mathrm{ch}\,\mathfrak{A}} |\delta_k(ax - Tx)| \le 2\varepsilon\|x\|.$$

Let us look at the case where $X = L^1(\mu)$: By the above, we can find an operator $S$ which is contained in $\mathrm{Mult}(X^*) = Z(X^*)$ such that

$$\|T^* - S\| \le 2\varepsilon.$$

But $S$ is a multiplication operator on $(L^1(\mu))^*$ (Example I.3.4(a)) and thus weak* continuous, hence the result. ☐

The way is now paved for the proof of Theorem 6.1(b). However, it turns out that more can be shown. We split this extended version of Theorem 6.1(b) into two pieces and begin with function algebras.

**Theorem 6.8** *Let $\mathfrak{A}$ be a function algebra on some compact Hausdorff space $K$. Then every $M$-ideal in $L(\mathfrak{A})$ is right inner. Furthermore, the $M$-ideals of $L(\mathfrak{A})$ are precisely the subspaces of the form*

$$L(\mathfrak{A})_J = \overline{\mathrm{lin}}\,\{M_a T \mid T \in L(\mathfrak{A}), a \in J\} = \{T \in L(\mathfrak{A}) \mid \limsup_{k \to k_0} \|T^*\delta_k\| = 0 \ \forall k_0 \in D\},$$

*where $J$ is an $M$-ideal of $\mathfrak{A}$ and $D$ is the $p$-set corresponding to $J$ according to Theorem 4.2.*

PROOF: That all subspaces of the above form are in fact $M$-ideals was shown in Proposition 3.5. We are thus left with showing that all $M$-ideals look like this.
So, let $\mathfrak{J}$ be an $M$-ideal in $L(\mathfrak{A})$ and denote by $P$ its defining $M$-projection $P : L(\mathfrak{A})^{**} \to \mathfrak{J}^{\perp\perp}$. Start with a net $(T_\alpha)$ in $\mathfrak{J}$ converging in the weak$^*$ topology to $P(e)$ and with the property that $V(T_\alpha) \subset R(\varepsilon_\alpha)$ for some net $(\varepsilon_\alpha)$ of positive numbers tending to zero; see Theorem 5.4. By Lemma 6.7, we may suppose without loss of generality that for some $a_\alpha \in \mathfrak{A}$ the operators $T_\alpha$ are of the form $M_{a_\alpha}$. But then it follows that

$$P(e) = w^*\text{-}\lim_\alpha M_{a_\alpha} \in (\mathrm{Mult}^l_{inn} L(\mathfrak{A}))^{\perp\perp}$$

whence by Lemma 3.4,

$$P(e) \in \mathrm{Mult}^{l,2}_{inn} L(\mathfrak{A}^{**}).$$

It follows that $P(e)$ is an $M$-projection. By injectivity of the map $\Delta$ (Theorem 2.1), we must have $P = L^2_{P(e)}$, and so, $\mathfrak{J}$ is inner. Since $P(e) \in \mathfrak{A}^{\perp\perp}$, when $\mathfrak{A}$ is identified with a subalgebra of $\mathrm{Mult}^l_{inn} L(\mathfrak{A})$, we may apply Proposition 3.3 to see that $P(e)$ gives rise to an inner $M$-ideal $J$ in $\mathfrak{A}$ as well. Next, observe that we are in a situation treated in Proposition 3.5. Consequently,

$$\mathfrak{J} = L(\mathfrak{A})_J = \overline{\mathrm{lin}}\,\{M_a T \mid T \in L(\mathfrak{A}), a \in J\},$$

since the inner projection pertaining to $L(\mathfrak{A})_J$ is also $P(e)$. It thus remains to verify that

$$L(\mathfrak{A})_J = \{T \in L(\mathfrak{A}) \mid \limsup_{k\to k_0} \|T^*\delta_k\| = 0 \ \forall k_0 \in D\}.$$

To this end, we observe that we have for an operator of the form $M_j T$ with $j \in J$ and for $k_0 \in D$,

$$\limsup_{k\to k_0} \|(M_j T)^*\delta_k\| = \limsup_{k\to k_0} \|j(k) T^*\delta_k\| \le \|T\| \limsup_{k\to k_0} |j(k)| = 0.$$

Now, recall that we saw in the proof of Theorem 4.2 that there is an approximate unit $(p_\alpha)_\alpha$ in $J$ with

$$\forall \varepsilon > 0\ \forall U \supset D,\ \ U \text{ open } \exists \alpha_0\ \forall \alpha > \alpha_0\ \left| (e - p_\alpha)_{|K\setminus U} \right| < \varepsilon$$

which, of course, may be chosen such that $M_{p_\alpha}$ converges to $P(e)$ in the $\sigma(L(\mathfrak{A})^{**}, L(\mathfrak{A})^*)$-topology. Hence, by Theorem 1.2(d),

$$\mathfrak{J} = \{T \in L(\mathfrak{A}) \mid \lim_\alpha M_{p_\alpha} T = T\},$$

and so, each $T$ in $L(\mathfrak{A})_J$ may be approximated uniformly by operators of the form $M_j T$, with $j \in J$. This yields

$$L(\mathfrak{A})_J \subset \{T \in L(\mathfrak{A}) \mid \limsup_{k\to k_0} \|T^*\delta_k\| = 0 \ \forall k_0 \in D\}.$$

On the other hand, for each $T \in L(\mathfrak{A})$ with $\limsup_{k\to k_0} \|T^*\delta_k\| = 0$ whenever $k_0 \in D$ we may, for all $\varepsilon > 0$, select an open neighbourhood $U$ of $D$ such that $\|T^*\delta_k\| < \varepsilon$ for all $k \in U$. Now, for sufficiently large $\alpha$,

$$\left|(e - p_\alpha)_{|K\setminus U}\right| < \varepsilon$$

and consequently we have, regarding elements of $L(\mathfrak{A})$ as vector valued functions on $K$,

$$\|M_{p_\alpha}T - T\| = \max\{\|(M_{e-p_\alpha}T)_{|U}\|, \|(M_{e-p_\alpha}T)_{|K\setminus U}\|\} < \varepsilon,$$

whence $\lim_\alpha M_{p_\alpha}T = T$ and $T \in L(\mathfrak{A})_J$. □

We next treat $M$-ideals in the space $L(X)$ in the case where $X$ is a predual of a function algebra.

