CHAPTER IV

Banach spaces which are L-summands in their biduals

IV.1 Basic properties

In this section we will mainly be concerned with the stability properties of Banach spaces which are L-summands in their biduals. Furthermore the relation of these spaces with M-embedded spaces is investigated, various examples and counterexamples are given, and some isometric properties (proximinality, extreme points) are studied.

Recall from Definition III.1.1 that an L- (M-) embedded space means a Banach space which is L-summand (M-ideal) in its bidual.
The spaces A and H_0^1 of analytic functions appearing in the following list of examples of L-embedded spaces have been defined on page 104. Furthermore recall that a von Neumann algebra M is a unital selfadjoint subalgebra of $L(H)$ (where H is a Hilbert space) which is closed for the weak (equivalently: strong) operator topology. It is easy to check that the unit ball of M is then compact with respect to the weak operator topology, hence the subspace

$$M_* = \{f \in M^* \mid f|_{B_M} \text{ is continuous for the weak operator topology}\}$$

of the dual M^* is a predual of M; that is, M is isometrically isomorphic to $(M_*)^*$ [592, p. 70], and the inclusion map coincides with the canonical embedding of M_* into its bidual M^*. Moreover, M_* is the only predual of M [592, p. 135]. A well known result in C^*-algebra theory states that conversely every C^*-algebra which is isometric to a dual Banach space (such an object is sometimes called a W^*-algebra) is a von Neumann algebra [592, p. 133].

Example 1.1 *The following (classes of) spaces are L-summands in their biduals:*

(a) $L^1(\mu)$*-spaces,*
(b) *preduals of von Neumann algebras,*
(c) *duals of M-embedded spaces,*
(d) *the Hardy space H_0^1, the dual of the disk algebra A^* and L^1/H_0^1.*

Proof: (a) We first present a Banach lattice argument and refer to [559] for unexplained notation on this; see also Example I.1.6(a) and the text thereafter. $L^1(\mu)$ is a projection band in its bidual, which is again an AL-space. So the defining norm-condition of AL-spaces shows the claim (cf. [559, Section V.8]).
The idea of orthogonality also emerges if one employs the following measure theoretic argument: Represent $(L^1(\mu))^*$ as $C(K)$, hence $(L^1(\mu))^{**}$ as $M(K)$. Then μ has a canonical extension to a measure $\widehat{\mu}$ on K (see the proof of Example III.1.4(h)). Therefore, the Lebesgue decomposition with respect to $\widehat{\mu}$ yields the desired L-projection from $M(K)$ onto the copy of $L^1(\mu)$, along the singular measures on K.

(b) The proof relies on some results to be presented in Chapter V which we shall accept for the time being. Let M be a von Neumann algebra and M_* its predual. We consider the annihilator of M_* in M^{**}:

$$(M_*)^\perp = \{F \in M^{**} \mid F|_{M_*} = 0\}.$$

It will be shown in Theorem V.1.10 that M^{**} is a C^*-algebra in its own right and that multiplication both from the right and the left is weak* continuous. The product in M^{**} is the (by virtue of Theorem V.1.10(a) unique) Arens product as defined in Definition V.1.1.
We wish to show that $(M_*)^\perp$ is a two-sided ideal in M^{**}. To this end it suffices, by weak* continuity of the multiplications, to check that $Fx \in (M_*)^\perp$ and $xF \in (M_*)^\perp$ provided $x \in M$ and $F \in (M_*)^\perp$. To show the former we must prove

$$(Fx)(f) = F(xf) = 0 \qquad \forall f \in M_*. \tag{$*$}$$

Now, by definition of the Arens product, $(xf)(y) = f(yx)$ for $y \in M$ from which we deduce $xf \in M_*$. Hence $(*)$ obtains. The proof that $xF \in (M_*)^\perp$ is similar.
Using Theorem V.4.4 we conclude that $(M_*)^\perp$ is a weak* closed M-ideal, and Corollary II.3.6(b) implies that M_* is an L-summand in M^*.

(c) Corollary III.1.3.

(d) The Hardy space H_0^1 is the dual of the M-embedded space $C(\mathbb{T})/A$; see Example III.1.4(h). (We shall give another proof in Example 3.6.)
As for A^*, note that $A^* \cong M(\mathbb{T})/A^\perp \cong M(\mathbb{T})/H_0^1$ by the F. and M. Riesz theorem, so the claim about A^* follows from Corollary 1.3 below and the fact that $M(\mathbb{T})$ and H_0^1 are L-embedded spaces. (Note that A^* has no M-embedded predual because it fails the RNP.)
The same reasoning applies to L^1/H_0^1. □

We mention in passing that $H^1 \cong H_0^1$ and $L^1/H^1 \cong L^1/H_0^1$ so that these spaces are L-embedded, too.

In example (a) above the complementary L-summand for $X = L^1(\mu)$ in X^{**} is the set of elements which are singular, i.e. lattice or measure theoretically orthogonal with respect to X. Adopting this viewpoint also in the general case we will write X_s for the complementary L-summand of a Banach space X which is L-summand in its bidual, i.e.

$$X^{**} = X \oplus_1 X_s.$$

It is not only a formal matter to imagine the elements in X_s as "singular" elements of X^{**}. Proposition 2.1 shows that $x^{**} \in X^{**}$ belongs to X_s if x^{**}, viewed as a function on (B_{X^*}, w^*), is "extremely discontinuous", and for $X = L^1(\mu)$, $X^{**} = ba(\mu)$ a finitely additive measure ν is in X_s if it is "extremely non-σ-additive", namely purely finitely additive (see the discussion at the beginning of Section IV.3.)

Unlike to what has been shown in Theorem III.1.5 for M-embedded spaces, L-embeddedness does not pass to subspaces or quotients. (This is clear for quotients as every Banach space is a quotient of an L^1-space and follows for subspaces e.g. from Corollary 1.15.) Let us have a closer look at subspaces of L-embedded spaces:

Theorem 1.2 *For a Banach space X which is an L-summand in its bidual (so that $X^{**} = X \oplus_1 X_s$) and a closed subspace Y of X the following are equivalent:*

(i) *Y is an L-summand in its bidual.*

(ii) $Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap X_s)$

(iii) $P\overline{Y}^{w*} = Y$, *where P is the L-projection from X^{**} onto X.*

(iii′) $P\overline{B_Y}^{w*} = B_Y$

PROOF: (i) ⇒ (ii): Embedding Y^{**} as $Y^{\perp\perp}$ in X^{**} we have

$$Y^{\perp\perp} = Y \oplus_1 Z \qquad (*)$$

for some subspace Z of X^{**} since by assumption $Y^{**} = Y \oplus_1 Y_s$ (of course $Z \subset Y^{\perp\perp}$). We want to show

$$Z = Y^{\perp\perp} \cap X_s.$$

The inclusion "$\supset$" is obvious. To prove "$\subset$" take $z \in Z$ and decompose $z = x + x_s$ in $X^{**} = X \oplus_1 X_s$. Because of $(*)$ we have for all $y \in Y$

$$\|y + z\| = \|y\| + \|z\| = \|y\| + \|x\| + \|x_s\|$$

and because of $X^{**} = X \oplus_1 X_s$

$$\|y + z\| = \|(y + x) + x_s\| = \|y + x\| + \|x_s\|,$$

hence

$$\|y + x\| = \|y\| + \|x\| \qquad \text{for all } y \in Y.$$

But then also

$$\|y^{**} + x\| = \|y^{**}\| + \|x\| \qquad \text{for all } y^{**} \in Y^{\perp\perp}.$$

(This follows from considering the space $G = Y \oplus_1 \mathbb{K}x$ and noting $G^{**} = Y^{\perp\perp} \oplus_1 \mathbb{K}x$.) Since $z \in Y^{\perp\perp}$ we get

$$\|x_s\| = \|z - x\| = \|z\| + \|x\| = 2\,\|x\| + \|x_s\|.$$

So $x = 0$, hence $y \in X_s$.

(ii) $\Rightarrow$ (iii): Obvious by $Y^{\perp\perp} = \overline{Y}^{w*}$.

(iii) $\Rightarrow$ (i): Since (iii) is equivalent to $PY^{\perp\perp} = Y$ it follows from Lemma I.1.15 that $P|_{Y^{\perp\perp}}$ is an L-projection from $Y^{\perp\perp} = Y^{**}$ onto Y.

(iii) $\Rightarrow$ (iii′): Trivial.

(iii′) $\Rightarrow$ (iii): Since $\overline{B_Y}^{w*} = B_{Y^{\perp\perp}}$ we have $\overline{Y}^{w*} = \bigcup_{n\in\mathbb{N}} n\overline{B_Y}^{w*}$. [Note, however, that for a subspace Y of an arbitrary dual space E^* the inclusion $\overline{Y}^{\sigma(E^*,E)} = \bigcup_n n\overline{B_Y}^{\sigma(E^*,E)}$ is *false* in general.] ☐

Corollary 1.3 *Let X and Y be Banach spaces which are L-summands in their biduals, Y a subspace of X. Then X/Y is an L-summand in its bidual.*

Proof: If P is the L-projection from X^{**} onto X we have by assumption and Theorem 1.2 $PY^{\perp\perp} \subset Y^{\perp\perp}$. Lemma I.1.15 shows now that

$$\begin{array}{rcccc} P/Y^{\perp\perp} & : & X^{**}/Y^{\perp\perp} & \longrightarrow & X^{**}/Y^{\perp\perp} \\ & & x^{**} + Y^{\perp\perp} & \longmapsto & Px^{**} + Y^{\perp\perp} \end{array}$$

is an L-projection onto $(X + Y^{\perp\perp})/Y^{\perp\perp} \cong X/(X \cap Y^{\perp\perp}) = X/Y$. Clearly $X^{**}/Y^{\perp\perp} \cong (X/Y)^{**}$. ☐

Easy examples show:

- *X and X/Y L-embedded spaces $\not\Rightarrow$ Y L-embedded.*
- *Y and X/Y L-embedded spaces $\not\Rightarrow$ X L-embedded.*

Before we proceed with the discussion of stability properties of L-embedded spaces we give a kind of quantitative version of Theorem 1.2 which will be used in the next section. Note that for L-embedded spaces X and Y, with Y a subspace of X and corresponding L-projections P and Q from the the biduals onto the spaces, Theorem 1.2 gives $P|_{Y^{\perp\perp}} = Q$.

Lemma 1.4 *Let X be an L-embedded space with L-projection P from X^{**} onto X. Let the subspace $Y \subset X$ be an almost L-summand in its bidual in the sense that there is a number $0 < \varepsilon < 1/4$ such that $Y^{**} = Y \oplus Y_s$ and $\|y + y_s\| \geq (1-\varepsilon)(\|y\| + \|y_s\|)$ for all $y \in Y$, $y_s \in Y_s$. Then $\|P|_{Y^{\perp\perp}} - Q\| \leq 3\varepsilon^{1/2}$, where Y^{**} and $Y^{\perp\perp} = \overline{Y}^{w*} \subset X^{**}$ are identified and Q denotes the projection from $Y^{\perp\perp}$ onto Y.*

Proof: By assumption there is a subspace $Z \subset X^{**}$ such that $Y^{**} \cong Y^{\perp\perp} = \overline{Y}^{w*} = Y \oplus Z$ with $\|y + z\| \geq (1-\varepsilon)(\|y\| + \|z\|)$. Because of $\|Py^{\perp\perp} - Q\,y^{\perp\perp}\| = \|P(y+z) - Q(y+z)\| = \|Pz\|$ and because of $(\varepsilon^{1/2} + 2\varepsilon)\|z\| \leq \frac{\varepsilon^{1/2}+2\varepsilon}{1-\varepsilon}\|y+z\| \leq 3\varepsilon^{1/2}\|y+z\|$ for any $y \in Y$, $z \in Z$ it is enough to show $\|Pz\| \leq (\varepsilon^{1/2} + 2\varepsilon)\|z\|$ for each $z \in Z$. Decompose

$z = x + x_s$ in $X^{**} = X \oplus_1 X_s$. Since we are done if $\|x\| = \|Pz\| \le \varepsilon^{1/2}\|z\|$, we assume $\|x\| > \varepsilon^{1/2}\|z\|$ from now on. We obtain

$$\begin{aligned} \|y + x\| &= \|(y + x) + x_s\| - \|x_s\| = \|y + z\| - \|x_s\| \\ &\ge (1 - \varepsilon)(\|y\| + \|z\|) - \|x_s\| \\ &= (1 - \varepsilon)(\|y\| + \|x\| + \|x_s\|) - \|x_s\| \\ &= (1 - \varepsilon)(\|y\| + \|x\|) - \varepsilon\|x_s\| \\ &\ge (1 - \varepsilon)(\|y\| + \|x\|) - \varepsilon\|z\| \\ &\ge (1 - \varepsilon)(\|y\| + \|x\|) - \varepsilon^{1/2}\|x\| \\ &\ge (1 - 2\varepsilon^{1/2})(\|y\| + \|x\|) \qquad (1) \end{aligned}$$

for all $y \in Y$, which extends to all $y^{\perp\perp} \in Y^{\perp\perp}$, as will be shown in a moment:

$$\|y^{\perp\perp} + x\| \ge (1 - 2\varepsilon^{1/2})(\|y^{\perp\perp}\| + \|x\|). \qquad (2)$$

For the time being we take (2) for granted and have in particular for $z \in Y^{\perp\perp}$

$$\|x_s\| = \| - z + x\| \ge (1 - 2\varepsilon^{1/2})(\|z\| + \|x\|) \ge (1 - 2\varepsilon^{1/2})(\|z\| + \varepsilon^{1/2}\|z\|)$$

and finally

$$\|Pz\| = \|x\| = \|z\| - \|x_s\| \le \|z\| - (1 - 2\varepsilon^{1/2})(1 + \varepsilon^{1/2})\|z\| = (\varepsilon^{1/2} + 2\varepsilon)\|z\|.$$

It remains to prove (2). We first observe that $x \notin Y$ because otherwise

$$0 = \| - x + x\| \ge (1 - 2\varepsilon^{1/2})(\| - x\| + \|x\|) > 0$$

by our standing assumption that $\|x\| > \varepsilon^{1/2}\|z\|$. Thus it makes sense to consider the direct sum $G = Y \oplus \mathbb{K}x \subset X$, and we let ι denote the identity from G onto $\widetilde{G} = Y \oplus_1 \mathbb{K}x$. Therefore $\widetilde{G}^{**} \cong Y^{\perp\perp} \oplus_1 \mathbb{K}x$ and $\|\iota\| \le (1 - 2\varepsilon^{1/2})^{-1}$ by (1). Inequality (2) follows now with $y^{\perp\perp} + x \in G^{\perp\perp}$ from

$$\|y^{\perp\perp}\| + \|x\| = \|\iota^{**}(y^{\perp\perp} + x)\| \le (1 - 2\varepsilon^{1/2})^{-1}\|y^{\perp\perp} + x\|. \qquad \square$$

We continue with two more stability properties of L-embedded spaces.

Proposition 1.5 *The class of L-embedded spaces is stable by taking*
(a) *1-complemented subspaces,*
(b) *ℓ^1-sums.*

Proof: (a) Let X be an L-summand in its bidual, $P : X^{**} \to X^{**}$ the L-projection onto X, Y a 1-complemented subspace of X, $Q : X \to Y$ the contractive projection (considered as a mapping onto Y), and $i : Y \to X$ the inclusion mapping. Then $Q^{**}Pi^{**}$ is the L-projection from Y^{**} onto Y.

(b) Let $(X_i)_{i\in I}$ be a family of L-embedded spaces and $X := (\oplus \sum X_i)_{\ell^1(I)}$. Then $X^* = (\oplus \sum X_i^*)_{\ell^\infty(I)}$. Putting

$$Y := \Big(\oplus \sum X_i^{**}\Big)_{\ell^1(I)} \quad \text{and} \quad Z := \Big\{x^{**} \in X^{**} \mid x^{**}|_{\left(\oplus \sum X_i^*\right)_{c_0(I)}} = 0\Big\}$$

it is standard to show that

- $Y \cap Z = \{0\}$,
- $\|(x_i^{**}) + x^{**}\| = \|(x_i^{**})\| + \|x^{**}\| \quad$ for $(x_i^{**}) \in Y$, $x^{**} \in Z$,
- $Y + Z = X^{**}$.

So we have the decomposition $X^{**} = Y \oplus_1 Z$. Using the assumption it is now easy to see that X is an L-summand in Y, hence X is an L-summand in X^{**}. □

Proposition 1.6 *Let X be an L-embedded space, Y_1, Y_2, $Y_i\,(i \in I)$ subspaces of X which are also L-embedded spaces. Then:*

(a) *$\bigcap_{i \in I} Y_i$ is an L-embedded space.*
(b) *$B_{Y_1} + B_{Y_2}$ is closed.*
(c) *If $Y_1 + Y_2$ is closed, then $Y_1 + Y_2$ is an L-embedded space.*

Proof: (a) follows from Theorem 1.2.

(b) By w^*-compactness and w^*-continuity of addition we have

$$\overline{B_{Y_1} + B_{Y_2}}^{w*} = \overline{B_{Y_1}}^{w*} + \overline{B_{Y_2}}^{w*},$$

hence by Theorem 1.2(iii′) (P is the L-projection from X^{**} onto X)

$$P(\overline{B_{Y_1} + B_{Y_2}}^{w*}) = P(\overline{B_{Y_1}}^{w*} + \overline{B_{Y_2}}^{w*}) = P\overline{B_{Y_1}}^{w*} + P\overline{B_{Y_2}}^{w*} = B_{Y_1} + B_{Y_2}.$$

This implies the closedness of $B_{Y_1} + B_{Y_2}$.

(c) If $Y_1 + Y_2$ is closed, a standard application of the open mapping theorem shows that there is $\alpha > 0$ such that

$$B_{Y_1+Y_2} \subset \alpha(B_{Y_1} + B_{Y_2}),$$

hence

$$P(\overline{B_{Y_1} + B_{Y_2}}^{w*}) \subset \alpha P(\overline{B_{Y_1}}^{w*} + \overline{B_{Y_2}}^{w*}) = \alpha(B_{Y_1} + B_{Y_2}) \subset Y_1 + Y_2.$$

An appeal to Theorem 1.2 concludes the proof. □

The next example shows that one can't expect more in Proposition 1.6(b). For part (b) recall that the bidual of an L^1-space [of the predual of a von Neumann algebra] is again an L^1-space [the predual of a von Neumann algebra].

Example 1.7

(a) *If X is an L-embedded space and Y_1, Y_2 are subspaces of X which are also L-embedded spaces, then $Y_1 + Y_2$ need not be closed and $\overline{Y_1 + Y_2}^{\|\,\|}$ need not be an L-embedded space.*
(b) *The bidual of an L-embedded space need not be an L-embedded space.*

PROOF: (a) By Proposition 1.10 below a subspace of $X := \ell^1 \cong \ell^1 \oplus_1 \ell^1$ is an L-embedded space if and only if it is w^*-closed. So take two w^*-closed subspaces Y_1 and Y_2 of X such that $\overline{Y_1 + Y_2}^{\|\ \|}$ is not w^*-closed. [To be specific, consider $Y_1 := \ell^1 \times \{0\}$ and $Y_2 = \operatorname{graph}(T)$ with

$$\begin{array}{rccl} T & : & \ell^1 & \longrightarrow & \ell^1 \\ & & (x_n) & \longmapsto & (-\sum \frac{x_k}{k}, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots). \end{array}$$

Note: T is w^*-continuous and $\overline{Y_1 + Y_2}^{\|\ \|} = \{(x,y) \mid \sum y_k = 0\}$, which is w^*-dense.]

(b) The space $X = (\oplus \sum \ell^\infty(n))_{\ell^1}$ is an L-embedded space by Proposition 1.5(b) and $X^* = \bigl(\oplus \sum \ell^1(n)\bigr)_{\ell^\infty}$. Now X^* contains a 1-complemented subspace isometric to ℓ^1 (see below), hence X^{**} contains an isometric copy of ℓ^∞ as a 1-complemented subspace. If X^{**} were an L-embedded space, then ℓ^∞ would be one by Proposition 1.5(a) as well – but ℓ^∞ has nontrivial M-structure.

For the convenience of the reader let us specify a contractive projection P from X^* onto an isometric copy of ℓ^1: Write the elements of X^* as $(x_{kn})_{k,n\in\mathbb{N}}$ with $x_{kn} = 0$ if $k > n$, i.e. $(x_{1n}, \ldots, x_{nn}, 0, \ldots) = (x_{1n}, \ldots, x_{nn}) \in \ell^1(n)$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and put for $k \in \mathbb{N}$

$$y_k := \lim_{n,\mathcal{U}} x_{kn} \qquad \text{and} \qquad y_{kn} = \begin{cases} 0 & \text{if } k > n \\ y_k & \text{if } k \le n. \end{cases}$$

Then $P : (x_{kn}) \mapsto (y_{kn})$ is the desired projection. (One needs a free ultrafilter to get that P maps onto the subspace $\{(y_{kn}) \mid (y_k) \in \ell^1\}$.) □

Concerning the first statement in Example 1.7 we remark that in every infinite dimensional L-embedded space X, which is the dual of an M-embedded space Z, there are subspaces Y_1 and Y_2 such that $Y_1 + Y_2$ is not closed. Indeed, by a construction of Wilansky [635, p. 12] there are closed subspaces Z_1 and Z_2 in Z with $Z_1 + Z_2$ not closed. Let $Y_i = {Z_i}^\perp$ and recall the result of Reiter mentioned in Lemma I.1.14(a).

Using L-embedded spaces we will now partially redeem what was promised in Remark (c) following Theorem I.1.9.

Example 1.8 *There is a nonreflexive L-embedded space Y such that Y^* contains no nontrivial M-ideals. Hence (putting $X := Y^*$): There is a Banach space X without nontrivial M-ideals such that X^* has a nontrivial L-summand.*

PROOF: Take $Y = A_*$, the predual of an infinite dimensional von Neumann algebra A which is a finite factor (see [165], [592] for definitions and examples). Then $Y^* = A$ has no nontrivial closed two-sided ideals ([165, p. 257, Cor. 3] or [592, p. 349, Cor. 4.7]), hence no nontrivial M-ideals either, by Theorem V.4.4 below. □

The next two results concern L-embedded spaces which are duals of M-embedded spaces. We characterise these spaces in the class of Banach spaces which are L-summands in their biduals and describe their subspaces which are L-embedded.

Proposition 1.9 *Let X be a Banach space which is an L-summand in its bidual.*

(a) *There is (up to isometric isomorphism) at most one predual of X which is an M-ideal in its bidual.*

(b) *There is a predual of X which is an M-ideal in its bidual if and only if X_s is w^*-closed in X^{**}.*

More generally:

(b′) *If Z is a subspace of X which is an L-summand in its bidual, then there is a predual of Z which is an M-ideal in its bidual if and only if $Z^{\perp\perp} \cap X_s$ is w^*-closed in X^{**}.*

Proof: (a) Let Y_1, Y_2 be M-embedded spaces with $Y_1^* \cong X \cong Y_2^*$ and $I : Y_1^* \longrightarrow Y_2^*$ an isometric isomorphism. We claim

$$\pi_{Y_1^*} = I^{**-1} \pi_{Y_2^*} I^{**} \qquad (*)$$

where $\pi_{Y_i^*}$ denotes the natural (L-) projection from Y_i^{***} to Y_i^*. Assuming $(*)$ we have $I^{**}\pi_{Y_1^*} = \pi_{Y_2^*} I^{**}$, but this is equivalent to the w^*-w^*-continuity of I. Hence $I = J^*$ for some isometric isomorphism $J : Y_2 \longrightarrow Y_1$.
To prove $(*)$ observe that $\pi_{Y_1^*}$ is an L-projection and $I^{**-1}\pi_{Y_2^*} I^{**}$ is a bicontractive projection (even an L-projection) with the same range, so they coincide by Proposition I.1.2.

(b) The "only if" part is obvious. So assume that $X_s = Y^\perp$ for some subspace Y of X^*, i.e.

$$X^{**} = X \oplus_1 Y^\perp. \qquad (\dagger)$$

Is is easy to show that

$$\begin{array}{rcl} I : X & \longrightarrow & Y^* \\ x & \longmapsto & i_x|_Y \end{array}$$

is an isometric isomorphism (note that because of $X \cap Y^\perp = \{0\}$ the subspace Y is total). The isometric isomorphism I^* maps $i_Y(Y)$ onto Y, which shows, since by $(\dagger)$ Y is an M-ideal in X^*, that $i_Y(Y)$ is an M-ideal in Y^{**}.

(b′) By assumption and Theorem 1.2 we have $Z_s = Z^{\perp\perp} \cap X_s$. Noting that (via $i^{**} : Z^{**} \longrightarrow X^{**}$; $i : Z \longrightarrow X$ the inclusion map) $\sigma(Z^{**}, Z^*) = \sigma(X^{**}, X^*)|_{Z^{\perp\perp}}$ we conclude using the special case (b): Z has an M-embedded predual iff Z_s is $\sigma(Z^{**}, Z^*)$-closed iff $Z^{\perp\perp} \cap X_s$ is $\sigma(X^{**}, X^*)|_{Z^{\perp\perp}}$-closed iff $Z^{\perp\perp} \cap X_s$ is $\sigma(X^{**}, X^*)$-closed. □

Proposition 1.10 *For a Banach space X which is an M-ideal in its bidual and a subspace Y of X^* the following are equivalent:*

(i) *Y is an L-summand in its bidual.*

(ii) *Y is the dual of a space which is an M-ideal in its bidual.*

(iii) *Y is $\sigma(X^*, X)$-closed.*

Proof: (i) ⇒ (iii): Since X is an M-embedded space, the L-projection P is the natural projection from X^{***} onto X^*, i.e., considered as a map to X^* it is i_X^*. By Theorem 1.2 we have $P\overline{B_Y}^{\sigma(X^{***}, X^{**})} = B_Y$. So by w^*-continuity of i_X^* and $\sigma(X^{***}, X^{**})$-compactness of $\overline{B_Y}^{\sigma(X^{***}, X^{**})}$ we infer that B_Y is $\sigma(X^*, X)$-compact, hence Y is $\sigma(X^*, X)$-closed by the Krein-Smulian theorem.

(iii) ⇒ (ii): If $Y = Z^\perp$ for some $Z \subset X$, we have that X/Z is an M-embedded space (Theorem III.1.6) and $(X/Z)^* \cong Z^\perp = Y$.

(ii) $\Rightarrow$ (i): Corollary III.1.3. □

In connection with Theorem 1.2 and Proposition 1.10 the following general fact seems worth remarking:

If X is a Banach space and P denotes the natural projection from X^{***} onto X^*, then for a subspace Y of X^* the following are equivalent:

(i) $PY^{\perp\perp} = Y$
(ii) $Y^{\perp\perp} = Y \oplus (Y^{\perp\perp} \cap X^{\perp})$
(iii) *Y is $\sigma(X^*, X)$-closed.*

[Proof: The equivalence of (i) and (ii) is straightforward. For "(i) $\Rightarrow$ (iii)" argue as in the above proof. For "(iii) $\Rightarrow$ (i)" with $Y = Z^{\perp}$ for some $Z \subset X$ one may consider the quotient map $q : X \longrightarrow X/Z$ and observe that $\operatorname{ran} q^* = Z^{\perp}$, $\operatorname{ran} q^{***} = Z^{\perp\perp\perp} = Y^{\perp\perp}$ and $Pq^{***} = q^{***}Q$, where Q is the natural projection from $(X/Z)^{***}$ onto $(X/Z)^*$.]

We conclude this section with the study of some isometric properties (proximinality, extreme points) of L-embedded spaces. We prepare our discussion of best approximation by the following simple lemma.

Lemma 1.11 *Let X be a Banach space and $P : X^{**} \longrightarrow X^{**}$ a contractive projection. Then every subspace Y of X for which $P\overline{Y}^{w*} \subset Y$ is proximinal in PX^{**}.*

Proof: Recall the well known fact that every w^*-closed subspace of a dual space is proximinal. Then take a best approximation y^{**} in $\overline{Y}^{w*}$ for $x \in PX^{**}$. We get

$$d(x, Y) \geq d(x, \overline{Y}^{w*}) = \|x - y^{**}\| \geq \|P(x - y^{**})\| = \|x - Py^{**}\| \geq d(x, Y)\ . \qquad (*)$$

Hence Py^{**} is a best approximation for x in Y and $d(x, Y) = d(x, \overline{Y}^{w*})$. □

Applying this to our present situation we get:

Proposition 1.12 *Let X be a Banach space which is an L-summand in its bidual and Y a subspace of X, which is also an L-summand in its bidual. Then Y is proximinal in X and $P_Y(x)$, the set of best approximations to x from Y, is weakly compact for all $x \in X$.*

Proof: The first part is obvious from the above lemma and Theorem 1.2. For the second part note that the last (in)equality in $(*)$ yields

$$\|x - Py^{**}\| = \|x - y^{**}\| = \|x - Py^{**} - (Id - P)y^{**}\| = \|x - Py^{**}\| + \|y^{**} - Py^{**}\|\ .$$

Hence $y^{**} = Py^{**}$, and $P_{\overline{Y}^{w*}}(x) = P_Y(x)$. But then (with $d := d(x, Y)$)

$$P_Y(x) = Y \cap B_X(x, d) = \overline{Y}^{w*} \cap B_{X^{**}}(x, d) = P_{\overline{Y}^{w*}}(x)$$

shows that the w^*-closure of $P_Y(x)$ stays in X and this gives the weak compactness. □

To put the next results into their proper perspective note the following two facts:

- *For a proximinal subspace Y of a Banach space X there is always a homogeneous lifting $f : X/Y \longrightarrow X$ of the quotient map q satisfying $\|f(\dot{x})\| = \|\dot{x}\|$.*
  (See Theorem II.1.9 and the text thereafter. An easy two-dimensional example shows that such a lifting need not be continuous.)

- *$L^1(\mu)$ contains a nontrivial finite-dimensional subspace with a continuous selection for the metric projection iff μ has atoms.* ([390, Th. 1.4])

Hence for an L-embedded space X with an L-embedded subspace Y one cannot expect a continuous lifting $f : X/Y \longrightarrow X$ in general.

Proposition 1.13 *Let X be an L-embedded Banach space and Y an L-embedded subspace of X. Then every lifting $f : X/Y \longrightarrow X$ of the quotient map q satisfying $\|f(\dot{x})\| = \|\dot{x}\|$ maps relatively $\|\,.\,\|$-compact sets onto relatively weakly compact sets. If Y is Chebyshev, f is even norm-weak continuous.*

PROOF: Let $K \subset X/Y$ be relatively $\|\,.\,\|$-compact and $f(\dot{x}_\alpha) \in f(K)$. Choose a $\|\,.\,\|$-convergent subnet $(\dot{x}_\beta)$ with $\dot{x}_\beta \longrightarrow \dot{x}$. Since $(f(\dot{x}_\beta))$ is bounded we find another subnet $(\dot{x}_\gamma)$ such that $f(\dot{x}_\gamma) \stackrel{w*}{\longrightarrow} x^{**} \in X^{**}$. We wish to show $x^{**} \in X$. Now $q^{**}(f(\dot{x}_\gamma)) \stackrel{w*}{\longrightarrow} q^{**}(x^{**})$ and $q^{**}(f(\dot{x}_\gamma)) = q(f(\dot{x}_\gamma)) = \dot{x}_\gamma$. From this we conclude

$$q^{**}(x^{**}) = \dot{x} = q(f(\dot{x})) = q^{**}(f(\dot{x})),$$

i.e. $x^{**} - f(\dot{x}) \in Y^{\perp\perp} = Y \oplus_1 Y_s$. Writing $x^{**} - f(\dot{x}) = y_a + y_s$ according to this decomposition, we get from the w^*-lower semicontinuity of the norm of X^{**}

$$\begin{aligned} \|\dot{x}\| &= \lim \|\dot{x}_\gamma\| = \lim \|f(\dot{x}_\gamma)\| \geq \|x^{**}\| \\ &= \|f(\dot{x}) + y_a\| + \|y_s\| \geq \|f(\dot{x}) + y_a\| \\ &\geq d(f(\dot{x}), Y) = \|q(f(\dot{x}))\| = \|\dot{x}\|. \end{aligned} \tag{1}$$

Hence $y_s = 0$, so $x^{**} \in X$ and the w^*-convergence of $f(\dot{x}_\gamma)$ to x^{**} is the desired w-convergence.
Concerning the second part of the statement note that $\|\dot{x}\| = \|f(\dot{x})\|$ means that 0 is a best approximation for $f(\dot{x})$, and the (in)equality (1) shows that $-y_a$ is another one. By uniqueness we get $y_a = 0$, and this implies $x^{**} = f(\dot{x})$. We leave it to the reader to adjust the beginning of the proof appropriately. ☐

It is obvious that we can't expect unique best approximation in general. However let us mention that by a result of Doob H^1_0 *is* Chebyshev in $L^1(\mathbb{T})$. Kahane has even characterised the translation invariant subspaces Y of $L^1(\mathbb{T})$ which admit unique best approximation: $Y = L^1_\Lambda(\mathbb{T})$ with Λ an infinite arithmetical progression with odd difference [362, p. 795]. In Section IV.4 we will see that they are all L-embedded (Proposition 4.4).
The weak compactness of $P_Y(x)$ has a nice consequence for the extreme points of the unit balls. For completeness we include the relevant result.