**Theorem 6.9** *Suppose that $X$ is a predual of a function algebra $\mathfrak{A}$ on some compact space $K$.*

(a) *If $\mathfrak{J}$ is an $M$-ideal in $L(X)$ then it is left inner and, furthermore, there is a $p$-set $D \subset K$ for $\mathfrak{A}$ such that*

$$\mathfrak{J} = \mathfrak{J}^{(D)} := \{T \in L(X) \mid \limsup_{k\to k_0} \|T^{**}\delta_k\| = 0 \ \forall k_0 \in D\}.$$

(b) *In the special case that $X = L^1(\mu)$, for each closed subset $D$ of $K$, where $C(K) = L^1(\mu)^*$, $\mathfrak{J}^{(D)}$ is an $M$-ideal.*

Proof: Denote by Adj the antihomomorphism that maps an operator $T \in L(X)$ to $T^* \in L(\mathfrak{A})$. Since Adj$^{**}$ is an antihomomorphism from $L(X)^{**}$ equipped with the first Arens product to $L(\mathfrak{A})^{**}$ when the latter is furnished with the second Arens product (see Theorem 1.2), we may identify $L(X)^{**}$ with a subalgebra of $L(\mathfrak{A})^{**}$.
Let $P : L(X)^{**} \to \mathfrak{J}^{\perp\perp}$ be the $M$-projection given by $\mathfrak{J}$. As in the proof of Theorem 6.8, we start with a net $(T_\alpha)$ in $B_{L(X)}$ converging to $P(e)$ in the weak$^*$ topology and satisfying

$$V(T_\alpha) \subset R(\varepsilon_\alpha)$$

for some net $(\varepsilon_\alpha)$ of real numbers converging to zero. Clearly, $(T_\alpha)$ converges also in the $\sigma(L(\mathfrak{A})^{**}, L(\mathfrak{A})^*)$-topology to $P(e)$. Moreover, by Lemma 6.7, we may disturb $T_\alpha^*$ slightly in norm to obtain a net $(M_{a_\alpha}) \in L(\mathfrak{A})$ still converging to $P(e)$. Following the lines of Theorem 6.8's proof, we see that

$$P(e) \in \mathrm{Mult}^{l,2}_{inn}(L(\mathfrak{A})^{**})$$

which amounts to

$$P(e) \in \mathrm{Mult}^{r,1}_{inn}(L(X)^{**}).$$

Hence, $\mathfrak{J}$ must be left inner. Applying Proposition 3.3, we get an $M$-ideal $\mathfrak{J}_1$ in $L(\mathfrak{A})$ with

$$\mathfrak{J}_1 \cap L(X) = \mathfrak{J}.$$

An application of Theorem 6.8 now yields (a).
To prove (b), let $X = L^1(\mu)$ and choose a closed subset $D$ of $K$. We must show that

$$\mathfrak{J}^{(D)} = \{T \in L(L^1(\mu)) \mid \limsup_{k\to k_0} \|T^{**}\delta_k\| = 0 \ \forall k_0 \in D\}$$

is an $M$-ideal. For this purpose, let $S_1, S_2, S_3 \in B_{\mathfrak{J}^{(D)}}$ and $T \in B_{L(L^1(\mu))}$ be given. Pick an open neighbourhood $U$ of $D$ such that $\|S_i^{**}\delta_t\| < \varepsilon$ for all $t \in U$ and a continuous function $\psi$ with $0 \le \psi \le 1$, $\psi|_D = 0$ and $\psi|_{K\setminus U} = 1$. Putting $S = TM_\psi$ we have $S \in \mathfrak{J}^{(D)}$. Since, in addition,

$$\begin{aligned} \|S_i + T - S\| &= \max\{\|(S_i + (1-\psi)T)|_U\|, \|(S_i + (1-\psi)T)|_{K\setminus U}\|\} \\ &\le 1 + \varepsilon, \end{aligned}$$

the claim follows from Theorem I.2.2. □

It is unknown how the sets appearing in part (a) of the above theorem may be characterised, nor is it clear whether *all* $p$-sets $D$ of $\mathfrak{A}$ arise in this way.
Let us come to our last theorem. Also here, the nature of the closed sets $D$ involved is somewhat obscure.

**Theorem 6.10** *If $X$ is a (complex) $L^1$-predual then all $M$-ideals of $L(X)$ are right inner and possess the form*

$$\mathfrak{J} = L(X) \cap \mathfrak{J}^{(D)},$$

*where $D$ denotes a closed subset in the compact space $K$ with $X^{**} = C(K)$ and $\mathfrak{J}^{(D)}$ is the $M$-ideal in $L(L^1(\mu))$ described in Theorem 6.9.*

The proof of this theorem can be given in analogy to the first part of the proof of Theorem 6.9.

## V.7 Notes and remarks

General remarks. Although the notion of an $M$-ideal was originally designed for real Banach spaces, it soon became clear that the case of complex scalars should be covered as well. This was first formulated in [311] for the explicit purpose of investigating $M$-ideals in function algebras. Some time afterwards the fundamental articles [574], [577] and [578] by R. Smith and J. D. Ward appeared, where a detailed analysis of $M$-ideals in Banach algebras and some applications are presented. In these papers the idea of associating the hermitian projection $P(e) \in \mathfrak{A}^{**}$ with an $M$-ideal $\mathfrak{J} \subset \mathfrak{A}$ was developed, which later led to the notion of an inner $M$-ideal in [630].
The background material on numerical ranges in Section V.1 can be found for example in the books [84], [85], and [86] by Bonsall and Duncan; for the Arens products we mention the useful survey [176]. Clearly, Theorem 1.10 is a standard result in the theory of $C^*$-algebras, but we decided to include a proof in order to be able to present a rather self-contained exposition of the $M$-structure of $C^*$-algebras. Actually, much more than the Arens regularity of a $C^*$-algebra is true. It is proved in [260] and [524] that *every*

bilinear operator from $X \times X$ to $X$ is Arens regular provided $X^*$ has property $(V^*)$; note that this applies in particular to Banach spaces with $L$-embedded duals (Theorem IV.2.7) and thus to $C^*$-algebras (Example IV.1.1(b)).

As we said, the fact (Theorem 2.1) that, for an $M$-projection $P$ associated with an $M$-ideal $\mathfrak{J} \subset \mathfrak{A}$, $\Delta_{\mathfrak{A}^{**}}(P)$ is a hermitian projection is a basic result from [577]; the sources of the remaining parts of Theorem 2.1 are [235] and [578] (part (c)), [630] (parts (d) and (f)) and [638] (part (e)). The rest of Section V.2 and the example at the end of Section V.3 are taken from [577]. Section V.3 is based on [630] where the phrase "inner $M$-ideal" is coined and this notion is investigated in detail. One aspect of the present chapter is to show that nearly everything that can be proved about $M$-ideals in Banach algebras has actually to do with inner $M$-ideals.

Theorem 4.1 is in essence a result of Smith [574] as is the equivalence (ii) $\iff$ (iii) of Theorem 4.2. The equivalence (i) $\iff$ (iii) of this theorem is due to Hirsberg [311] who found it as a special case of a much more general result characterising $M$-ideals in subspaces of $C(K)$ containing the constants in terms of so-called $M$-sets; see also [35, Chap. 4] for a presentation of this approach and [399, Th. 7.6] for another proof of the implication in question. Here we have tried to reconcile the two equivalences in a perspicuous manner; for the special case of the disk algebra cf. [130]. We shall discuss the noncommutative version of Theorem 4.1 in the next subsection.