Proposition 1.14 *Let Y be a proximinal subspace of a Banach space X for which $P_Y(x)$ is weakly compact for all $x \in X$. Then*

$$\operatorname{card}(\operatorname{ex} B_{X/Y}) \leq \operatorname{card}(\operatorname{ex} B_X)\ .$$

PROOF: We first claim:

For every $\dot{x} \in \operatorname{ex} B_{X/Y}$ *there is* $y \in P_Y(x)$ *such that* $z = x - y \in \operatorname{ex} B_X$.

To see this observe that the weakly compact set $x - P_Y(x)$ has an extreme point $z = x - y$ by the Krein-Milman theorem. Now $x - P_Y(x) = B_X \cap (x + Y)$ and the latter set is a face of B_X (since $\dot{x} \in \operatorname{ex} B_{X/Y}$), i.e. $z = x - y \in \operatorname{ex} B_X$.
The mapping $\dot{x} \mapsto z$ from $\operatorname{ex} B_{X/Y}$ to $\operatorname{ex} B_X$, defined by the claim, is clearly injective, hence the proposition is proved. □

Combining Propositions 1.12 and 1.14 with the well known fact that $\operatorname{ex} B_{L^1(\mu)} = \emptyset$ if μ has no atoms, we get:

Corollary 1.15 *If* μ *has no atoms then no proper finite-codimensional subspace* Y *of* $L^1(\mu)$ *is an L-summand in its bidual. In particular, no such subspace is the range of a contractive projection in* $L^1(\mu)$.

We remark that a complete description of the ranges of contractive projections in $L^1(\mu)$-spaces is given by the Douglas-Ando theorem: these are the L^1-spaces over σ-subrings and measures absolutely continuous with respect to μ, cf. [385, p. 162]. In Section IV.3 we will characterise the L-embedded subspaces of $L^1(\mu)$.

The next result was first proved in [21, p. 37].

Corollary 1.16 $\operatorname{ex} B_{L^1/H^1_0} = \emptyset$

PROOF: Propositions 1.12 and 1.14, Example 1.1(d). □

Remark 1.17 (Smoothness and rotundity in L-embedded spaces)
(a) In Corollary III.2.12 we have proved using a renorming of M-embedded spaces:

There are nonreflexive strictly convex L-embedded spaces.

Concerning smoothness the following example is worth mentioning:

The Hardy space H^1_0 *is smooth; hence there are nonreflexive smooth L-embedded spaces.*

[PROOF: We consider H^1_0 as a real Banach space. Take an $f \in S_{H^1_0}$. Then $\mu\{z \in \mathbb{T} \mid f(z) = 0\} = 0$ (e.g. [316, p. 52]). Now also in $\mathbb{C}$-valued $L^1(\mu)$-spaces – considered as real Banach spaces – one has for $f \in S_{L^1(\mu)}$ the equivalence "f is a smooth point iff $\mu\{f = 0\} = 0$". So f is a smooth point even in $L^1(\mathbb{T})$, all the more in H^1_0.]

Note that this gives an easy and "natural" example of the following situation:

There is a Banach space X *such that* X *is smooth and* X^* *is not strictly convex.*

[In fact ${H^1_0}^* \cong L^\infty/H^\infty$ is not strictly convex, because it is the bidual of the M-embedded space $C(\mathbb{T})/A$; use Proposition I.1.7.] The first example of the above phenomenon was constructed by Klee in [379, Prop. 3.3]; another example is due to Troyanski [601]. We also note explicitly:

The space $C(\mathbb{T})/A$ is a strictly convex M-embedded space.

In fact, its dual H_0^1 is smooth.

(b) Passing to stronger smoothness properties we observe:

The norm of a nonreflexive L-embedded space is nowhere Fréchet-differentiable.

[Proof: The following assertion is well known and easy to prove:

If the norm of a Banach space X is F-differentiable at x, then the norm of the bidual X^{**} is F-differentiable at i_x.

So assuming F-differentiability at x we would have in particular that i_x is a smooth point in X^{**} – but for nonreflexive L-embedded spaces this is clearly false.]

IV.2 Banach space properties of L-embedded spaces

We will start this section with two proofs of the weak sequential completeness of L-embedded spaces X. Both reveal interesting aspects: the first one the extreme discontinuity of the elements of X_s, the second one a weak form of w^*-closedness of arbitrary L-summands in dual spaces. We will then prove Pełczyński's property (V^*) for L-embedded spaces. Although this property is stronger than weak sequential completeness, its proof requires to have the latter in advance. We will conclude with a study of the Radon-Nikodým property for spaces which are L-summands in their biduals.

The following proposition provides the link between isometric and isomorphic properties and is the essential step for the first proof that Banach spaces X which are L-summands in their biduals are weakly sequentially complete. It characterises elements of X_s as maximally discontinuous, when considered as functions on (B_{X^*}, w^*). To measure the discontinuity of a function $f : K \to \mathbb{K}$ at a point $x \in K$ we will *not* work with the usual definition

$$\operatorname{osc}(f, x) := \inf_{U \in \mathfrak{U}(x)} \sup_{y,z \in U} |f(y) - f(z)|$$

where $\mathfrak{U}(x)$ denotes the neighbourhood system of x. Although this makes perfect sense for $\mathbb{K} = \mathbb{C}$, it causes – for complex scalars – some problems in the proof of the following proposition (see the Notes and Remarks). Instead we will use

$$MC(f, x) := \bigcap_{U \in \mathfrak{U}(x)} \overline{f(U)}.$$

For bounded functions this definition yields the usual properties of a "modulus of continuity":

- *f is continuous at x* $\iff$ $MC(f, x) = \{f(x)\}$
- *If f is continuous at x and $g : K \to \mathbb{K}$ is arbitrary, then*

$$MC(f + g, x) = f(x) + MC(g, x).$$

We omit the standard proofs of these statements. Note, however, that the above is false for unbounded functions and false for bounded functions if the closure of $f(U)$ is omitted.

Proposition 2.1 *Let X be a Banach space and consider $x^{**} \in X^{**}$ as a function on (B_{X^*}, w^*). The following statements are equivalent:*

(i) $\|x^{**} + x\| = \|x^{**}\| + \|x\|$ *for all* $x \in X$.

(ii) $MC(x^{**}, x^*) = B_{\mathbb{K}}(0, \|x^{**}\|)$ *for all* $x^* \in B_{X^*}$.

In particular: For a Banach space X which is an L-summand in its bidual all the $x^{**} \in X^{**} \setminus X$ are everywhere discontinuous on (B_{X^*}, w^*).

Proof: (i) $\Rightarrow$ (ii): Since $x^{**}(U) \subset x^{**}(B_{X^*}) \subset B_{\mathbb{K}}(0, \|x^{**}\|)$ the inclusion "$\subset$" is clear. To prove "$\supset$" we have to show that for all $x^* \in B_{X^*}$ and for all w^*-neighbourhoods $U = \{y^* \in B_{X^*} \mid |x^*(x_k) - y^*(x_k)| < \eta,\ 1 \le k \le n\}$ of x^* the set $x^{**}(U)$ is dense in $B(0, \|x^{**}\|)$. So take $\theta \in B(0, \|x^{**}\|)$ and $\varepsilon > 0$. We are looking for a $y^* \in U$ such that $|x^{**}(y^*) - \theta| < \varepsilon$. Define a linear functional by

$$\begin{array}{rcrcl} \varphi & : & X + \mathbb{K}x^{**} & \longrightarrow & \mathbb{K} \\ & & x + \lambda x^{**} & \longmapsto & x^*(x) + \lambda\theta. \end{array}$$

Then by assumption

$$|\varphi(x + \lambda x^{**})| = |x^*(x) + \lambda\theta| \le \|x^*\|\|x\| + |\lambda||\theta| \le \|x\| + |\lambda|\|x^{**}\| = \|x + \lambda x^{**}\|,$$

hence $\|\varphi\| \le 1$. Let $\overline{\varphi} \in B_{X^{***}}$ be a Hahn-Banach extension of φ. By Goldstine's theorem we find $y^* \in B_{X^*}$ such that

$$|x^{**}(y^*) - \overline{\varphi}(x^{**})| < \varepsilon \tag{1}$$

$$|y^*(x_k) - \overline{\varphi}(x_k)| < \eta, \quad 1 \le k \le n. \tag{2}$$

Because $\overline{\varphi}(x^{**}) = \varphi(x^{**}) = \theta$ the inequality (1) implies $|x^{**}(y^*) - \theta| < \varepsilon$. By definition $\overline{\varphi}(x_k) = \varphi(x_k) = x^*(x_k)$, so inequality (2) gives $y^* \in U$.

(ii) $\Rightarrow$ (i): Take $x \in X$ and $\varepsilon > 0$. Choose $x^* \in B_{X^*}$ such that $x^*(x) = \|x\|$ and put $U := \{y^* \in B_{X^*} \mid |x^*(x) - y^*(x)| < \varepsilon/2\}$. For all $y^* \in U$ this yields $\mathrm{Re}\, y^*(x) \ge \|x\| - \varepsilon/2$. By assumption we have $\overline{x^{**}(U)} = B_{\mathbb{K}}(0, \|x^{**}\|)$, so we find $y^{**} \in U$ such that $\mathrm{Re}\, x^{**}(y^*) > \|x^{**}\| - \varepsilon/2$. Combining these we get

$$\|x^{**} + x\| \ge |(x^{**} + x)(y^*)| \ge \mathrm{Re}\,(x^{**} + x)(y^*) \ge \|x^{**}\| + \|x\| - \varepsilon,$$

hence $\|x^{**} + x\| = \|x^{**}\| + \|x\|$.

Since $x^{**} = x + x_s$ is in $X^{**} \setminus X$ iff $x_s \neq 0$, the second statement follows from the first and the properties of the modulus of continuity MC. ☐

Theorem 2.2 *Every Banach space X which is an L-summand in its bidual is weakly sequentially complete.*

Proof: For a weak Cauchy sequence (x_n) in X an element $x^{**} \in X$ is defined by $x^{**}(x^*) := \lim x^*(x_n),\ x^* \in X^*$. This means $x_n \stackrel{w*}{\longrightarrow} x^{**}$, hence x^{**} is the pointwise limit of a sequence of continuous functions, i.e. of the first Baire class on (B_{X^*}, w^*). By Baire's theorem (see e.g. [157, p. 67]) x^{**} has a point of continuity on B_{X^*}, so by Proposition 2.1 x^{**} belongs to X and (x_n) converges to x^{**} weakly. □

By the above result and Example 1.1(c) we retrieve the Mooney-Havin theorem which asserts that L^1/H^1_0 is weakly sequentially complete. We refer to [489, Corollary 8.1, Theorem 7.1 and Proposition 6.1] for a "classical" proof.

Corollary 2.3 *Every nonreflexive subspace of a Banach space X which is an L-summand in X^{**} contains an isomorphic copy of ℓ^1.*

PROOF: Let Y be a nonreflexive subspace of X. By the Eberlein-Smulian theorem B_Y is not weakly sequentially compact, i.e. there is a sequence (y_n) in B_Y which doesn't have a weakly convergent subsequence. Theorem 2.2 shows that there are no weakly Cauchy subsequences either. Now Rosenthal's ℓ^1-theorem [421, Theorem 2.e.5] applies. □

For an extension, see Corollary 2.8.

Remark 2.4 Of course the conclusion of Corollary 2.3 holds in all weakly sequentially complete Banach spaces. We would like to mention, however, a local reflexivity argument of a more geometrical flavour to show that nonreflexive L-embedded spaces contain ℓ^1 isomorphically:
Take $x_1 \in S_X$ and $x_s \in S_{X_s}$. Then $\operatorname{lin}(x_1, x_s) \cong \ell^1(2)$ and by local reflexivity we find $x_2 \in X$ such that $E_2 := \operatorname{lin}(x_1, x_2) \overset{1+\varepsilon_1}{\simeq} \operatorname{lin}(x_1, x_s)$. In the next step we start from $\operatorname{lin}(x_1, x_2, x_s) \overset{1+\varepsilon_1}{\simeq} \ell^1(3)$ and find $x_3 \in X$ such that $E_3 := \operatorname{lin}(x_1, x_2, x_3) \overset{1+\varepsilon_2}{\simeq} \operatorname{lin}(x_1, x_2, x_s)$. The induction is now clear and a suitable choice of ε_n makes the sequence (x_n) equivalent to the unit vector basis of ℓ^1.

The second proof of the weak sequential completeness of L-embedded spaces will be an easy consequence of the following result, which we propose to call the "ace of ♢ lemma". It may be regarded as a weak substitute for the lacking w^*-closedness of L-summands in dual spaces (cf. Example I.1.6(b) and Theorem I.1.9). Its proof uses nothing but the interaction of the w^*-lower semicontinuity of the dual norm and the ace of ♢ shape of B_{X^*}, i.e. the L-decomposition. In particular it does not rely on a Baire argument.

Proposition 2.5 *Let U be an L-summand in a dual space X^* and (x^*_k) a weak Cauchy sequence in U. Then the w^*-limit x^* of (x^*_k) belongs to U, too.*

PROOF: Let V be the complementary L-summand for U, i.e. $X^* = U \oplus_1 V$. Note that by the Banach-Steinhaus theorem the w^*-limit x^* of (x^*_k) exists in X^* and $\|x^*\| \le \sup \|x^*_k\| < \infty$. Decompose $x^* = u^* + v^*$ in $X^* = U \oplus_1 V$. Passing to $(x^*_k - u^*)$ we may assume that (x^*_k) is a weak Cauchy sequence in U which is w^*-convergent to $v^* \in V$, and we want to show that $v^* = 0$.
Assuming $v^* \neq 0$ we will inductively construct a subsequence $(x^*_{k_n})$ which is equivalent to the standard ℓ^1-basis:

$$m \sum_{i=1}^{n} |\alpha_i| \le \left\| \sum_{i=1}^{n} \alpha_i x^*_{k_i} \right\| \le M \sum_{i=1}^{n} |\alpha_i| \quad \forall\, n \in \mathbb{N} \quad \forall\, \alpha_i \in \mathbb{K}$$

where $m = \|v^*\|/2$ and $M = \sup \|x_k^*\|$. This then gives a contradiction because such a sequence $(x_{k_n}^*)$ is not weakly Cauchy.
Choose a sequence of positive numbers ε_n such that $\sum_{n=1}^{\infty} \varepsilon_n < \|v^*\|/2$. By w^*-lower semicontinuity of the dual norm we have $\|v^*\| \le \liminf \|x_k^*\| \le M$. Thus there is an $x_{k_1}^*$ such that $\|x_{k_1}^*\| \ge \|v^*\| - \varepsilon_1$, which settles the beginning of the induction.
Assume we have found $x_{k_1}^*, \dots, x_{k_n}^*$ satisfying

$$\left(\|v^*\| - \sum_{i=1}^{n} \varepsilon_i \right) \sum_{i=1}^{n} |\alpha_i| \le \left\| \sum_{i=1}^{n} \alpha_i x_{k_i}^* \right\|.$$

Choose a finite (ε_{n+1}/M)-net $(\alpha^l)_{l \le L}$ in the unit sphere of $\ell^1(n+1)$ such that $\alpha_{n+1}^l \neq 0$ for all $l \le L$. The w^*-convergence (for $k \to \infty$) of $\left(\sum_{i=1}^{n} \alpha_i^l x_{k_i}^*\right) + \alpha_{n+1}^l x_k^*$ to $\left(\sum_{i=1}^{n} \alpha_i^l x_{k_i}^*\right) + \alpha_{n+1}^l v^*$ yields

$$\begin{aligned}
\liminf_k \left\| \left(\sum_{i=1}^{n} \alpha_i^l x_{k_i}^* \right) + \alpha_{n+1}^l x_k^* \right\| &\ge \left\| \left(\sum_{i=1}^{n} \alpha_i^l x_{k_i}^* \right) + \alpha_{n+1}^l v^* \right\| \\
&= \left\| \sum_{i=1}^{n} \alpha_i^l x_{k_i}^* \right\| + \|\alpha_{n+1}^l v^*\| \\
&\ge \left(\|v^*\| - \sum_{i=1}^{n} \varepsilon_i \right) \sum_{i=1}^{n} |\alpha_i^l| + |\alpha_{n+1}^l| \|v^*\| \\
&> \left(\|v^*\| - \sum_{i=1}^{n} \varepsilon_i \right) \sum_{i=1}^{n+1} |\alpha_i^l| \\
&= \|v^*\| - \sum^{n} \varepsilon_i.
\end{aligned}$$

Because only finitely many sequences are involved we find $x_{k_{n+1}}^*$ such that

$$\left\| \sum_{i=1}^{n+1} \alpha_i^l x_{k_i}^* \right\| \ge \|v^*\| - \sum_{i=1}^{n} \varepsilon_i \quad \forall\, l \le L.$$

For arbitrary $\alpha \in S_{\ell^1(n+1)}$ choose α^l such that $\|\alpha - \alpha^l\|_1 \le \varepsilon_{n+1}/M$. Then

$$\begin{aligned}
\left\| \sum_{i=1}^{n+1} \alpha_i x_{k_i}^* \right\| &= \left\| \sum_{i=1}^{n+1} \alpha_i^l x_{k_i}^* + (\alpha_i - \alpha_i^l) x_{k_i}^* \right\| \\
&\ge \left\| \sum_{i=1}^{n+1} \alpha_i^l x_{k_i}^* \right\| - \left\| \sum_{i=1}^{n+1} (\alpha_i - \alpha_i^l) x_{k_i}^* \right\| \\
&\ge \left(\|v^*\| - \sum_{i=1}^{n} \varepsilon_i \right) - \|\alpha - \alpha^l\|_1 M
\end{aligned}$$

$$
\begin{aligned}
&\geq \quad \|v^*\| - \sum_{i=1}^{n+1} \varepsilon_i \\
&= \quad \left(\|v^*\| - \sum_{i=1}^{n+1} \varepsilon_i \right) \sum_{i=1}^{n+1} |\alpha_i|.
\end{aligned}
$$

This clearly extends to all $\alpha \in \ell^1$ and thus finishes the induction and the proof. □

We have actually proved: If (x_k^*) is a w^*-Cauchy sequence in U with a non-zero w^*-limit in V, then (x_k^*) has a subsequence equivalent to the standard ℓ^1-basis. (The existence of the w^*-limit in X^* is again clear.) Note, however, that this does *not* show the w^*-sequential completeness of L-summands in dual spaces, since we need a *weak* Cauchy sequence to get a contradiction.
We will mention another proof of Proposition 2.5 in the Notes and Remarks section.

Second proof of Theorem 2.2: Let (x_n) be a weak Cauchy sequence in the L-embedded space X. Because then (x_n) is also a weak Cauchy sequence in X^{**}, Proposition 2.5 yields that the $\sigma(X^{**}, X^*)$-limit of (x_n) lies in X so that (x_n) is weakly convergent. □

We remark that the ace of diamonds lemma yields that an element of X_s cannot be of the first Baire class relative to (B_{X^*}, w^*). This should be compared to Proposition 2.1.

We now come to property (V^*) (see III.3.3) for L-embedded spaces. This was an open problem for quite a while and has only recently been proved in [492]. Property (V^*) was known to hold for all the concrete spaces in Example 1.1, yet the proofs also in these special cases are not at all easy. We again stress that we need the weak sequential completeness of L-embedded spaces, though formally a consequence of property (V^*) by virtue of Proposition III.3.3.F, as an instrumental ingredient in the proof of Theorem 2.7.
We first provide a technical characterisation of property (V^*):

Lemma 2.6 *For a Banach space X the following assertions are equivalent:*

(i) *X has property (V^*), that is for each set $K \subset X$ which is not relatively w-compact there is a wuC-series $\sum x_i^*$ in X^* such that $\sup_{x \in K} |x_i^*(x)| \not\to 0$.*

(ii) *X is w-sequentially complete and for any (for some) number δ with $0 < \delta < 1$ the following holds: if (y_k) satisfies $(1-\delta)\sum |\alpha_k| \leq \|\sum \alpha_k y_k\| \leq \sum |\alpha_k|$ for all finite scalar sequences (α_k), then there are a subsequence (y_{k_n}), positive numbers $\rho = \rho(\delta, (y_k))$, $M = M(\delta, (y_k)) < \infty$ and for each $n \in \mathbb{N}$ there is a finite sequence $(y_i^{n*})_{i=1}^n$ in X^* such that*

$$
\begin{aligned}
|y_i^{n*}(y_{k_i})| &> \rho \quad \forall\, i \leq n \\
\left\| \sum_{i=1}^n \alpha_i y_i^{n*} \right\| &\leq M \max_{i \leq n} |\alpha_i| \quad \forall\, (\alpha_i) \subset \mathbb{K}.
\end{aligned}
$$

Proof: (i) $\Rightarrow$ (ii): Let δ be any number with $0 < \delta < 1$ and (y_k) an ℓ^1-basis in X satisfying $(1-\delta)\sum|\alpha_k| \le \|\sum \alpha_k y_k\| \le \sum|\alpha_k|$ for all (α_k). Since $K = \{y_k \mid k \in \mathbb{N}\}$ is not relatively w-compact, by (i) there are a wuC-series $\sum x_i^*$, a subsequence (y_{k_n}) and a number $\rho > 0$ such that $|x_n^*(y_{k_n})| > \rho$ for all $n \in \mathbb{N}$. Set $y_i^{n*} = x_i^*$ for all $i, n \in \mathbb{N}$. That property (V^*) implies w-sequential completeness was noted in Proposition III.3.3.F.
(ii) $\Rightarrow$ (i): Suppose (ii) holds for a fixed number δ with $0 < \delta < 1$. Let $K \subset X$ be not relatively w-compact. If K is not bounded, there are $x_n \in K$ and $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and $2^{-n}x_n^*(x_n) > 1$ for each $n \in \mathbb{N}$, and $\sum 2^{-n}x_n^*$ is trivially a wuC-series. Therefore we assume K to be bounded. Since X is weakly sequentially complete, by Rosenthal's ℓ^1-theorem K contains an ℓ^1-basis (x_n), i.e. $c\sum|\alpha_n| \le \|\sum \alpha_n x_n\| \le \sum|\alpha_n|$ for some $c > 0$. By James' distortion theorem [421, Prop. 2.e.3] (but for the present formulation the original reference [334] is slightly more appropriate) there are pairwise disjoint finite sets $A_k \subset \mathbb{N}$ and a sequence (λ_n) of scalars such that the sequence $y_k = \sum_{A_k} \lambda_n x_n$ satisfies

$$(1-\delta)\sum|\alpha_k| \le \Big\|\sum \alpha_k y_k\Big\| \le \sum|\alpha_k|, \qquad \sum_{n\in A_k}|\lambda_n| < \frac{1}{c} \quad \forall k \in \mathbb{N}. \tag{$*$}$$

Now observe that $(y_i^{n*})_n$ is a bounded sequence in X^*, for each i. Fix a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and let $x_i^* = w^*\text{-}\lim_{n\in\mathfrak{U}} y_i^{n*}$. It is clear that $x_i^*(y_{k_i}) \ge \rho$ for all i, and $\sum x_i^*$ is a wuC-series by virtue of

$$\begin{aligned}\Big\|\sum_{i=1}^k \alpha_i x_i^*\Big\| &= \Big\|w^*\text{-}\lim_{n\in\mathfrak{U}}\sum_{i=1}^k \alpha_i y_i^{n*}\Big\| \\ &\le \Big\|\liminf \sum_{i=1}^k \alpha_i y_i^{n*}\Big\| \\ &\le M\max|\alpha_i|.\end{aligned}$$

By $(*)$ there is x_{n_i} such that $|x_i^*(x_{n_i})| > \rho\, c$ for each $i \in \mathbb{N}$, because otherwise we would have

$$|x_i^*(y_{k_i})| \le \sum_{n\in A_{k_i}} |\lambda_n||x_i^*(x_n)| \overset{(*)}{<} \frac{1}{c}\rho c = \rho.$$

This proves (i). □

Theorem 2.7 *Every L-embedded Banach space X has property (V^*).*

Proof: We will verify (ii) of Lemma 2.6 in order to prove property (V^*). We already know from Theorem 2.2 that X is w-sequentially complete.
Let ε, δ be numbers such that $0 < \varepsilon < 1/4$, $0 < \delta < \varepsilon^2/9^2$, and let (y_k) be an ℓ^1-basis in X as in (ii). We will find the subsequence (y_{k_n}) and the (y_i^{n*}) required in Lemma 2.6 by induction using the principle of local reflexivity. To make the construction work we need an $x_s \in X_s$ which is "near" to an accumulation point of the y_k. More precisely:

Claim: There is $x_s \in X_s$ such that

$$\|x_s\| \ge 1 - 4\delta^{1/2} \tag{1}$$

and for all $\eta > 0$, all $x^* \in X^*$ and all $k_0 \in \mathbb{N}$ there is $k \geq k_0$ with

$$|x_s(x^*) - x^*(y_k)| \leq 3\delta^{1/2}\|x^*\| + \eta. \tag{2}$$

Proof of the claim: Denote the usual basis of ℓ^1 by (e_n^*) and denote by π_{ℓ^1} the canonical projection from $(\ell^1)^{**} = \ell^1 \oplus_1 {c_0}^\perp$ onto ℓ^1. The w^*-closure of the set $\{e_n^* \mid n \in \mathbb{N}\} \subset \ell^1$ in the bidual of ℓ^1 contains an accumulation point $\mu \in {c_0}^\perp = (\ell^1)_s$ of norm $\|\mu\| = 1$. (Actually, every such accumulation point has these properties.) We will map μ into X_s.
Put $Y = \overline{\text{lin}}\,\{y_k \mid k \in \mathbb{N}\}$. The canonical isomorphism $S : Y \to \ell^1$ satisfies $\|y^{**}\| \leq \|S^{**}y^{**}\| \leq \frac{1}{1-\delta}\|y^{**}\|$ for all $y^{**} \in Y^{**}$. In particular $1 - \delta \leq \|z_s\| \leq 1$ for $z_s = (S^{**})^{-1}(\mu)$. Consider $z_s \in X^{**}$ via the identification of Y^{**} and $Y^{\perp\perp} \subset X^{**}$. Denote by Q the canonical projection from $Y^{\perp\perp}$ onto Y (i.e. $Q = (S^{**})^{-1}\pi_{\ell^1}S^{**}$). Then $z_s \in Y_s = \ker Q$ follows from $\mu \in \ker \pi_{\ell^1}$, and z_s is a $\sigma(X^{**}, X^*)$-accumulation point of the set $\{y_k \mid k \in \mathbb{N}\}$ in Y_s. Finally put $x_s = (Id_{X^{**}} - P)(z_s) \in \ker P = X_s$, where P is as usual the L-projection from $X^{**} = X \oplus_1 X_s$ onto X.
For the decomposition $y^{\perp\perp} = y + y_s$ in $Y^{\perp\perp} = Y \oplus Y_s$ of any element $y^{\perp\perp} \in Y^{\perp\perp}$ we have

$$\begin{aligned} \|y + y_s\| &\geq (1-\delta)\|S^{**}y + S^{**}y_s\| \\ &= (1-\delta)(\|S^{**}y\| + \|S^{**}y_s\|) \\ &\geq (1-\delta)(\|y\| + \|y_s\|). \end{aligned}$$

Since $\delta < 1/4$ the assumptions of Lemma 1.4 are satisfied. We deduce from this result

$$\|x_s - z_s\| = \|Pz_s\| = \|Pz_s - Qz_s\| \leq 3\delta^{1/2}\|z_s\|. \tag{3}$$

Hence

$$\|x_s\| = \|z_s - Pz_s\| = \|z_s\| - \|Pz_s\| \geq (1 - 3\delta^{1/2})\|z_s\| \geq 1 - 4\delta^{1/2},$$

which is the first assertion of the claim. We use that z_s is an $\sigma(X^{**}, X^*)$-accumulation point of the y_k and (3) to find

$$\begin{aligned} |x_s(x^*) - x^*(y_k)| &\leq |x_s(x^*) - z_s(x^*)| + |z_s(x^*) - x^*(y_k)| \\ &\leq 3\delta^{1/2}\|x^*\| + \eta\,. \end{aligned}$$

This ends the proof of the claim.

Choose a sequence (ε_n) of positive numbers such that $\prod_{n\geq 1}(1-\varepsilon_n) \geq 1-\varepsilon$ and $\prod_{n\geq 1}(1+\varepsilon_n) \leq 1+\varepsilon$. We will construct by induction over $n = 1, 2, \dots$ finite sequences $(y_i^{n*})_{i=1}^n \subset X^*$ and a subsequence (y_{k_n}) of (y_k) such that

$$|y_i^{n*}(y_{k_i})| > 1 - 9\delta^{1/2} \qquad \forall i \leq n \tag{4}$$

$$\begin{aligned} \Big(\prod_{i=1}^n (1-\varepsilon_i)\Big) \max_{i\leq n} |\alpha_i| &\leq \Big\|\sum_{i=1}^n \alpha_i y_i^{n*}\Big\| \\ &\leq \Big(\prod_{i=1}^n (1+\varepsilon_i)\Big) \max_{i\leq n} |\alpha_i| \qquad \forall (\alpha_i) \subset \mathbb{K}. \end{aligned} \tag{5}$$

(This means that we will fulfill (ii) of Lemma 2.6 with $\rho = 1 - 9\delta^{1/2}$ and $M \le 1 + \varepsilon$.)
For $n = 1$ we set $k_1 = 1$ and choose $y_1^{1^*}$ such that $\|y_1^{1^*}\| = 1$ and $y_1^{1^*}(y_{k_1}) = \|y_{k_1}\| \ge 1 - \delta$; $y_1^{1^*}$ satisfies (5), too.
For the induction step $n \mapsto n+1$ we observe that $P^*|_{X^*}$ is an isometric isomorphism from X^* onto $X_s^\perp$, that $X^{***} = X^\perp \oplus_\infty X_s^\perp$ and that $(P^* x^*)|_X = x^*|_X$ for all $x^* \in X^*$. Choose $t \in \ker P^* \subset X^{***}$ such that

$$\|t\| = 1 \qquad \text{and} \qquad t(x_s) = \|x_s\|.$$

Put $E = \operatorname{lin}(\{P^* y_i^{n^*} \mid i \le n\} \cup \{t\}) \subset X^{***}$ and $F = \operatorname{lin}(\{y_{k_i} \mid i \le n\} \cup \{x_s\}) \subset X^{**}$. An application of the principle of local reflexivity provides us with an operator $R : E \to X^*$ such that

$$(1 - \varepsilon_{n+1})\|e^{***}\| \le \|Re^{***}\| \le (1 + \varepsilon_{n+1})\|e^{***}\| \qquad \text{for all } e^{***} \in E$$

and

$$f^{**}(Re^{***}) = e^{***}(f^{**}) \qquad \text{for all } f^{**} \in F.$$

Let $y_i^{n+1^*} = RP^* y_i^{n^*}$ for $i \le n$ and let $y_{n+1}^{n+1^*} = Rt$. Then the $(y_i^{n+1^*})_{i=1}^{n+1}$ fulfill (5,n+1) by

$$\begin{aligned}
\Big(\prod_{i=1}^{n+1}(1-\varepsilon_i)\Big) \max_{i \le n+1} |\alpha_i| &\le (1-\varepsilon_{n+1}) \max\left(\Big(\prod_{i=1}^{n}(1-\varepsilon_i)\Big) \max_{i\le n}|\alpha_i|, |\alpha_{n+1}|\right) \\
&\le (1-\varepsilon_{n+1}) \max\left(\left\|\sum_{i=1}^n \alpha_i y_i^{n^*}\right\|, \|\alpha_{n+1} t\|\right) \\
&= (1-\varepsilon_{n+1}) \max\left(\left\|\sum_{i=1}^n \alpha_i P^* y_i^{n^*}\right\|, \|\alpha_{n+1} t\|\right) \\
&= (1-\varepsilon_{n+1}) \left\|\left(\sum_{i=1}^n \alpha_i P^* y_i^{n^*}\right) + \alpha_{n+1} t\right\| \\
&\le \left\|\sum_{i=1}^{n+1} \alpha_i y_i^{n+1^*}\right\| \\
&\le (1+\varepsilon_{n+1}) \left\|\left(\sum_{i=1}^n \alpha_i P^* y_i^{n^*}\right) + \alpha_{n+1} t\right\| \\
&= (1+\varepsilon_{n+1}) \max\left(\left\|\sum_{i=1}^n \alpha_i P^* y_i^{n^*}\right\|, \|\alpha_{n+1} t\|\right) \\
&= (1+\varepsilon_{n+1}) \max\left(\left\|\sum_{i=1}^n \alpha_i y_i^{n^*}\right\|, \|\alpha_{n+1} t\|\right) \\
&\le (1+\varepsilon_{n+1}) \max\left(\Big(\prod_{i=1}^{n}(1+\varepsilon_i)\Big) \max_{i\le n}|\alpha_i|, |\alpha_{n+1}|\right) \\
&\le \Big(\prod_{i=1}^{n+1}(1+\varepsilon_i)\Big) \max_{i \le n+1} |\alpha_i|.
\end{aligned}$$