As already mentioned in the Notes and Remarks to Chapter I, the coincidence of $M$-ideals and closed two-sided ideals in $C^*$-algebras can be traced back to work by Effros [186] and Prosser [509]. For the real Banach space of self-adjoint elements of a $C^*$-algebra it was noted explicitly by Alfsen and Effros [11, Prop. 6.18]. Their proof used only the by then well known correspondence between weak$^*$ closed ideals and central projections in von Neumann algebras, and so applied equally well in the complex case. Indeed, the complex case was attributed to them without further comment in [8, p. 237]. Another proof of the complex case (applicable to general Banach algebras) was later given by Smith and Ward [577]; yet another proof appeared in [589]. Our proof that $M$-ideals are ideals is taken from [477], for other arguments see [577], [579] or the following subsection. The proof of Proposition 4.6 was shown to us by R. Smith, and that of Theorem 4.7 by A. Rodríguez-Palacios; the latter theorem has its origin in [11, Cor. 6.17]. Proposition 4.8 and its corollaries are taken from [121].

The results in Section V.5 aim at approximating elements $a^{**} \in \mathfrak{A}^{**}$ by those of $\mathfrak{A}$ respecting norm and numerical range conditions. Clearly the simplest result of this type is Goldstine's theorem. A far more elaborate result is the principle of local reflexivity. The version we employ, Theorem 1.4, is proved in [58]; for a proof of the classical version see [152] and for various refinements [62]. Proposition 5.1 is a new result building on ideas from [119] and [578]. Theorem 5.3 is due to Smith [576], but our proof is taken from [434] where this result is proved in the abstract framework of numerical range spaces. In fact, its assertion holds with $\varepsilon_\alpha = 0$ if $v(F)$ has nonempty interior. A related result is discussed in [129]: If $\mathfrak{A}$ is a unital Banach algebra and $\mathfrak{J} \subset \mathfrak{A}$ is a two-sided ideal which is an $M$-ideal, then each $\xi \in \mathfrak{A}/\mathfrak{J}$ whose "essential" numerical range $v(\xi, \mathfrak{A}/\mathfrak{J})$ has nonempty interior has a representative $a \in \mathfrak{A}$ with $v(a) = v(\xi, \mathfrak{A}/\mathfrak{J})$ and $\|a\| = \|\xi\|$. The norm approximation in Theorem 5.4 is due to Smith and Ward [578] who offer an entirely different proof. Here are some applications proved in that paper. Suppose $\mathfrak{J}$ is a two-sided inner $M$-ideal in the unital Banach algebra $\mathfrak{A}$; for instance any $M$-ideal in a commutative Banach algebra (Theorem 4.1), any closed two-sided ideal in a $C^*$-algebra (Theorem 4.4) or the ideal $K(\ell^p)$ in $L(\ell^p)$, $1 < p < \infty$ (Theorem 6.1 and Example VI.4.1). Then, for the natural quotient homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$, the formula

$$\|\pi(a^n)\| = \inf_{y \in \mathfrak{J}} \|(a+y)^n\| \qquad \forall a \in \mathfrak{A},\ n \in \mathbb{N}$$

holds, as does the spectral radius formula

$$r(\pi(a)) = \inf_{y \in \mathfrak{J}} r(a+y) \qquad \forall a \in \mathfrak{A}.$$

The same formula holds for an ideal in a commutative Banach algebra possessing a bounded approximative unit. In the context of $C^*$-algebras, these results are due to Pedersen [482].

Theorem 6.1(a) is the core of the proof from [119] that $M$-ideals in $L(X)$ are left ideals for uniformly convex $X$ (Corollary 6.5). This result extends work in [578] on $L(\ell^p)$ (or $L(L^p(\mu))$) drawing on Clarkson's inequalities instead of Lemma 6.4. Corollary 6.6 comes from [578]; there is another argument in [234], completing a preliminary attack in [579]. It is still an open question whether $L(L^p[0,1])$ contains any nontrivial $M$-ideals; in [119] a uniformly convex Banach space $X$ of the form $X = \ell^r(\ell^p(n_i))$ is constructed such that $K(X)$ is an $M$-ideal in $L(X)$, but $L(X)$ contains a closed ideal which is not an $M$-ideal. In fact, the closure of the ideal of operators on $X$ which factor through a subspace of some $L^r(\mu)$-space works; though this is quite concrete an example, its verification remains involved. Theorem 6.1(b) is proved for $X = \ell^1$ in [575] and for $X = C(K)$ in [235]. This paper also covers the case of operators on $C_0(\Omega)$ for certain locally compact $\Omega$. The remaining cases of Theorem 6.1(b), presented in detail in Theorems 6.8–6.10, are considered in [630] as is part (d). For part (c) see [628].

Subalgebras of $C^*$-algebras; nest algebras. Suppose $\mathfrak{A}$ is a unital (nonselfadjoint) subalgebra of a $C^*$-algebra and thus of $L(H)$. Such an algebra can be regarded as the noncommutative analogue of a function algebra. Adopting this point of view it seems natural to inquire whether a characterisation of the $M$-ideals of $\mathfrak{A}$ which parallels the corresponding result on function algebras (Theorem 4.2) can be obtained. This is indeed possible.

Theorem. *Let $\mathfrak{A} \subset L(H)$ be a unital subalgebra. Then a closed subspace $J \subset \mathfrak{A}$ is an $M$-ideal if and only if it is a two-sided ideal containing a two-sided 1-approximative unit.*

This is proved in [195], and in [504] it is observed that the result extends easily to nonunital algebras. Let us sketch an argument, suggested by D. Yost, to show that the $M$-ideals of $\mathfrak{A}$ are actually ideals: We assume, without loss of generality, that $J$ is an $M$-summand. Denote by $P$ the $M$-projection from $\mathfrak{A}$ onto $J$, and let $Q = Id - P$ and $p = P(Id)$. Since $P^*$ is an $L$-projection, one observes that $P^*f \ge 0$ and $Q^*f \ge 0$ whenever $f \in \mathfrak{A}^*$, $f \ge 0$ (i.e., $\|f\| = f(Id)$). Then one deduces that $p$ must be a positive element of $L(H)$, and we know from Theorem 2.1 that $p$ is a projection. Now the crucial step consists in proving that $ap \in J$ for all $a \in \mathfrak{A}$. To show this let $f \in \mathfrak{A}^*$, $f \ge 0$, and let $g \in L(H)^*$ be a positive norm-preserving extension of $Q^*f$. It follows that

$$\begin{aligned} |f(Q(ap))| &= |(Q^*f)(ap)| = |g(ap)| \\ &\leq |g(a^*a)|^{1/2} |g(p^2)|^{1/2} = 0, \end{aligned}$$

where we used the Cauchy-Schwarz inequality for $C^*$-algebras (note that $a^*$ exists in $L(H)$, but not necessarily in $\mathfrak{A}$) and observed that

$$g(p^2) = g(p) = (Q^*f)(p) = f(Qp) = 0$$

since $Qp = 0$. An application of Theorem 1.5 yields that $f(Q(ap)) = 0$ for *all* $f \in \mathfrak{A}^*$ so that $Q(ap) = 0$ and $ap \in J$. Likewise, $pa \in J$, and $J$ is a two-sided ideal.

A special class of subalgebras of $L(H)$ which has attracted some interest is the class of nest algebras. In the following, $H$ denotes "a" separable infinite dimensional complex Hilbert space, and for convenience we put $\mathcal{K} = K(H)$ and $\mathcal{L} = L(H)$. A *nest* $\mathcal{N}$ is a strongly closed totally ordered set of orthogonal projections on $H$ containing 0 and $Id$. The corresponding *nest algebra* $\mathcal{A} = \mathcal{A}(\mathcal{N})$ consists of all those operators on $H$ that leave ran$(P)$ invariant for each $P \in \mathcal{N}$. Examples include the nest of coordinate projections $x \mapsto \sum_{k=1}^{n} \langle x, e_k \rangle e_k$ with respect to a fixed orthonormal basis, in which case the corresponding nest algebra consists of those operators which have an upper triangular representation for that basis, and the Volterra nest, consisting of the projections $P_t(f) = \chi_{[0,t]} f$ on $L^2[0,1]$. Such algebras consist of operators which have, in some sense, a triangular representation, and are generally considered as noncommutative variants of the Hardy space $H^\infty$. For information on the theory of nest algebras we recommend the survey [507] and the monograph [147].

A crucial fact about nest algebras is that the unit ball of $\mathcal{A} \cap \mathcal{K}$ is dense in that of $\mathcal{A}$ for the weak operator topology ([507, Th. 2.1] or [147, p. 36]). A variety of corollaries can be deduced from this result. First of all, one can show that $\mathcal{A} + \mathcal{K}$ is closed; operators in $\mathcal{A} + \mathcal{K}$, being compact perturbations of operators in $\mathcal{A}$, are called quasi-triangular. Clearly, this is the formal analogue of Sarason's theorem that $H^\infty + C(\mathbb{T})$ is a closed subalgebra of $L^\infty(\mathbb{T})$. Then, $\mathcal{A}$ is canonically isometric to the bidual of $\mathcal{A} \cap \mathcal{K}$, and thus, by Theorem III.1.6, $\mathcal{A} \cap \mathcal{K}$ is an $M$-ideal in $\mathcal{A}$. Note that the corresponding commutative result about $H^\infty \cap C$ ($= A$, the disk algebra) is false.

However, the result that $C/A$ is isometric to $(H^\infty + C)/H^\infty$ which is an $M$-ideal in $L^\infty/H^\infty$ (Example III.1.4(h)) extends: $\mathcal{K}/(\mathcal{A} \cap \mathcal{K})$ is isometric to $(\mathcal{A} + \mathcal{K})/\mathcal{A}$ which is an $M$-ideal in $\mathcal{L}/\mathcal{A}$ [225]. The latter fact is most effectively proved using Proposition I.1.16, cf. [148]. One just has to observe that assumption ($\mathrm{b}_1$) of that proposition, with $J = \mathcal{K}$, $X = \mathcal{L}$ and $Y = \mathcal{A}$, is fulfilled by the above-quoted density theorem. Since $\mathcal{L}/\mathcal{A}$ happens to be the bidual of $(\mathcal{A} + \mathcal{K})/\mathcal{A} \cong \mathcal{K}/(\mathcal{A} \cap \mathcal{K})$, the above result also follows from the stability of the class of $M$-embedded spaces with respect to quotients. It is noteworthy that the proof of this noncommutative $M$-ideal result is far easier than in the commutative case, since $\mathcal{K}$ is an $M$-ideal in $\mathcal{L}$; in the commutative case we had to deal with an $L$-projection in $(L^\infty)^*$ that did not derive from an $M$-ideal.

We just suggested using Proposition I.1.16 in the analysis of the quotient space $\mathcal{L}/\mathcal{A}$. The fact that one has ($\mathrm{b}_1$) and thus (a) of this proposition might be considered as a version of the F. and M. Riesz theorem. For a deeper theorem of this type we refer to [209].

We wish to discuss one more aspect of the formal resemblance between $C$, $H^\infty$, $L^\infty$ and $\mathcal{K}$, $\mathcal{A}$, $\mathcal{L}$. In these strings of spaces the space $L^1$ is considered to be the counterpart of $N(H)$, the nuclear operators, and the Hardy space $H^1$ corresponds to $\mathcal{A}^1 := \mathcal{A} \cap N(H)$. The space $H^1_0$ teams up with the subspace $\mathcal{A}^1_0 \subset \mathcal{A}^1$ of "strictly" triangular operators, meaning

$$\mathcal{A}^1_0 = \{T \in \mathcal{A}^1 \mid (Id - P^-)TP = 0 \quad \forall P \in \mathcal{N}\}$$

where $P^- = \sup\{E \in \mathcal{N} \mid E < P\}$. Then one has indeed that $(\mathcal{K}/(\mathcal{A} \cap \mathcal{K}))^* \cong \mathcal{A}^1_0$ and $N(H)/\mathcal{A}^1_0 \cong (\mathcal{A} \cap \mathcal{K})^*$, see [507]. A lot of similarities between $H^1_0$ and $\mathcal{A}^1_0$ have been pointed out in [33], among them the noncommutative Havin theorem that $N(H)/\mathcal{A}^1_0$ is weakly sequentially complete; this holds true because $\mathcal{A} \cap \mathcal{K}$, being a subspace of $\mathcal{K}$, is $M$-embedded, see Corollary III.3.7(b).
For $M$-ideals and more general "analytic" subalgebras of $L(H)$ see [504].