We use (2) with $\eta \leq 3\delta^{1/2}\varepsilon_{n+1}$ and $x^* = y_{n+1}^{n+1^*}$ in order to find $y_{k_{n+1}}$ such that

$$\left| x_s(y_{n+1}^{n+1^*}) - y_{n+1}^{n+1^*}(y_{k_{n+1}}) \right| \leq 3\delta^{1/2}\|y_{n+1}^{n+1^*}\| + \eta \leq 3\delta^{1/2}(1+2\varepsilon_{n+1}) \leq 5\delta^{1/2};$$

here $\|R\| \leq 1+\varepsilon_{n+1}$ enters. Then (4,n+1) for $i = n+1$ follows from

$$\begin{aligned} |y_{n+1}^{n+1^*}(y_{k_{n+1}})| &\geq |x_s(y_{n+1}^{n+1^*})| - 5\delta^{1/2} \\ &\overset{\text{loc. refl.}}{=} |t(x_s)| - 5\delta^{1/2} \\ &\overset{(1)}{\geq} (1-4\delta^{1/2}) - 5\delta^{1/2} = 1 - 9\delta^{1/2}. \end{aligned}$$

Recall $P^*x^*(x) = x^*(x)$ to see that $y_i^{n+1^*}(y_{k_i}) = (P^*y_i^{n^*})(y_{k_i}) = y_i^{n^*}(y_{k_i})$. (4,n+1) for $i \leq n$ is consequence of this and (4,n). This completes the induction and the proof. □

We note the following consequence of the above theorem, Corollary III.3.3.C, and Proposition III.3.3.E:

Corollary 2.8 *Every nonreflexive subspace of an L-embedded space contains a complemented subspace isomorphic to ℓ^1.*

In the remainder of this section we will study the Radon-Nikodým property for L-embedded spaces (although this may seem not very promising at a first glance at the Examples 1.1). By Theorem III.3.1 L-embedded spaces X which are duals of M-embedded spaces have the RNP and for $L^1(\mu)$-spaces X this is even characteristic – use the well-known fact that $L^1(\mu)$ *has the RNP iff μ is purely atomic*. This carries over to the noncommutative case (cf. Example 1.1(b)).

Proposition 2.9 *Let X be the predual of a von Neumann algebra. Then X has the RNP if and only if X is the dual of a space which is an M-ideal in its bidual.*

Proof: This is essentially a result of Chu who proved in Theorem 4 of [127] that if X has the RNP then X^* is a direct sum of type I factors, so

$$X^* \cong \Big(\oplus \sum L(H_i)\Big)_{\ell^\infty(I)} \qquad \text{and} \qquad X \cong \Big(\oplus \sum N(H_i)\Big)_{\ell^1(I)}.$$

The last-named space has the M-embedded space $(\oplus \sum K(H_i))_{c_0(I)}$ as a predual (see Example III.1.4(f) and Theorem III.1.6(c)). □

The following question is now tempting to ask.

Question. *Is it true for an L-embedded space X that*

X has the RNP $\iff$ X is the dual of an M-embedded space?

We will see in Section IV.4 that the answer is **no**. [This can't be decided with the examples given so far, for instance the non-dual space L^1/H_0^1 (Corollary 1.16) fails the RNP.]

Remarks 2.10 (a) In spite of the negative answer to the above question there are some positive results. Note that by Proposition 1.9(b) the problem translates into

$$X \text{ } L\text{-embedded space, } X_s \text{ not } w^*\text{-closed} \quad \overset{?}{\Longrightarrow} \quad X \text{ fails the RNP}$$

Assuming even more than "X_s is w^*-dense" one gets:

If X is an L-summand in X^{**} and B_{X_s} is w^*-dense in $B_{X^{**}}$ then B_X has no strongly exposed point (in particular, X fails the RNP).

Proof: If $x \in B_X$ is strongly exposed then $Id : (B_{X^{**}}, w^*) \to (B_{X^{**}}, \|\,.\,\|)$ is continuous at i_x (see e.g. [249, proof of Theorem 10.2]). By w^*-density there is a net (x_α^{**}) in B_{X_s} such that $x_\alpha^{**} \overset{w*}{\longrightarrow} x$. Hence $x_\alpha^{**} \overset{\|\,\|}{\longrightarrow} x$ and $x \in X_s$. See [158, p. 202] for the second statement of the proposition. □

Note that we only used the existence of a subspace $Y \subset X^{**}$ such that $X \cap Y = \{0\}$ and B_Y is w^*-dense in $B_{X^{**}}$.

(b) We mention a condition which guarantees the above w^*-density (of course the conclusion about the Radon-Nikodým property is trivial in this case).

If X is an L-embedded space with $\operatorname{ex} B_X = \emptyset$, then B_{X_s} is w^*-dense in $B_{X^{**}}$.

Proof: We have by Lemma I.1.5 that $\operatorname{ex} B_{X^{**}} = \operatorname{ex} B_X \cup \operatorname{ex} B_{X_s}$, so by assumption and the Krein-Milman theorem

$$B_{X^{**}} = \overline{\text{co}}^{w*} \operatorname{ex} B_{X_s} = \overline{B_{X_s}}^{w*}. \qquad □$$

In this connection it is worth mentioning that for a nonreflexive L-embedded space X one has $\operatorname{ex} B_{X_s} \neq \emptyset$. [Otherwise, $\operatorname{ex} B_{X^{**}} = \operatorname{ex} B_X$ by Lemma I.1.5. Hence every $x^* \in X^*$ attains its norm on $\operatorname{ex} B_X$, so X is reflexive by James' theorem.]

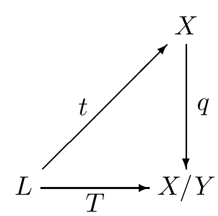
There is another extreme point argument which has a consequence for the RNP.

Proposition 2.11 *If X is an L-embedded space with $\operatorname{ex} B_X = \emptyset$ (e.g. $X = L^1(\mu)$, μ without atoms) and $Y \subset X$ an L-embedded subspace, then X/Y fails the RNP.*

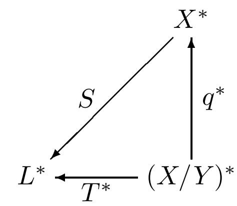
Proof: By Propositions 1.12 and 1.14 we have $\operatorname{ex} B_{X/Y} = \emptyset$, so X/Y fails the RNP (see [158, p. 202]). □

The next result prepares a certain converse of the above proposition, but it is also of independent interest.

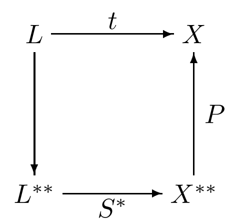
Proposition 2.12 *Let L be a Banach space such that L^* is injective, and X and Y L-embedded spaces with $Y \subset X$. Then every operator $T : L \to X/Y$ factors through the quotient map $q : X \to X/Y$:*



Proof: Since q^* is a metric injection and L^* is injective there is an operator $S \in L(X^*, L^*)$ such that $Sq^* = T^*$:



Put $t := PS^*|_L$ where P is the L-projection from X^{**} onto X:



We will show that $T = qt$ is the wanted factorisation. Since

$$(S^* - t^{**})(L) = (S^* - t)(L) = (S^* - PS^*)(L) = (Id - P)S^*(L) \subset X_s$$

and

$$q^{**}(X_s) = (X_s + Y^{\perp\perp})/Y^{\perp\perp} = (X/Y)_s$$

(observe $q^{**}P = (P/Y^{\perp\perp})q^{**}$; Corollary 1.3) we get

$$q^{**}(S^* - t^{**})(L) \subset (X/Y)_s.$$

On the other hand $Sq^* = T^*$ gives $q^{**}S^* = T^{**}$, hence $q^{**}(S^* - t^{**}) = T^{**} - q^{**}t^{**}$. So

$$q^{**}(S^* - t^{**})(L) = (T^{**} - q^{**}t^{**})(L) = (T - qt)(L) \subset X/Y.$$

We obtain $(T - qt)(L) = \{0\}$, i.e. $T = qt$. □

The first part of the next result says that every $L^1(\mu)$-subspace of X/Y "comes from X".

Corollary 2.13 *If X and Y are L-embedded spaces with $Y \subset X$ and X/Y contains a subspace V isomorphic (isometric) to $L^1(\mu)$, then there is a subspace U of X such that $q(U) = V$ and $q|_U$ is an isomorphism (an isometric isomorphism). In case $X/Y \simeq L^1(\mu)$ ($X/Y \cong L^1(\mu)$) there is a (norm-1-) projection Q on X with $Y = \ker Q$.*

Proof: Let T be the isomorphism from $L^1(\mu)$ onto V and $T = qt$ according to the above proposition. Put $U := t(L^1(\mu))$. It is easy to see that U is closed and $q|_U$ is bijective from U to V. If T is isometric we may choose the extension S of T^* in the proof of Proposition 2.12 with $\|S\| = 1$; recall that $L^\infty(\mu)$ is a $\mathcal{P}_1$-space. Then

$$\|tl\| = \|PS^*l\| \le \|l\| \qquad \text{and} \qquad \|l\| = \|Tl\| = \|qtl\| \le \|tl\|$$

show that t is isometric. From

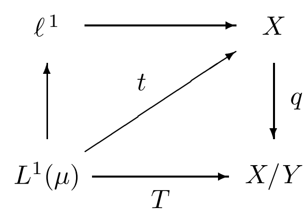
$$\|qu\| = \|qtl\| = \|Tl\| = \|l\| = \|tl\| = \|u\|$$

we obtain that $q|_U$ is isometric.
If T is onto then $Q := tT^{-1}q$ is the desired projection. □

Finite-dimensional examples show that Y need not be the range of a norm-1-projection in X if $X/Y \cong L^1(\mu)$.

Corollary 2.14 *Let X and Y be L-embedded spaces with $Y \subset X$. If X has the RNP then X/Y has the RNP.*

Proof: By the Lewis-Stegall theorem [158, p. 66] a Banach space Z has the RNP iff every operator from $L^1(\mu)$ to Z factors through ℓ^1. So the assumption and Proposition 2.12 give the following diagram:



Now the proof is completed by another application of the Lewis-Stegall theorem. □

IV.3 Subspaces of $L^1(\mu)$ which are L-summands in their biduals

In this section we characterise the spaces appearing in its title and, among them, those which are duals of M-embedded spaces. The essential tool for doing this is the topology of convergence in measure on the space $L^1(\mu)$. Though not always easy to visualize and not very popular in (isometric) Banach space theory, it is this topology which allows the internal characterisation. We give various examples and applications.

Let us recall some facts about convergence in measure. Our setting in this section will always be a *finite* complete measure space (S, Σ, μ), that is a set S, a σ-algebra Σ of subsets of S, a finite nonnegative countably additive measure μ on Σ such that $B \in \Sigma$, $A \subset B$ and $\mu(B) = 0$ imply $A \in \Sigma$. (Completing the measure space has no effect on the Banach space $L^1(\mu)$ and avoids difficulties in the description of $L^1(\mu)^{**}$. See the Notes and Remarks for infinite measure spaces.) We denote by $L^0(\mu) := L^0(S, \Sigma, \mu)$ the space of $\mathbb{K}$-valued measurable functions on S with identification of equivalent (meaning

μ-almost everywhere equal) functions. A sequence (f_n) in $L^0(\mu)$ is said to *converge in measure* to $f \in L^0(\mu)$ ($f_n \stackrel{\mu}{\longrightarrow} f$) if

$$\mu\{|f_n - f| \geq a\} \longrightarrow 0 \qquad \text{for all } a > 0.$$

Put for $f, g \in L^0(\mu)$

$$\|f\|_0 := \int \frac{|f|}{1+|f|}\, d\mu, \qquad d_0(f,g) := \|f-g\|_0$$

$$\|f\|_\mu := \inf\{a > 0 \mid \mu\{|f| \geq a\} \leq a\}, \qquad d_\mu(f,g) := \|f-g\|_\mu .$$

Then both d_0 and d_μ are invariant metrics on $L^0(\mu)$ for which $L^0(\mu)$ is complete and

$$f_n \stackrel{d_0}{\longrightarrow} f \quad \iff \quad f_n \stackrel{d_\mu}{\longrightarrow} f \quad \iff \quad f_n \stackrel{\mu}{\longrightarrow} f, \quad f_n, f \in L^0(\mu)$$

(Note however that the metrics d_0 and d_μ are not Lipschitz equivalent, although there is $c > 0$ such that $\|\cdot\|_0 \leq c\|\cdot\|_\mu$, but the quotient $\frac{\|\cdot\|_\mu}{\|\cdot\|_0}$ is unbounded in general.) It is a standard fact that a sequence converges in measure if it converges almost everywhere, and if it converges in measure, then some subsequence converges almost everywhere. (Recall that we are dealing with finite measures.) Hence the space $L^0(\mu)$ is the topological vector space best suited to describe almost everywhere convergence. Mostly we will not work with an explicit metric on $L^0(\mu)$, but simply use that convergence in measure is convergence in a complete metric topological linear space. – All this can be found in text-books on measure theory.

Let us also recall that the bidual of $L^1(\mu)$ is isometrically isomorphic to

$$ba(\mu) := \{\nu \in ba(S,\Sigma) \mid \nu(E) = 0 \text{ if } E \in \Sigma \text{ and } \mu(E) = 0\},$$

see [178, Th. IV.8.16] or [646, Th. 2.2]. The embedded copy of $L^1(\mu)$ is the set of countably additive members of $ba(\mu)$, and $L^1(\mu)_s$ corresponds to the purely finitely additive measures [646, Th. 2.6].

In the following we collect some tools for proving the bidual characterisation of convex bounded subsets of $L^1(\mu)$ which are closed with respect to convergence in measure.