$JB^*$-ALGEBRAS AND $JB^*$-TRIPLES. In an attempt to give an algebraic formalisation of the axioms of quantum mechanics Jordan, von Neumann and Wigner introduced the notion of a Jordan algebra in the thirties. Infinite-dimensional versions of their results were, however, first proved as late as 1978 when the landmark paper [13] by Alfsen, Shultz and Størmer appeared. This paper deals with a class of real algebras called $JB$-algebras; subsequently Kaplansky introduced their complex counterparts, the $JB^*$-algebras (see [644]). Here is the definition: A complex Banach space $X$, equipped with a bilinear map $(x, y) \mapsto x \circ y$ (the "Jordan product") and an isometric algebra involution $x \mapsto x^*$, is called a $JB^*$-algebra if

$$\begin{aligned} x \circ y &= y \circ x && (1)\\ (x^2 \circ y) \circ x &= x^2 \circ (y \circ x) && (2)\\ \|x \circ y\| &\leq \|x\|\,\|y\| && (3)\\ \|\{xxx\}\| &= \|x\|^3 && (4) \end{aligned}$$

for all $x$, $y \in X$, where

$$\{xyz\} = x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*). \qquad (5)$$

Note that the product is not supposed associative – (2) being a weak substitute – so that $(X, \circ)$ is not a Banach algebra proper. The main examples of $JB^*$-algebras are the norm-closed subspaces of $C^*$-algebras which are closed under the product

$$x \circ y = (xy + yx)/2.$$

(Here $xy$ denotes the original product in the $C^*$-algebra.) But apart from these so-called special $JB^*$-algebras there are others, notably the properly complexified 27-dimensional algebra of self-adjoint $3 \times 3$-matrices over the Cayley numbers. This object is often denoted by $C_6$. Because any classification of $JB^*$-algebras has to take care of $C_6$, one cannot expect that the theory of irreducible representations of $C^*$-algebras extends literally to $JB^*$-algebras. All the same, many of the ideas figuring so prominently in $C^*$-algebra theory can be adapted to the nonassociative setting. In this regard factor representations, corresponding to the irreducible representations in $C^*$-algebra theory, ideal theory and $JBW^*$-algebras (these are $JB^*$-algebras which are dual Banach spaces) have been studied; basic references are [184] and [644].

Payá, Pérez and Rodríguez [477] used $M$-ideals in a decisive manner to prove that every $JB^*$-algebra admits a faithful family of factor representations. For instance they prove that the weak* closed ideals in a $JBW^*$-algebra are in one-to-one correspondence with its $M$-summands, and that they are determined by central projections (cf. our Proposition 4.5). As a result, the closed ideals in a $JB^*$-algebra are in one-to-one correspondence with its $M$-ideals, and consequently the closed two-sided ideals in a $C^*$-algebra coincide with its closed Jordan ideals; this is a well-known result due to Civin and Yood. (We also mention the paper [284] for a geometric description of the ideals of a $JB$-algebra in terms of split faces.) To construct a factor representation the authors of [477] look at the largest $M$-ideal $M_p \subset \ker p$ for an extreme functional $p \in \operatorname{ex} B_{X^*}$ (a device introduced by Alfsen and Effros under the name primitive $M$-ideal) and prove that the $M$-summand orthogonal to $M_p^{\perp\perp}$ is a factor, i.e. has no weak* closed ideals, equivalently: no $M$-summands. They also classified the factors arising in their approach [478].

Another result relevant to $M$-structure theory in [477] is the $L$-embeddedness of preduals of $JBW^*$-algebras, which extends our Example IV.1.1(b). A complete list of those $JB^*$-algebras which are $M$-embedded, due to A. Rodríguez, can be found in [394, Th. 14], cf. Proposition III.2.9 for the $C^*$-algebra case. Actually, all the results given above were derived for noncommutative $JB^*$-algebras, here (1) is replaced by the requirement that

$$(x \circ y) \circ x = x \circ (y \circ x). \tag{1'}$$

Also, Rodríguez has proved [523] that the unital noncommutative $JB^*$-algebras are the largest class of (nonassociative) algebras for which the Vidav-Palmer theorem is valid, i.e. which are spanned by their hermitian elements.

A more general structure than a $JB^*$-algebra is that of a $JB^*$-triple which arises naturally in finite and infinite dimensional holomorphy. $JB^*$-triples were first used by Koecher and his school in order to give an alternative approach to the classification of symmetric domains in the finite dimensional case; their importance for infinite dimensions was realized by W. Kaup. Omitting all details – we refer to [25] and [614] for a more extended presentation and more on the relevant literature – we would like to briefly sketch the basic idea.

Denote by $D$ a bounded symmetric domain in a (complex) Banach space $X$, which means that for all $x \in D$ there is an involutory and biholomorphic map $\varphi$ which has $x$ as an isolated fixed point. Extending a classical result of Cartan to this more general setting, Upmeier and Vigué independently proved that the group $\operatorname{Aut} D$ of all biholomorphic automorphisms naturally bears the structure of a real Banach Lie group, whose Lie algebra $\operatorname{aut} D$ can concretely be realized as the space of all *complete* holomorphic vector fields $F : D \to X$ by the mapping

$$\xi \in \operatorname{aut} D \longmapsto \left( x \in X \mapsto \lim_{t \to 0} \frac{\Phi_\xi(t)x - x}{t} \right)$$

where $\Phi_\xi(t)$ is the one-parameter subgroup of $\operatorname{Aut} D$ associated with $\xi \in \operatorname{aut} D$ via the exponential mapping. (We should add that a holomorphic vector field $F : D \to X$ is called complete if for each $u_0 \in D$ the initial value problem

$$\begin{aligned} y(0) &= u_0 \\ y'(t) &= F(y(t)) \end{aligned}$$

has a global solution $y : \mathbb{R} \to D$.) The point now is that there is no loss in generality to assume that $D$ is actually balanced, and that for these domains all complete holomorphic vector fields are polynomials of degree at most 2. Moreover, if $F = p_0 + p_1 + p_2$ is the Taylor expansion of such a polynomial, then $p_0 + p_2$ and $p_1$ both belong to $\operatorname{aut} D$, and $p_0 + p_2$ is uniquely determined by the constant term $p_0 \in X$. Also, the symmetry of $D$ implies that $\operatorname{Aut} D$ acts transitively on $D$ and that consequently *each* $p_0 \in X$ gives rise to a (unique) vector field $p_0 + p_2$. Hence, with each $y \in X$ one can associate a unique symmetric bilinear map $P_y : X \times X \to X$ leading to the triple product

$$\{x, y, z\} := P_y(x, z)$$

which is bilinear and symmetric in $x$ and $z$ and conjugate linear in $y$. It furthermore satisfies the Jordan triple identity

$$\{\{uvw\}yx\} + \{\{uvx\}yw\} - \{uv\{wyx\}\} = \{w\{vuy\}x\},$$

and it can be shown that the operators $x \,\square\, x : z \mapsto \{xxz\}$ are bounded, linear and hermitian. In addition, these operators have positive spectrum and satisfy the "$C^*$-condition"

$$\|x \,\square\, x\| = \|x\|^2,$$

which in turn is equivalent to the "$JB^*$-condition"

$$\|\{xxx\}\| = \|x\|^3.$$

Stated in a short and more notational way: A Banach space containing a bounded symmetric balanced domain can be given the structure of a *$JB^*$-triple* – the defining conditions for the triple product are listed above. Also the converse holds, and both statements together yield part of an important result of Kaup ([370], [371]) stating that, up to biholomorphic equivalence, bounded symmetric domains in complex Banach spaces are the open unit balls of $JB^*$-triples and that two such domains are equivalent iff the corresponding $JB^*$-triples are isometrically isomorphic (in which case they are isomorphic with respect to their $JB^*$-structure – this latter statement involves a result due to Kaup and Upmeier [373]).
There are prominent examples of $JB^*$-triples whose triple product does not refer explicitly to the analytic structure of the underlying space. Under the triple product (5) a $JB^*$-algebra becomes a $JB^*$-triple, in particular, $\{xyz\} = (xy^*z + zy^*x)/2$ introduces the structure of a $JB^*$-triple on a $C^*$-algebra. Further examples are the spaces $L(H, K)$ of bounded linear operators between (different) Hilbert spaces $H$ and $K$, with the triple product $\{RST\} = (RS^*T + TS^*R)/2$. An important feature of $JB^*$-triples is the stability of this class under contractive projections ([236], [372]). This provides more examples, since in particular the range of a norm-one projection on a $C^*$-algebra, though generally

not a $C^*$-algebra, is a $JB^*$-triple. (It is a $C^*$-algebra if the projection is completely positive.)

Similar to the case of a $JB^*$-algebra, the underlying geometrical structure often helps in understanding the rather puzzling algebraic behaviour of a given $JB^*$-triple. Of interest in this respect are results concerning the $M$-structure of $JB^*$-triples which again provides a coupling of algebraic and geometrical concepts. A subspace $J$ of a $JB^*$-triple is called an ideal if $\{xyz\} \in J$ whenever $x$, $y$ or $z$ is in $J$. Horn [323] proves that the weak$^*$ closed ideals in a $JBW^*$-triple (a $JB^*$-triple which is a dual Banach space) coincide with its $M$-summands, and Barton and Timoney [43] deduce that the closed ideals of a $JB^*$-triple coincide with its $M$-ideals. These papers also contain proofs of the strong uniqueness of the predual of a $JBW^*$-triple and its $L$-embeddedness. Moreover, the bidual of a $JB^*$-triple $X$ is a $JBW^*$-triple with separately weak$^*$ continuous triple product extending the one of $X$; it is this crucial result due to Dineen [160] that links the ideals of $X$ to the weak$^*$ closed ideals of $X^{**}$ and makes the approach to ideals via the bidual possible.

With these theorems in hand the theory of factor representations of [477] lends itself to generalisation. This is carried out in [43], too; and a Gelfand-Naimark theorem for $JB^*$-triples is achieved in [237] and [324]. For an $M$-ideal approach to the latter see [161, Example 38]. Dineen and Timoney characterise in [162] the centralizer of a $JB^*$-triple as its centroid, consisting of those bounded linear operators for which $T\{x,y,z\} = \{Tx,y,z\}$.

Several authors ([42], [101], [102], [128]) investigate the RNP for preduals of $JBW^*$-triples. Also here $M$-ideal methods turn out to be helpful; moreover one can read from the above papers that a $JB^*$-triple is $M$-embedded if and only if it is the $c_0$-sum of so-called elementary triples.

Finally, we mention the monographs [285] and [605] and the expository lectures [526] and [606] as general references on the subject. Also the recent paper [227] contains a survey of the general theory of $JB^*$-algebras.

Completely bounded maps and $M$-ideals. In recent years the study of "complete" notions has become a major subject in $C^*$-algebra theory. Here is the basic principle. Every (concrete) $C^*$-algebra $\mathfrak{A}$ of operators on a Hilbert space $H$ gives rise to a sequence of matrix algebras as follows. Denote by $M_n$ the space of complex $n \times n$-matrices and by $M_n(\mathfrak{A})$ the space of $n \times n$-matrices with entries from $\mathfrak{A}$. These spaces are endowed with $C^*$-algebra structures in a canonical (and unique) fashion, in that $M_n \cong L(\ell^2(n))$ and $M_n(\mathfrak{A})$ embeds into $M_n(L(H)) \cong L(H \oplus_2 \ldots \oplus_2 H)$. Now, given two $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and a bounded linear map $T : \mathfrak{A} \to \mathfrak{B}$, $T$ gives rise to a string of bounded linear operators $T_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$, defined by $T_n((x_{ij})) = (Tx_{ij})$. The operator $T$ is called completely bounded if $\sup_n \|T_n\| < \infty$, and the completely bounded norm $\|T\|_{cb}$ is defined to be this supremum. In general, given a property (P) such an operator might or might not have, $T$ is said to have "complete (P)" whenever all the $T_n$ fulfill (P); thus completely contractive, completely isometric, completely positive operators etc. are defined. For a detailed account and precise bibliographical references we refer to Paulsen's monograph [475].

In his address to the 1986 International Congress of Mathematicians [191] E. G. Effros suggested a research program entitled "Quantized Functional Analysis" to investigate matricially normed spaces and completely bounded maps from an abstract point of view.

Many steps in this program have been carried out so far (e.g. [80], [194], [196], [197]); here we would like to comment on $M$-ideal techniques employed in this setting. We first have to give some definitions.

Not only a $C^*$-algebra $\mathfrak{A}$ is equipped with a sequence of matrix norms, but also each of its closed subspaces. In addition, it is easily checked that for $X = (x_{ij}) \in M_n(\mathfrak{A})$ and $Y = (y_{ij}) \in M_m(\mathfrak{A})$ the block diagonal matrix $X \oplus Y := \operatorname{diag}(X, Y) \in M_{n+m}(\mathfrak{A})$ satisfies

$$\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\}.$$

This leads to the definition of an abstract operator space: An abstract operator space is a vector space $V$, together with a sequence of distinguished norms $\|\,.\,\|_n$ on each of the matrix spaces $M_n(V)$ such that

$$\|AXB\|_n \le \|A\|\,\|X\|_n\,\|B\| \quad \forall A, B \in M_n,\ X \in M_n(V),$$

$$\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\} \quad \forall X \in M_n(V),\ Y \in M_m(V).$$

Every normed space $(V, \|\,.\,\|)$ can be turned into an abstract operator space; but there are various ways to do this. One way is to equip $M_n(V)$, which can algebraically be identified with $M_n \otimes V$, with the injective tensor norm; another, different, method is to let

$$\|(v_{ij})\|_n^{\max} = \sup \|(\varphi(v_{ij}))\| \tag{1}$$

where the supremum ranges over all Hilbert spaces $H$ and all contractive $\varphi : V \to L(H)$, the norm on the right hand side of (1) referring to the canonical $C^*$-norm of $M_n(L(H))$. The above considerations show that a subspace of a $C^*$-algebra is an abstract operator space. A fundamental result due to Ruan [537] states that, conversely, every abstract operator space is completely isometric to a subspace of $L(H)$, endowed with its $C^*$-matricial norms; see [199] for a simpler proof. This result can be regarded as a "quantized" version of the result from ordinary "commutative" functional analysis that every normed space is isometric to a subspace of some $C(K)$-space.