Lemma 3.1 *For every net $(f_\alpha)_{\alpha\in \mathsf{A}}$ in $L^1(\mu)$ w^*-converging to $\nu \in L^1(\mu)^{**}$ there is a sequence $e_n \in \operatorname{co}\{f_\alpha \mid \alpha \in \mathsf{A}\}$ such that $e_n \to P\nu$ $\mu -$ a.e. where P denotes the L-projection from $L^1(\mu)^{**}$ onto $L^1(\mu)$. Moreover, given $\alpha_n \in \mathsf{A}$ we may assume $e_n \in \operatorname{co}\{f_\alpha \mid \alpha \geq \alpha_n\}$.*

Proof: Let $f := P\nu$ and $\pi := \nu - f = (Id - P)\nu$. Then $\pi \in L^1(\mu)_s$, i.e. π is purely finitely additive. Put

$$F := \{g \in L^\infty(\mu) \mid |\pi|(|g|) = 0\}.$$

(Lattice-theoretically F is the absolute kernel of π.) Then F is a closed subspace of $\ker\pi$ and an order ideal in $L^\infty(\mu)$, i.e.

$$|h| \leq g, \;\; h \in L^\infty(\mu), \;\; g \in F \quad \Longrightarrow \quad h \in F. \qquad (*)$$

For $g \in F$ we have

$$\int f_\alpha g \, d\mu = g(f_\alpha) \longrightarrow \nu(g) = (f+\pi)(g) = f(g) = \int fg \, d\mu,$$

that is $f_\alpha \overset{\sigma(L^1,F)}{\longrightarrow} f$.
There is a $g_0 \in F$ such that $g_0 > 0$ μ-a.e. [By [646, Th. 1.19] there is for every $n \in \mathbb{N}$ an $A_n \in \Sigma$ such that $\mu(A_n) < \frac{1}{n}$ and $|\pi|(\complement A_n) = 0$. Assume without restriction that the A_n are decreasing and put $C_n := \complement A_n$. Then $g_0 := \chi_{C_1} + \sum_{n\geq 2} \frac{1}{n}\chi_{C_n \setminus C_{n-1}}$ has the desired properties.] Put $\lambda := g_0\mu$. The formal identity operator $T : L^1(\mu) \longrightarrow L^1(\lambda)$, $Tf = f$ is continuous, and looking at the involved dualities one finds $T^*h = hg_0$ for $h \in L^\infty(\lambda)$. Because of the ideal property $(*)$ of F one gets $\operatorname{ran} T^* \subset F$, hence $f_\alpha \longrightarrow f$ with respect to $\sigma(L^1(\lambda), L^\infty(\lambda))$. In particular

$$f \in \overline{\operatorname{co}}^{\sigma(L^1(\lambda),L^\infty(\lambda))}\{f_\alpha \mid \alpha \geq \alpha_n\} = \overline{\operatorname{co}}^{\|\ \|_{L^1(\lambda)}}\{f_\alpha \mid \alpha \geq \alpha_n\} \qquad \text{for all } n.$$

So we find $e_n \in \operatorname{co}\{f_\alpha \mid \alpha \geq \alpha_n\}$ such that $\|e_n - f\|_{L^1(\lambda)} \to 0$. Then we get for some subsequence (e_{n_k}) $e_{n_k} \to f$ λ-a.e., so also $e_{n_k} \to f$ μ-a.e. □

Corollary 3.2 *If $f, f_n \in L^1(\mu)$ and (f_n) converges to f almost everywhere, then for every w^*-accumulation point ν of (f_n) one obtains $P\nu = f$ where P denotes the L-projection from $L^1(\mu)^{**}$ onto $L^1(\mu)$.*

PROOF: There is a subnet, i.e. a directed set A and a monotone cofinal map $\alpha \mapsto n_\alpha$, such that (f_{n_α}) is w^*-convergent to ν. For every n choose α_n such that $n_{\alpha_n} \geq n$. By Lemma 3.1 there are $e_n \in \operatorname{co}\{f_{n_\alpha} \mid \alpha \geq \alpha_n\} \subset \operatorname{co}\{f_m \mid m \geq n\}$ with $e_n \overset{\text{a.e.}}{\longrightarrow} P\nu$. Hence $f = P\nu$. □

REMARK: There is a proof of Corollary 3.2 which avoids dealing with subnets and using the properties of purely finitely additive measures we employed in establishing Lemma 3.1. One may argue as follows:
Translation by f shows that it is sufficient to prove that

$$f_n \in L^1(\mu),\ f_n \overset{\text{a.e.}}{\longrightarrow} 0,\ \nu \text{ a } w^*\text{-accumulation point of } (f_n) \quad \Longrightarrow \quad \nu \in L^1(\mu)_s.$$

In this situation write $\nu = g + \pi \in L^1(\mu) \oplus_1 L^1(\mu)_s$. We claim $g = 0$. Let $\varepsilon > 0$ and put for $n \in \mathbb{N}$

$$A_n := \bigcap_{k \geq n} \{|f_k| \leq \varepsilon\}.$$

Then $A_n \subset A_{n+1}$ $(n \in \mathbb{N})$ and, because of the pointwise convergence, $S \setminus \bigcup A_n$ is a null set. Fix $n \in \mathbb{N}$ and denote by P_{A_n} the L-projection $h \longmapsto \chi_{A_n} h$ on $L^1(\mu)$. Since

$$\nu \in \{f_k \mid k \geq n\}^{-w^*}$$

and $P_{A_n}{}^{**}$ is w^*-continuous we get

$$P_{A_n}{}^{**}\nu \in \{P_{A_n} f_k \mid k \geq n\}^{-w^*}.$$

The set

$$K := \{h \in L^1(\mu) \mid |h| \le \varepsilon \chi_{A_n}\}$$

is w-compact in $L^1(\mu)$, so w^*-compact in $L^1(\mu)^{**}$. Since K contains $P_{A_n} f_k$ $(k \ge n)$ we obtain

$$P_{A_n}{}^{**}\nu \in \overline{\{P_{A_n} f_k \mid k \ge n\}} \subset K \subset L^1(\mu).$$

Since $P_{A_n}{}^{**}P = PP_{A_n}{}^{**}$ by Theorem I.1.10 we get

$$P_{A_n} g = P_{A_n}{}^{**}P\nu = PP_{A_n}{}^{**}\nu = P_{A_n}{}^{**}\nu \in K,$$

which means $|g|_{A_n}| \le \varepsilon$. Hence $|g| \le \varepsilon$ a.e. and thus $g = 0$. □

We will be concerned with subsets of $L^1(\mu)$ which are closed in coarser topologies (e.g. in the topology of $L^0(\mu)$). The following proposition summarizes some easy observations in this connection.

Proposition 3.3 *For a bounded subset C of $L^1(\mu)$ the following are equivalent:*

(i) *C is closed in $L^1(\mu)$ with respect to convergence in measure.*

(ii) *C is closed in $L^0(\mu)$ with respect to convergence in measure.*

(iii) *C is closed in $L^1(\mu)$ with respect to convergence in $\|\,.\,\|_p$ for one (for all) $p \in (0,1)$.*

Proof: (ii) ⇒ (i) is trivial.

(i) ⇒ (ii): Assume $f_n \in C$, $f \in L^0(\mu)$ and $f_n \xrightarrow{\mu} f$. Then there is a subsequence (f_{n_k}) of (f_n) which converges almost everywhere. Hence by Fatou's Lemma

$$\int |f|\, d\mu = \int \underline{\lim}\, |f_{n_k}|\, d\mu \le \underline{\lim} \int |f_{n_k}|\, d\mu = \underline{\lim}\, \|f_{n_k}\|_1 < \infty$$

so that $f \in L^1(\mu)$. By assumption (i) we conclude now that $f \in C$.

(i) ⇒ (iii): This is a simple consequence of the Chebyshev-Markov inequality

$$\mu\{|f| \ge a\} \le \frac{1}{a^p} \int |f|^p \; d\mu, \quad f \in L^p(\mu)$$

(which shows that the $\|\,.\,\|_p$-topology is finer than the topology of convergence in measure).

(iii) ⇒ (i): If for $M > 0$ we have $C \subset MB_{L^1(\mu)}$, $f_n \in C$, $f \in L^1(\mu)$ and $f_n \xrightarrow{\mu} f$ we get $f_n \xrightarrow{\|\,\|_p} f$ by (use Hölder's inequality with exponents $\frac{1}{1-p}$ and $\frac{1}{p}$)

$$\begin{aligned}
\|f_n - f\|_p^p &= \int |f_n - f|^p \; d\mu \\
&= \int_{\{|f_n - f| \le a\}} |f_n - f|^p \; d\mu + \int_{\{|f_n - f| > a\}} 1 \cdot |f_n - f|^p \; d\mu \\
&\le \mu(S) a^p + \left(\int_{\{|f_n - f| > a\}} 1 \; d\mu \right)^{1-p} \left(\int_{\{|f_n - f| > a\}} |f_n - f| \; d\mu \right)^p \\
&\le \mu(S) a^p + (2M)^p \mu\{|f_n - f| > a\}^{1-p}
\end{aligned}$$

and suitably choosing a. Use the hypothesis (iii) to conclude that $f \in C$. □

Because of the equivalence of (i) and (iii) in the above proposition we will henceforth simply write *μ-closed* for "closed in the (relative) topology of $L^0(\mu)$ (on $L^1(\mu)$)".
The next theorem, due to Buhvalov and Lozanovskii, is one of the main results from [99]; there a general lattice-theoretic approach in certain ideal function spaces is given, too (see also [368, Chapter X.5]).

Theorem 3.4 *For a convex bounded subset C of $L^1(\mu)$ the following are equivalent:*

(i) $P\overline{C}^{w*} = C$*, where P is the L-projection from $L^1(\mu)^{**}$ onto $L^1(\mu)$.*

(ii) *C is closed in $L^1(\mu)$ with respect to convergence in measure.*

PROOF: (i) $\Rightarrow$ (ii): Take $f_n \in C$, $f \in L^1(\mu)$ with $f_n \xrightarrow{\mu} f$. We have to show that $f \in C$. Passing to a subsequence we may assume $f_n \xrightarrow{\text{a.e.}} f$. Since $\overline{C}^{w*}$ is w^*-compact there is a w^*-accumulation point $\nu \in \overline{C}^{w*}$ of (f_n). By Corollary 3.2 and the assumption (i) we have $f = P\nu \in P\overline{C}^{w*} = C$.

(ii) $\Rightarrow$ (i): The inclusion $C \subset P\overline{C}^{w*}$ is trivial. So we assume $\nu \in \overline{C}^{w*}$ and we have to show that $f := P\nu$ belongs to C. Take a net (f_α) in C such that $f_\alpha \xrightarrow{w*} \nu$. By Lemma 3.1 there is a sequence $e_n \in \operatorname{co}(f_\alpha) \subset C$ such that $e_n \xrightarrow{\text{a.e.}} f$. Since μ is finite this implies $e_n \xrightarrow{\mu} f$, so the μ-closedness of C gives $f \in C$. (Note that we didn't use the boundedness of C in this part of the proof.) □

Our aim for the first part of this section, the internal characterisation of subspaces of $L^1(\mu)$ which are L-embedded spaces, is now achieved.

Theorem 3.5 *For a subspace X of $L^1(\mu)$ the following are equivalent:*

(i) *X is an L-summand in its bidual.*

(ii) *B_X is μ-closed.*

(iii) *B_X is $\|\,.\,\|_p$-closed for one (for all) $p \in (0,1)$.*

PROOF: This follows from Proposition 3.3, Theorem 3.4 and Theorem 1.2 with $C = B_X$. (Note, of course, that $L^1(\mu)$ is an L-summand in its bidual.) □

We stress that statement (i) of the above theorem is independent of the embedding of the space X in an L^1-space, whereas the properties in (ii) and (iii) refer to the surrounding measure space and the particular embedding.
As a first application we will reprove the L-embeddedness of the Hardy space H_0^1 with the techniques of this section. Part (b) is included to show that the implication "(i) $\Rightarrow$ (ii)" in Theorem 3.4 doesn't hold for unbounded C; m denotes the normalized Lebesgue measure on $\mathbb{T}$.

Example 3.6

(a) *$B_{H_0^1}$ is m-closed (hence H_0^1 is an L-summand in its bidual).*

(b) *H_0^1 is m-dense in $L^1(\mathbb{T})$.*

PROOF: (a) We will prove a stronger statement which will be useful in the next section. To give the precise formulation we anticipate some notation from harmonic analysis (cf. Section IV.4): For $k \in \mathbb{Z}$ and $z \in \mathbb{T}$ put $\gamma_k(z) := z^k$. The k-th Fourier coefficient of $f \in L^1(\mathbb{T})$ is then

$$\widehat{f}(k) = \int_{\mathbb{T}} f\,\overline{\gamma_k}\,dm.$$

If we write for a subset Λ of $\mathbb{Z}$

$$L^1_\Lambda := \{f \in L^1(\mathbb{T}) \mid \widehat{f}(k) = 0 \quad \text{for } k \notin \Lambda\}$$

we have $H^1_0 = L^1_{\mathbb{N}}$.

CLAIM: *$B_{L^1_\Lambda}$ is $\|\,.\,\|_p$-closed $(0 < p < 1)$ in $L^1(\mathbb{T})$ for all subsets Λ of $\mathbb{N}$.*

For the proof it is appropriate to make a distinction between these spaces of classes of functions on $\mathbb{T}$ and the corresponding spaces of holomorphic functions on the open unit disk $\mathbb{D}$. We put

$$H^1(\mathbb{D}) := \left\{F : \mathbb{D} \longrightarrow \mathbb{C} \;\middle|\; F \text{ holomorphic, } \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})|\,dt < \infty\right\}$$

and recall that the Poisson transform $f \mapsto Pf$ yields a surjective isometry between H^1 and $H^1(\mathbb{D})$, the inverse being $F \mapsto F^*$, where $F^* \in L^1(\mathbb{T})$ denotes the boundary value function of F (see [544, Chapter 17]). Since $Pf(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n$ for $z \in \mathbb{D}$ and $f \in H^1$, the space L^1_Λ corresponds to

$$H^1_\Lambda(\mathbb{D}) := \{F \in H^1(\mathbb{D}) \mid F^{(n)}(0) = 0 \quad \text{for } n \notin \Lambda\}.$$

Any $F \in H^1(\mathbb{D})$ is the Cauchy integral of its boundary values F^* (see [544, Th. 17.11]), so for $r < 1$

$$\sup_{|z| \le r} |F(z)| \le \|F^*\|_1 (1-r)^{-1}$$

and thus the unit ball of $H^1(\mathbb{D})$ is uniformly bounded on compact subsets of $\mathbb{D}$.
To prove the claim, let $(f_n) = (F_n^*)$ be a sequence in the unit ball of $L^1_\Lambda \cong H^1_\Lambda(\mathbb{D})$ which $\|\,.\,\|_p$-converges to $f \in L^1$. We have to show $f \in L^1_\Lambda$. By the above remark and Montel's theorem there is a subsequence (F_{n_k}) which converges uniformly on compact subsets of $\mathbb{D}$ (=: with respect to τ_K) to some holomorphic function G on $\mathbb{D}$. It is then easy to see that $G \in H^1(\mathbb{D})$ and $G^{(n)}(0) = 0$ for $n \notin \Lambda$, hence $G \in H^1_\Lambda(\mathbb{D})$. So proving $G^* = f$ will show the claim.
If we let $H_k := F_{n_k} - G$ we have

$$H_k \xrightarrow{\tau_K} 0 \quad \text{and} \quad H_k^* \xrightarrow{\|\,\|_p} f - G^*.$$

For all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\int_0^{2\pi} |H_k^*(e^{it}) - H_l^*(e^{it})|^p\,dt < \varepsilon \quad \text{for all } k, l \ge N.$$

This implies

$$\int_0^{2\pi} |H_k(re^{it}) - H_l(re^{it})|^p \, dt < \varepsilon \quad \text{for all } k, l \geq N \text{ and } r < 1$$

since the left-hand side increases monotonically for $0 < r < 1$. Letting l tend to infinity this shows

$$\int_0^{2\pi} |H_k(re^{it})|^p \, dt \leq \varepsilon \quad \text{for all } k \geq N \text{ and } r < 1.$$

If now r tends to 1, we get

$$\int_0^{2\pi} |H_k^*(e^{it})|^p \, dt = \|H_k^*\|_p^p \leq \varepsilon \quad \text{for all } k \geq N.$$

This says $H_k^* \xrightarrow{\|\,\|_p} 0$, and we conclude $f = G^*$.