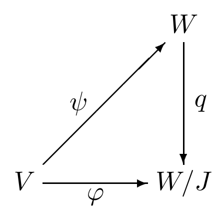
For a map $T$ between abstract operator spaces $(V, \{\|\,.\,\|_n\})$ and $(W, \{\|\,.\,\|_n\})$ complete boundedness is defined as above; we denote by $CB(V, W)$ the space of completely bounded maps and by $\|\,.\,\|_{cb}$ the completely bounded norm. One of the most important results on completely bounded maps is the Arveson-Wittstock Hahn-Banach theorem [475, p. 100] which can be rephrased as follows: For an abstract operator space $(V, \{\|\,.\,\|_n\})$, a subspace $S \subset V$ and a completely bounded map $\varphi : S \to L(H)$ there is an extension $\psi : V \to L(H)$ such that $\|\psi\|_{cb} = \|\varphi\|_{cb}$. A simple proof of this theorem, building on factorisation techniques, was given by Pisier [502].

The dual of an abstract operator space $(V, \{\|\,.\,\|_n\})$ can be given the structure of an operator space as well. For this one identifies $M_n(V^*)$ with $CB(V, M_n)$ and gives it the corresponding completely bounded norm. We shall always tacitly assume that this dual structure is imposed on $V^*$. Then it turns out that the canonical embedding from $V$ into $V^{**}$ is completely isometric; for details see [80] or [198].

Effros and Ruan [197] define and study complete $L$- and $M$-projections on abstract operator spaces, the definition being of course that all the $P_n$ are $L$- (resp. $M$-) projections on the $M_n(V)$. A complete $L$- (resp. $M$-) summand is the range of a complete $L$- (resp. $M$-) projection. A closed subspace $J \subset V$ is called a complete $M$-ideal if $J^\perp$ is a complete

$L$-summand in $V^*$. This can be shown to be equivalent with the requirement that all the $M_n(J)$ be $M$-ideals in $M_n(V)$. Effros and Ruan also give an example of an $M$-ideal which fails to be a complete $M$-ideal. In fact, if $\ell^\infty(3)$ is given an operator space structure by means of (1), then the $M$-summand $\ell^\infty(2)$ is not a complete $M$-ideal. (On the other hand, every $M$-ideal is complete for the operator space structure derived from $M_n \otimes_\varepsilon V$; this is a special case of Proposition VI.3.1 below. This is also the case in the $C^*$-algebra setting since the $M_n(J)$ are closed two-sided ideals if $J$ is.)
The motivation to consider complete $M$-ideals lies in the first place in their use in the proof of lifting theorems, see Theorem II.2.1. The corresponding result for abstract operator spaces can, however, only be proved for those operator spaces for which a certain approximation condition is fulfilled. Since this condition corresponds to the equation $L(E, X^{**}) = L(E, X)^{**}$ for finite dimensional $E$ in the Banach space setting and thus to the validity of the principle of local reflexivity, such operator spaces are termed locally reflexive; in the $C^*$-algebra case such a condition is investigated in [192] under the name property $(C'')$. So, the lifting theorem of [197], which can be regarded as a general form of the ones in [120] and [192], states:

> Theorem. *If $V$ and $W$ are abstract operator spaces with $V$ separable and $W$ locally reflexive, if $J \subset W$ is a complete $M$-ideal, $q : W \to W/J$ the (complete) quotient map and $\varphi : V \to W/J$ a complete contraction, and finally if $V$ enjoys the completely metric approximation property, then there exists a complete contraction $\psi : V \to W$ such that $q\psi = \varphi$:*




We also mention the paper [538] which studies uniqueness of preduals of abstract operator spaces from the $M$-structure point of view.

More on cohomology of operator algebras. The origins of homology theory are usually attributed to the paper "Analysis situs" by H. Poincaré [503]. Originally designed as a tool to distinguish between different topological spaces, this theory developed a purely algebraic branch on the basis of work by E. Noether, H. Hopf, L. Vietoris and W. Mayer. The diversity of algebraic (co)homology theories that emerged in the 40s and 50s found a unified and axiomatic foundation in the book of H. Cartan and Eilenberg [112]. For a somewhat more detailed exposition of this story we recommend Dieudonné's monograph [159].
The cohomology theory which is presented in Section V.4 was initiated in 1945 by Hochschild in [314]. (Hochschild himself attributes his construction to unpublished work of Eilenberg and MacLane, who had used similar constructions for groups.) One of his primary interests was an extension of Wedderburn's third structure theorem – an algebra $\mathfrak{A}$ splits as $\mathfrak{A} = \mathfrak{R} \oplus \mathfrak{A}_1$, where $\mathfrak{R}$ denotes the radical and $\mathfrak{A}_1$ is a *subalgebra* – in terms of

$H^2(\mathfrak{A},\mathfrak{R})$. He in fact proved that all singular extensions (i.e. extensions where $\mathfrak{R}^2 = 0$) of $\mathfrak{A}$ by $\mathfrak{R}$ are essentially parametrized by $H^2(\mathfrak{A},\mathfrak{R})$, and that a Wedderburn-type result holds for all extensions of $\mathfrak{A}$ iff $H^2(\mathfrak{A},\mathfrak{K}) = 0$ for every two-sided $\mathfrak{A}$-module $\mathfrak{K}$.

It is possible to relate low dimensional Hochschild groups to other known invariants. So $H^1(\mathfrak{A},\mathfrak{X})$ measures how many derivations from $\mathfrak{A}$ to $\mathfrak{X}$ exist which are not inner, and $H^2(\mathfrak{A},\mathfrak{X})$ is connected to extensions of $\mathfrak{A}$ by $\mathfrak{X}$ and to liftings of derivations. For $n = 3$, however, an appropriate interpretation is quite involved (see [315]) and a concrete meaning of $H^n(\mathfrak{A},\mathfrak{X})$ for $n > 3$ seems to be unknown. (It is, however, possible to "reduce dimensions" via the canonical isomorphism $C^{n+1}(\mathfrak{A},\mathfrak{X}) \cong C^n(\mathfrak{A}, C^1(\mathfrak{A},\mathfrak{X}))$ which yields $H^{n+1}(\mathfrak{A},\mathfrak{X}) \cong H^n(\mathfrak{A}, C^1(\mathfrak{A},\mathfrak{X}))$.) We should also mention that there is a corresponding sequence of homology groups $H_k(\mathfrak{A},\mathfrak{X})$, which are defined in terms of $n$-fold tensor-products $C_k(\mathfrak{A},\mathfrak{X}) = \mathfrak{X} \otimes_{\mathfrak{A}} \mathfrak{X} \ldots \otimes_{\mathfrak{A}} \mathfrak{X}$ in such a way that the adjoint of the boundary mapping $\delta_k : C_k(\mathfrak{A},\mathfrak{X}) \to C_{k-1}(\mathfrak{A},\mathfrak{X})$ equals $\delta^k$ as defined in Section V.4.

One of the first important results on Hochschild cohomology within the category of Banach algebras is due to Kadison [357] and, independently, Sakai [553], who showed that any derivation on a von Neumann algebra $\mathfrak{M}$ is inner or, equivalently, that $H^1(\mathfrak{M},\mathfrak{M}) = 0$. The whole theory was set to work within this frame mainly by Kadison, Ringrose and B. E. Johnson in a number of papers [346], [350], [358], [359]. (See also [347] or [522] for a survey of these results.) In the case of Banach algebras the idea to pass to multilinear operators which are *continuous* with respect to some natural topology suggests itself. (That it might be appropriate to use e.g. the weak$^*$ topology for dual modules is illustrated by the Johnson-Kadison-Ringrose theorem that was used in the proof of Corollary 4.10. It is taken from [350] where the authors apply this result in order to show that $H^k(\mathfrak{R},\mathfrak{M}) = 0$ whenever $\mathfrak{R}$ is a $C^*$-algebra acting on a Hilbert space $H$ and $\mathfrak{M}$ is a type I or hyperfinite von Neumann algebra which is a dual $\mathfrak{R}$-module.) Nevertheless, there is a dilemma: If one proceeds in this way, the cohomology theory thus obtained will suffer from the fact that continuous complementability of submodules is in general not to be expected and, consequently, that difficulties arise in connection with the existence of long exact sequences. On the other hand, the purely algebraic construction is difficult to handle in infinite dimensions. (This is also inherent in Helemskii's alternative approach to Hochschild groups in terms of admissible projective resolutions – see Chapter III, especially Theorem 4.9 therein, of the monograph [301], which is the most exhaustive source on the topic we are aware of.)

Applications of continuous Hochschild cohomology cover – besides the already mentioned characterisation of amenability of topological groups – Wedderburn decompositions of Banach algebras with finite dimensional radicals [346] as well as deformations of Banach algebras [349], [513]. It is furthermore connected to cyclic cohomology, see below.

More recent progress on the calculation of cohomology groups for operator algebras was made by Christensen, Effros and Sinclair [124] by use of the completely bounded version of this theory. They show that $H^k_{cb}(\mathfrak{A},\mathfrak{M})$, the cohomology group based on completely bounded multilinear maps, vanishes whenever $\mathfrak{A}$ is a $C^*$-subalgebra of an injective von Neumann algebra $\mathfrak{M}$. The point here is that in some cases these groups coincide with the (norm-) continuous Hochschild groups. This permits one to show that in the above situation, with $\mathfrak{A} \subset \mathfrak{M} \subset L(H)$ and under the additional assumption that the weak closure of $\mathfrak{A}$ is either properly infinite or else isomorphic to a von Neumann tensor product $\mathfrak{R}_0 \widehat{\otimes} \mathfrak{N}$, where $\mathfrak{N}$ is the (Murray-von Neumann) hyperfinite factor of type $\mathrm{II}_1$, it

is also true that $H^k(\mathfrak{A}, \mathfrak{M}) = 0$.
In another direction, Christensen and Sinclair proved that $H^k(\mathfrak{A}, \mathfrak{A}^*)$ vanishes for all $k$ whenever the $C^*$-algebra $\mathfrak{A}$ is nuclear or has no bounded traces [125].
We would finally like to remark that the groups $H^k(\mathfrak{A}, \mathfrak{A}^*)$ are of particular interest, since they are related to Connes' cyclic cohomology (cf. [136, pp. 100–102, II.4] and [113] for a general survey), as follows. A $(k+1)$-form $\omega$ with entries in $\mathfrak{A}$ is called *cyclic* if

$$\omega(a_1, \ldots, a_k, a_0) = (-1)^k \omega(a_0, a_1, \ldots, a_k),$$

and the space of cyclic $(k+1)$-forms is denoted by $C^k_\lambda(\mathfrak{A})$. By

$$\begin{aligned}\delta^k_\lambda \omega(a_0, a_1, \ldots, a_{k+1}) &= \sum_{\kappa=0}^{k} (-1)^\kappa \omega(a_0, \ldots, a_\kappa a_{\kappa+1}, \ldots, a_{k+1}) \; + \\ &\quad + (-1)^{k+1} \omega(a_{k+1} a_0, a_1, \ldots, a_k)\end{aligned}$$

one defines linear mappings $C^k_\lambda(\mathfrak{A}) \to C^{k+1}_\lambda(\mathfrak{A})$ which satisfy $\delta^k_\lambda \delta^{k-1}_\lambda = 0$. The resulting cohomology groups are denoted by $H^k_\lambda(\mathfrak{A})$. Since there is an obvious one-to-one correspondence between $(k+1)$-forms on $\mathfrak{A}$ and elements of $C^k(\mathfrak{A}, \mathfrak{A}^*)$, it can easily be shown that under this identification the $C^k_\lambda(\mathfrak{A})$ are a subcomplex of the Hochschild complex. Furthermore, there is a long exact sequence

$$\begin{aligned}0 \to H^0_\lambda(\mathfrak{A}) \to H^0(\mathfrak{A}, \mathfrak{A}^*) \to H^{-1}_\lambda(\mathfrak{A}) \to H^1_\lambda(\mathfrak{A}) \to H^1(\mathfrak{A}, \mathfrak{A}^*) \to H^0_\lambda(\mathfrak{A}) \to H^2_\lambda(\mathfrak{A}) \to \cdots \\ \cdots \to H^n_\lambda(\mathfrak{A}) \to H^n(\mathfrak{A}, \mathfrak{A}^*) \to H^{n-1}_\lambda(\mathfrak{A}) \to H^{n+1}_\lambda(\mathfrak{A}) \to H^{n+1}(\mathfrak{A}, \mathfrak{A}^*) \to \cdots\end{aligned}$$

connecting Hochschild groups with cyclic cohomology groups [136, p. 119]. The latter is very important for the calculation of $H^k_\lambda(\mathfrak{A})$ in general. It permits, for instance, using the Christensen-Sinclair result to prove that $H^k_\lambda(\mathfrak{A}) = 0$ for all $k$ when $\mathfrak{A}$ has no bounded traces, and that $H^k_\lambda(\mathfrak{A}) = 0$ ($k$ odd), respectively $H^k_\lambda(\mathfrak{A}) = H^0_\lambda(\mathfrak{A})$ for even $k$, whenever $\mathfrak{A}$ is nuclear.