(b) Take functions $\varphi_n : \mathbb{T} \longrightarrow (0, \infty)$ such that

$$\varphi_n,\ \log \varphi_n \in L^1(\mathbb{T}), \qquad \int \log \varphi_n \, dm = 0, \quad \text{and} \qquad \varphi_n \xrightarrow{m} 0$$

(e.g. $\varphi_n = \frac{1}{n}\chi_{\mathbb{T}\setminus E_n} + a_n \chi_{E_n}$ with $m(E_n) = \frac{1}{n}$ and suitable a_n). Let Q_n be the outer function for φ_n, i.e.

$$Q_n(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \varphi_n(e^{it}) \, dt \right\}, \quad z \in \mathbb{D}$$

then (see [544, Th. 17.16]) $Q_n \in H^1(\mathbb{D})$, $|Q_n^*| = \varphi_n$, and

$$\log Q_n(0) = \frac{1}{2\pi} \int_0^{2\pi} \log \varphi_n(e^{it}) \, dt = 0.$$

Hence $Q_n^* \xrightarrow{m} 0$ and $Q_n(0) = 1$. Since H^1-functions are the Poisson integrals of their boundary values, we get

$$1 = Q_n(0) = \frac{1}{2\pi} \int_0^{2\pi} Q_n^*(e^{it}) \, dt = \widehat{Q_n^*}(0)$$

where $\widehat{Q_n^*}$ denotes the Fourier transform of Q_n^* – see Section IV.4. This yields

$$f_n := Q_n^* - \widehat{Q_n^*}(0)\, \mathbf{1} \in H_0^1 \quad \text{and} \quad f_n \xrightarrow{m} -\mathbf{1}.$$

Hence $H^1 \subset \overline{H_0^1}^m$. Since $f_n \gamma_{-1} \in H^1$ and $f_n \gamma_{-1} \xrightarrow{m} -\gamma_{-1}$, we find

$$\overline{\text{lin}}\, \{\gamma_k \mid k \geq -1\} \subset \overline{H^1}^m \subset \overline{H_0^1}^m.$$

Inductively one now proves easily that $\gamma_k \in \overline{H_0^1}^m$ for all $k \in \mathbb{Z}$, hence assertion (b). $\square$

The above proof of Example 3.6(a) uses only elementary properties of H^1-functions. Allowing less trivial, yet standard, tools from H^p-theory, namely $H^p \cap L^1 \subset H^1$ (see

[243, Cor. II.4.3]) one gets its conclusion immediately. Also the stronger claim in the above proof can be obtained this way by observing that the inequality (see [180, p. 36])

$$|F(z)| \leq 2^{\frac{1}{p}} \|F^*\|_p (1-r)^{-\frac{1}{p}}$$

for $F \in H^p(\mathbb{D})$, $r < 1$ and $|z| = r$ shows that the topology τ_K is coarser than the $\|\,.\,\|_p$-topology, hence $H^p_\Lambda(\mathbb{D})$ is $\|\,.\,\|_p$-closed in $H^p(\mathbb{D})$.
Our next aim is to single out those X among the L-embedded subspaces of $L^1(\mu)$ which are even duals of M-embedded spaces. The extra condition on X is again given in terms of the topology of convergence in measure and is prepared by the following lemma and Proposition 3.9.

Lemma 3.7 *If $f_\alpha \in L^1(\mu)$, $\pi \in L^1(\mu)_s$, $\|f_\alpha\| \leq \|\pi\|$ and $f_\alpha \xrightarrow{w^*} \pi$, then $f_\alpha \xrightarrow{\mu} 0$.*

Proof: For $a > 0$ we want to show that $\mu\{|f_\alpha| \geq a\} \to 0$. Let $\varepsilon > 0$. Since π is purely finitely additive there is, by [646, Th. 1.19], $E \in \Sigma$ such that $\mu(E) < \frac{\varepsilon}{2}$ and $|\pi|(E) = \|\pi\|$. Then also $\|\chi_E \pi\| = \|\pi\|$. Since $f_\alpha \xrightarrow{w*} \pi$ we have $\chi_E f_\alpha \xrightarrow{w*} \chi_E \pi$. But this, together with $\|\chi_E f_\alpha\| \leq \|f_\alpha\| \leq \|\pi\| = \|\chi_E \pi\|$ and the w^*-lower semicontinuity of the norm implies that $\|\chi_E f_\alpha\| \to \|\pi\|$. So there is α_0 such that $|\,\|\pi\| - \|\chi_E f_\alpha\|\,| < a\frac{\varepsilon}{2}$ for all $\alpha \geq \alpha_0$. We have for these α

$$\int_{\complement E} |f_\alpha|\, d\mu = \|f_\alpha\| - \int_E |f_\alpha|\, d\mu \leq \|\pi\| - \|\chi_E f_\alpha\| \leq a\frac{\varepsilon}{2},$$

so

$$\begin{aligned} \mu\{|f_\alpha| \geq a\} &= \mu\left(E \cap \{|f_\alpha| \geq a\}\right) + \mu\left(\complement E \cap \{|f_\alpha| \geq a\}\right) \\ &\leq \frac{\varepsilon}{2} + \frac{1}{a}\int_{\complement E \cap \{|f_\alpha| \geq a\}} |f_\alpha|\, d\mu \\ &\leq \varepsilon. \end{aligned}$$

□

Definition 3.8

(a) *If $S \subset L^0(\mu)$ and $f : S \to \mathbb{C}$ is a function, then f is called* μ-continuous *if it is continuous in the topology of convergence in measure.*

(b) *For a subspace X of $L^1(\mu)$ put*

$$X^\sharp := \{x^* \in X^* \mid x^*|_{B_X} \text{ is } \mu\text{-continuous}\}.$$

A standard $\frac{\varepsilon}{3}$-argument shows that $X^\sharp$ is a closed subspace of X^*. This space $X^\sharp$ is our candidate for an M-embedded predual of X. To get acquainted with the above definition we note:

Remark.

(a) $L^1[0,1]^\sharp = \{0\}$

(b) *If X is nonreflexive, then $X^\sharp \neq X^*$.*

PROOF: (a) This follows from the following proposition and Remark 2.10(b), but a simple direct argument is possible, too.

(b) We show first:

CLAIM: A subspace X of $L^1(\mu)$ with $X^\sharp = X^*$ has the following property:

(∗) Every bounded sequence (f_n) in X has a subsequence (g_n) which has weakly convergent arithmetic means $\big(\frac{1}{n}\sum_{k=1}^n g_k\big)$.

To prove this we need the following result of Komlós [381, Theorem 1a].

> THEOREM: *For every bounded sequence (f_n) in $L^1(\mu)$ there are an $f \in L^1(\mu)$ and a subsequence (g_n) of (f_n) such that the sequence of arithmetic means $\big(\frac{1}{n}\sum_{k=1}^n g_k\big)_n$ converges to f almost everywhere. Moreover, the conclusion remains true for every subsequence of (g_n).*

So, given a bounded sequence (f_n) in X we find by Komlós' theorem a subsequence (g_n) such that $h_n := \frac{1}{n}\sum_{k=1}^n g_k$ converges to some $f \in L^1(\mu)$ μ-a.e., hence in measure. Thus (h_n) is Cauchy in $L^0(\mu)$ so that $h_{n_{k+1}} - h_{n_k} \xrightarrow{\mu} 0$ for all subsequences (n_k). Now the assumption $X^\sharp = X^*$ gives $x^*(h_{n_{k+1}} - h_{n_k}) \longrightarrow 0$ for all $x^* \in X^*$. Thus (h_n) is weakly Cauchy, hence weakly convergent (by weak sequential completeness).
A well-known application of James' theorem (cf. [154, pp. 82–83]) shows that an arbitrary Banach space X which has the property (∗) from the claim has to be reflexive. [Indeed, for $x^* \in S_{X^*}$ choose $x_n \in B_X$ such that $x^*(x_n) \to \|x^*\| = 1$. By (∗) there is a subsequence (y_n) of (x_n) and $x \in X$ with

$$\frac{1}{n}\sum_{k=1}^n y_k \xrightarrow{w} x.$$

In particular

$$x^*\left(\frac{1}{n}\sum_{k=1}^n y_k\right) \longrightarrow x^*(x).$$

But

$$x^*\left(\frac{1}{n}\sum_{k=1}^n y_k\right) = \frac{1}{n}\sum_{k=1}^n x^*(y_k) \longrightarrow 1.$$

So x^* attains its norm at $x \in B_X$.] □

Because of the similarity of the above arguments with the Banach-Saks property it seems worth recalling that, in view of Szlenk's result that $L^1(\mu)$ *has the weak Banach-Saks property* (cf. [154, p. 85]), for subspaces X of $L^1(\mu)$ the Banach-Saks property is equivalent to reflexivity.
Actually, the converse to assertion (b) holds as well. This is a particular consequence of the following proposition. For more on Remark (b) see the Notes and Remarks section.

Proposition 3.9 *For a subspace X of $L^1(\mu)$ one has*

$$X^\sharp = (X^{\perp\perp} \cap L^1(\mu)_s)_\perp.$$

Proof: "$\supset$": Assume there is $x^* \in (X^{\perp\perp} \cap L^1(\mu)_s)_\perp$ which is not continuous on B_X with respect to convergence in measure. Then there are $f_n, f \in B_X$ such that $f_n \xrightarrow{\mu} f$ and $x^*(f_n) \not\to x^*(f)$. We find a subsequence (g_n) of $(f_n - f)$ such that $g_n \xrightarrow{\text{a.e.}} 0$ and $x^*(g_n) \to a \neq 0$. Since (g_n) is a bounded sequence in X there is a w^*-accumulation point ν of (g_n) in $X^{\perp\perp}$. By Corollary 3.2 we also have $\nu \in L^1(\mu)_s$, so $\nu(x^*) = 0$ by assumption on x^*.
If now (g_α) is a subnet of (g_n) which converges to ν with respect to $\sigma(L^1(\mu)^{**}, L^1(\mu)^*)$, hence with respect to $\sigma(X^{**}, X^*)$, we have in particular $x^*(g_\alpha) \to \nu(x^*) = 0$. But since $(x^*(g_\alpha))$ is a subnet of $(x^*(g_n))$, we also have $x^*(g_\alpha) \to a \neq 0$. This contradiction shows the incorrectness of our assumption.

"$\subset$": Take $x^* \in X^\sharp$ and $\nu \in X^{\perp\perp} \cap L^1(\mu)_s$. We claim $\nu(x^*) = 0$. Since $\nu \in \|\nu\| B_{X^{\perp\perp}} = \|\nu\| B_{X^{**}}$, we find a net (f_α) in X such that $\|f_\alpha\| \le \|\nu\|$ and $f_\alpha \to \nu$ with respect to $\sigma(X^{**}, X^*)$, hence $f_\alpha \to \nu$ with respect to $\sigma(L^1(\mu)^{**}, L^1(\mu)^*)$. Since $\nu \in L^1(\mu)_s$ we infer from Lemma 3.7 that $f_\alpha \xrightarrow{\mu} 0$. So $x^* \in X^\sharp$ yields $x^*(f_\alpha) \to 0$. But the $\sigma(X^{**}, X^*)$-convergence of (f_α) to ν implies in particular $x^*(f_\alpha) \to \nu(x^*)$, hence the claim. ◻

The promised characterisation is the following.

Theorem 3.10 *For a subspace X of $L^1(\mu)$ the following are equivalent:*

(i) *X is isometrically isomorphic to the dual of a Banach space Y which is an M-ideal in Y^{**}.*

(ii) *B_X is μ-closed and $X^\sharp$ separates X.*

Moreover, if (i), (ii) *are satisfied Y is isometrically isomorphic to $X^\sharp$.*

Proof: (i) $\Rightarrow$ (ii): By Corollary III.1.3 X is an L-summand in X^{**}, so B_X is μ-closed by Theorem 3.5 and $X^{\perp\perp} = X \oplus_1 (X^{\perp\perp} \cap L^1(\mu)_s)$ by Theorem 1.2. Neglecting for simplicity the isometric isomorphism, i.e. assuming $X = Y^*$, we get from Proposition III.1.2 that $X^{**} = Y^{***} = Y^* \oplus_1 Y^\perp$, hence $Y^\perp = X^{\perp\perp} \cap L^1(\mu)_s$. So by Proposition 3.9

$$Y = (Y^\perp)_\perp = (X^{\perp\perp} \cap L^1(\mu)_s)_\perp = X^\sharp.$$

Since Y separates Y^*, we get that $X^\sharp$ separates X, and the predual Y is $X^\sharp$.

(ii) $\Rightarrow$ (i): If B_X is μ-closed, X is an L-embedded space by Theorem 3.5 and

$$X^{\perp\perp} = X \oplus_1 (X^{\perp\perp} \cap L^1(\mu)_s) \qquad (*)$$

by Theorem 1.2. We know that $X^\sharp = (X^{\perp\perp} \cap L^1(\mu)_s)_\perp$ from Proposition 3.9, hence

$$(X^\sharp)^\perp = ((X^{\perp\perp} \cap L^1(\mu)_s)_\perp)^\perp = (X^{\perp\perp} \cap L^1(\mu)_s)^{-w^*}. \qquad (**)$$

But since $X^\sharp$ separates X we have that $(X^\sharp)^\perp \cap X = \{0\}$. This yields, together with $(*)$ and $(**)$, that $X^{\perp\perp} \cap L^1(\mu)_s$ is w^*-closed, hence X is the dual of an M-embedded space Y and $Y \cong (X^{\perp\perp} \cap L^1(\mu)_s)_\perp = X^\sharp$ by Proposition 1.9. ◻

IV.4 Relations with abstract harmonic analysis

In this section we will relate subsets Λ of the dual group Γ of a compact abelian group G with Banach space properties of certain spaces of Λ-spectral functions. In our context the spaces L^1_Λ (see below) appear as a natural class of nontrivial subspaces of $L^1(G)$. Most of the following material is based on Godefroy's important paper [257].

Let us fix our notation and recall some facts from harmonic analysis. The symbol G always denotes a compact abelian group with group operation written as multiplication and normalized Haar measure m. We write $\Gamma = \widehat{G}$ for the (discrete) dual group of G consisting of the continuous group homomorphisms $\gamma : G \to \mathbb{T}$, the so-called characters, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the circle group. The spaces $L^1 := L^1(G) := L^1(G, \mathrm{Bor}(G), m)$ and $M := M(G)$ are commutative Banach algebras with respect to convolution. If we identify $f \in L^1$ with the m-continuous measure fm with density f, then $L^1(G)$ turns out to be an ideal in $M(G)$. [See [541, 1.1.7, 1.3.2 and 1.3.5] or [308]; these two books are our main references on harmonic analysis.] The Fourier (-Stieltjes) transform is the mapping

$$\begin{array}{rcll} \widehat{\ } : & M(G) & \longrightarrow & \ell^\infty(\Gamma) \\ & \mu & \longmapsto & \widehat{\mu} \qquad \text{with} \quad \widehat{\mu}(\gamma) := \int_G \overline{\gamma(x)}\, d\mu(x). \end{array}$$

$\widehat{\ }$ is an injective, multiplicative ($\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$) and continuous ($\|\widehat{\mu}\| \leq \|\mu\|$) operator [541, 1.3.3, 1.7.3]. If we identify Γ with the spectrum (= maximal ideal space) of $L^1(G)$ by $\gamma \longleftrightarrow \varphi_\gamma$ where

$$\begin{array}{rcll} \varphi_\gamma & : & L^1(G) & \longrightarrow \ \mathbb{C} \\ & & f & \longmapsto \ \varphi_\gamma(f) := \widehat{f}(\gamma) := \int f(x)\overline{\gamma(x)}\, dm(x) \end{array}$$

(see [541, 1.2.2]), then the restriction of $\widehat{\ }$ to $L^1(G)$ is the Gelfand transform of $L^1(G)$, [541, 1.2.3 and 1.2.4]. We denote by $T := T(G) := \mathrm{lin}\,\Gamma$ ($\subset C(G) \subset L^1(G)$) the space of so-called "trigonometric polynomials" on G.

Definition / Proposition 4.1 *For $\Lambda \subset \Gamma$ we write*

$$L^1_\Lambda := L^1_\Lambda(G) := \{f \in L^1(G) \mid \widehat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$

for the space of Λ-spectral functions in L^1. T_Λ, C_Λ and M_Λ are defined similarly.

(a) $T_\Lambda = \mathrm{lin}\,\Lambda$.
(b) *C_Λ, L^1_Λ and M_Λ are closed ideals in the corresponding convolution algebras.*
(c) *T_Λ is $\|\ \|_\infty$-dense in C_Λ and $\|\ \|_1$-dense in L^1_Λ.*
(d) $L^1_\Lambda = M_\Lambda \cap L^1$.
(e) $(C/C_{\Gamma\setminus\Lambda})^* \cong M_{\Lambda^{-1}}$.

Proof: See [309, 35.7] for (a) – (c). (d) is clear from the definition and (e) follows from (c) by standard duality arguments. □

It is well-known in harmonic analysis that the spaces L^1_Λ are precisely the closed translation invariant subspaces and precisely the closed ideals of the convolution algebra $L^1(G)$ [309, Theorem 38.7].

Before we define the four types of subsets of Γ we will be concerned with, let us agree on some further notation: we write M_a $(= L^1)$ and M_s for the subspaces of m-absolutely continuous and m-singular measures in M, P_a and P_s $(= Id - P_a)$ denote the corresponding L-projections.

Definition 4.2 *A subset Λ of Γ will be called*

(a) nicely placed *if L^1_Λ is an L-summand in its bidual,*
(b) *a* Shapiro set *if all subsets Λ' of Λ are nicely placed,*
(c) Haar invariant *if $P_a M_\Lambda \subset M_\Lambda$,*
(d) *a* Riesz set *if $M_\Lambda = L^1_\Lambda$.*

G. Godefroy coined the first two terms in [257]. He used "sous-espace bien disposé" in [256] for subspaces X of $L^1(\mu)$ which are L-summands in their biduals. By Theorem 1.2 this means a nice, compatible placement of X^{**} in the bidual of $L^1(\mu)$. However we have avoided this expression in order to emphasize that being bien disposé is a property of the space X itself and not of the embedding of X in $L^1(\mu)$. (X is "nicely placed" in every space $L^1(\nu)$ which contains it – see the remark following Theorem 3.5.) Here we consider the fixed "embedding" $\Lambda \subset \Gamma$ and the use of the expression "nicely placed" seems appropriate by the above.

Let us recall the reason for the name "Riesz set": The F. and M. Riesz theorem says: If $\mu \in M(\mathbb{T})$ is such that $\widehat{\mu}(n) = 0$ for all $n < 0$, then μ is absolutely continuous with respect to the Haar measure m on $\mathbb{T}$. Using the above notations this translates into: $\mu \in M_{\mathbb{N}_0}(\mathbb{T})$ implies $\mu = fm$ for some $f \in L^1(\mathbb{T})$, i.e. $M_{\mathbb{N}_0}(\mathbb{T}) = L^1_{\mathbb{N}_0}(\mathbb{T})$. Riesz sets were studied systematically for the first time in [441].

Noting that Λ is a Riesz set if and only if $P_s M_\Lambda = \{0\}$, we see that Riesz sets are Haar invariant. To study the relation between these two families of subsets more closely we prove:

Lemma 4.3 *A subset Λ of Γ such that Λ' is Haar invariant for all $\Lambda' \subset \Lambda$ is a Riesz set.*

PROOF: Take $\mu \in M_\Lambda$. Since Λ is Haar invariant we have $\mu_s \in M_\Lambda$, i.e. $\widehat{\mu}_s(\gamma) = 0$ for all $\gamma \notin \Lambda$. Take now $\gamma \in \Lambda$ and consider $\Lambda' := \Lambda \setminus \{\gamma\}$ and $\nu := \mu - \widehat{\mu}(\gamma)\gamma$. We get $\nu \in M_{\Lambda'}$, hence by assumption $\nu_s \in M_{\Lambda'}$. But $\nu_s = \mu_s$, so $0 = \widehat{\nu}_s(\gamma) = \widehat{\mu}_s(\gamma)$. We have proved that $\widehat{\mu}_s = 0$, hence $\mu_s = 0$. □

In the next proposition we collect some easy examples and counterexamples of Riesz and nicely placed sets, as well as their basic stability properties.

Proposition 4.4

(a) *Finite sets Λ are Riesz and nicely placed.*
(b) *Cofinite sets $\Lambda \neq \Gamma$ are not Riesz and not nicely placed if G is infinite.*
(c) *If Λ is a Riesz resp. nicely placed subset, then Λ^{-1} and $\lambda\Lambda$ $(\lambda \in \Gamma)$ are Riesz resp. nicely placed.*
(d) *Subgroups Λ of Γ are nicely placed.*
(e) *$n\mathbb{Z}$ is not a Riesz subset of $\mathbb{Z}$ $(n \neq 0)$.*

PROOF: (a) is easy.

(b) For $\Lambda = \Gamma \setminus \{\gamma_1, \dots, \gamma_n\}$ consider $\mu = \delta_e - (\gamma_1 + \dots + \gamma_n)m$, $e \in G$ the neutral element, for the Riesz part. The space L^1_Λ is finite codimensional in L^1 in this case, so Corollary 1.15 shows that Λ is not nicely placed.

(c) We have $\widehat{\mu}(\gamma^{-1}) = \overline{\widehat{\overline{\mu}}(\gamma)}$, where $\overline{\mu}(E) := \overline{\mu(E)}$ (as a measure) or $\overline{\mu}(g) := \overline{\mu(\overline{g})}$ (as a functional). So $\mu \in M_{\Lambda^{-1}}$ implies $\overline{\mu} \in M_\Lambda$. Now Λ is Riesz, and this gives $\overline{\mu} \ll m$. Consequently $\mu \ll m$. For $f \in L^1$ one obtains $\widehat{f}(\lambda\gamma) = \widehat{f\overline{\lambda}}(\gamma)$, hence $f \in L^1_{\lambda\Lambda}$ iff $f\overline{\lambda} \in L^1_\Lambda$. So $M_{\overline{\lambda}} : L^1 \to L^1$, $f \mapsto \overline{\lambda} f$ is a surjective isometry which maps $L^1_{\lambda\Lambda}$ onto L^1_Λ. Thus $L^1_{\lambda\Lambda}$ is L-embedded since L^1_Λ is. The other two statements are proved in a similar way.

(d) Put $H := \Lambda_\perp := \{x \in G \mid \gamma(x) = 1 \text{ for all } \gamma \in \Lambda\}$. By the "bipolar theorem for groups" [541, 2.1.3] it follows that $L^1_\Lambda = \{f \in L^1 \mid f = f_x \text{ for all } x \in H\}$ where $f_x(y) := f(xy)$. The translations $T_x : L^1 \to L^1$, $f \mapsto f_x$ are isometric with respect to d_m (the metric of convergence in m-measure – see the beginning of Section IV.3), so $\{f \in L^1 \mid f = f_x\} = (Id - T_x)^{-1}(\{0\})$ is m-closed in L^1, hence L^1_Λ is m-closed. But then $B_{L^1_\Lambda}$ is m-closed.

(e) For $z \in \mathbb{T}$ and $k \in \mathbb{Z}$ we have $\widehat{\delta_z}(k) = z^{-k}$. Put $z := e^{-\frac{2\pi i}{n}}$ and $\mu := \frac{1}{n}(\delta_1 + \delta_z + \dots + \delta_{z^{n-1}})$. Then $\mu \perp m$ and $\widehat{\mu} = \chi_{n\mathbb{Z}}$. □

Proposition 4.5 *For a subset Λ of Γ,*

(a) *if Λ is nicely placed then Λ is Haar invariant,*

(b) *if Λ is a Shapiro set then Λ is a Riesz set.*

PROOF: (a) We need the following result of Boclé ([82, Th. II], see also [564, Lemma 1.1]).

> LEMMA: *For a symmetric, open neighbourhood V of e in G put $u_V := \frac{1}{m(V)}\chi_V$. If $\nu \in M_s$ then $(u_V * \nu)_V$ converges in Haar measure to zero.*

[We wish to indicate the proof of this lemma in the classical case of the circle group $G = \mathbb{T}$ which we identify with the interval $]-\pi, \pi]$. If we let $F(t) = \int_0^t d\nu$, then a straightforward computation yields that

$$\left(\frac{1}{2\varepsilon}\chi_{[-\varepsilon,\varepsilon]} * \nu\right)(t) = \frac{1}{2\varepsilon}(F(t+\varepsilon) + F(t-\varepsilon))$$

which tends to 0 a.e. since by Lebesgue's differentiation theorem $F'(t)$ exists and equals 0 a.e. as ν is singular.]

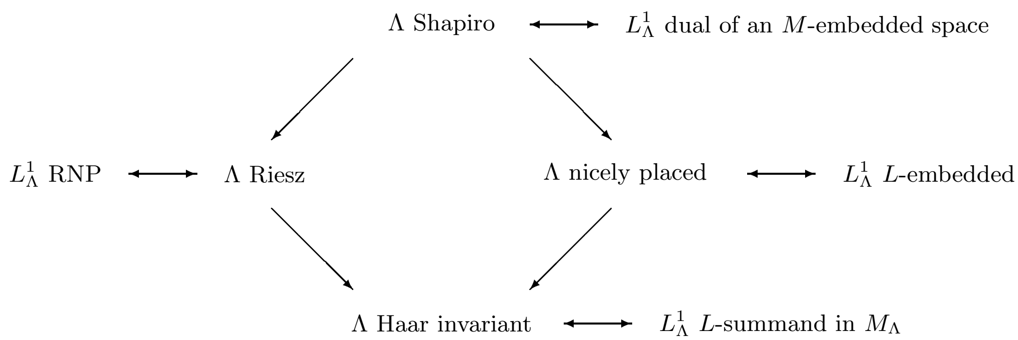
If now $\Lambda \subset \Gamma$ is nicely placed, $\mu \in M_\Lambda$ and $\mu = f + \mu_s$, we have to show $f \in L^1_\Lambda$. Since (u_V) is an approximate unit in L^1 (see [541, 1.1.8]), we get $u_V * f \xrightarrow{\|\ \|_1} f$, hence $u_V * f \xrightarrow{m} f$. The lemma yields $u_V * \mu_s \xrightarrow{m} 0$, so

$$u_V * \mu = u_V * f + u_V * \mu_s \xrightarrow{m} f.$$

Because L^1 and M_Λ are ideals, $(u_V * \mu)$ is a (bounded) net in L^1_Λ. By assumption and Theorem 3.5 $B_{L^1_\Lambda}$ is m-closed, hence $f \in L^1_\Lambda$.

Part (b) follows from Lemma 4.3 and part (a). □

The following diagram contains not only the relations between the four types of sets established so far, but also their Banach space characterisations which we will prove next.



Proposition 4.6 *A subset Λ of Γ is Haar invariant if and only if L^1_Λ is an L-summand in M_Λ.*

Proof: The "only if" part is trivial: Since $P_a M_\Lambda \subset M_\Lambda$, $P_a|_{M_\Lambda}$ is an L-projection in M_Λ with range L^1_Λ.
To prove the "if" part we consider the subspace M_Λ of the abstract L-space M. By Proposition I.1.21 M_Λ is invariant under the band projection P_B onto the band B generated by L^1_Λ in M. Since we may assume $\Lambda \neq \emptyset$ there is $\gamma \in L^1_\Lambda$ with $|\gamma(x)| = 1$ for all $x \in G$. So the closed (order) ideal generated by L^1_Λ in L^1 is L^1. But L^1 is a band in M, thus $B = L^1$. Of course $P_B = P_a$. □

There is no M- or L-structure characterisation of Riesz sets, however:

Theorem 4.7 *For a subset Λ of Γ the following are equivalent:*

(i) *Λ is a Riesz set.*
(ii) *L^1_Λ has the Radon-Nikodým property.*
(iii) *$L^1_{\Lambda'}$ is a separable dual space for all countable subsets Λ' of Λ.*

Proof: (i) ⇒ (iii): Every subset Λ' of Λ is Riesz, hence $L^1_{\Lambda'} = M_{\Lambda'} \cong (C/C_{\Gamma\setminus\Lambda'^{-1}})^*$ by Proposition 4.1(e). If Λ' is countable then $L^1_{\Lambda'}$ is separable.

(iii) ⇒ (ii): By [158, Th. III.3.2, p. 81] it is enough to show that every separable subspace X of L^1_Λ has the RNP. But every such subspace X is contained in some $L^1_{\Lambda'}$, Λ' countable. [Proof: If (x_n) is dense in X, find $t_{n,k} \in T_\Lambda$ with $t_{n,k} \longrightarrow x_n$ $(k \to \infty)$ by Proposition 4.1(c). This yields countable subsets $\Lambda_n \subset \Lambda$ such that $x_n \in L^1_{\Lambda_n}$. Then $\Lambda' := \bigcup \Lambda_n$ will do.] So X has the RNP as a subspace of a separable dual space [158, Th. III.3.1, p. 79 and Th. III.3.2, p. 81].

(ii) $\Rightarrow$ (i): We have to show that every measure $\mu \in M_\Lambda$ is absolutely continuous with respect to the Haar measure m. By Proposition 4.1 the following mapping is well-defined for such a μ:

$$\begin{array}{rcrcl} F & : & \mathrm{Bor}(G) & \longrightarrow & L^1_\Lambda \\ & & E & \longmapsto & F(E) := \chi_E * \mu. \end{array}$$

Now F is easily seen to be finitely additive and

$$\|F(E)\| = \|\chi_E * \mu\| \le \|\chi_E\| \|\mu\| = \|\mu\| m(E)$$

for $E \in \mathrm{Bor}(G)$ shows that F is m-continuous and of bounded variation. By the very definition of the Radon-Nikodým property there is a Bochner integrable function (!) $g : G \to L^1_\Lambda$ such that

$$F(E) = \int_E g\, dm \qquad \forall E \in \mathrm{Bor}(G).$$

We claim that there is a null set N such that

$$g(y)_y = g(x)_x \qquad \text{for } x, y \notin N. \tag{1}$$

(Recall that f_x denotes the translate of f by x, i.e. $f_x(y) = f(xy)$; $T_x : f \mapsto f_x$ is the corresponding operator on L^1.) Fix $z \in G$. Omitting the m for integration with respect to the Haar measure we obtain for $E \in \mathrm{Bor}(G)$

$$\begin{aligned} \int_E g(x)\, dx &= F(E) = \chi_E * \mu = (\chi_{zE} * \mu)_z = \Big(\int_{zE} g(x)\, dx\Big)_z \\ &= T_z\Big(\int \chi_{zE}(x) g(x)\, dx\Big) = \int T_z(\chi_{zE}(x) g(x))\, dx \\ &= \int \chi_{zE}(x) g(x)_z\, dx = \int \chi_E(z^{-1}x) g(x)_z\, dx \\ &= \int \chi_E(x) g(zx)_z\, dx = \int_E g(zx)_z\, dx. \end{aligned}$$

The uniqueness of the density function of a vector measure (see [158, Cor. II.2.5, p. 47]) now shows that $g(x) = g(zx)_z$ for $x \notin N_z$, a null set depending on z. Since $(x, z) \mapsto g(zx)_z$ is Bochner integrable, we deduce by a Fubini type argument the existence of a null set N such that $g(x) = g(zx)_z$ for almost all z if $x \notin N$. [In fact,

$$\int_E \int_F (g(x) - g(zx)_z)\, dz\, dx = \int_F \int_E (g(x) - g(zx)_z)\, dx\, dz = 0 \qquad \forall E, F \in \mathrm{Bor}(G)$$

so that off a null set N

$$\int_F (g(x) - g(zx)_z)\, dz = 0 \qquad \forall F \in \mathrm{Bor}(G),$$

whence the assertion.] This proves that for $x \notin N$

$$g(y)_y = g(x)_x \qquad \text{for almost every } y$$

and thus our claim.
So let us define
$$f := g(x_0)_{x_0} \in L^1(G)$$
where $x_0 \notin N$ is arbitrary. Hence $g(x)_x = f_{x^{-1}}$ for almost every x. We will now show $\mu = fm$ by proving
$$\chi_E * \mu = \chi_E * f \qquad \text{for all } E \in \mathrm{Bor}(G). \tag{2}$$
Indeed, if (2) holds then $\varphi * \mu = \varphi * f$ for all simple functions φ, hence – by continuity of multiplication (= convolution) in $M(G)$ – it holds for all $\varphi \in L^1(G)$; but then for all $\varphi \in C(G)$. Noting $C(G) * M(G) \subset C(G)$ we evaluate $\varphi * \mu$ and $\varphi * f$ at the neutral element and obtain
$$\int \varphi(y^{-1})\, d\mu(y) = \int \varphi(y^{-1})\, d(fm)(y),$$
hence $\mu = fm$, and μ is m-continuous.
In order to see (2) recall
$$\chi_E * \mu = F(E) = \int_E g(x)\, dx = \int_E f_{x^{-1}}\, dx$$
and
$$(\chi_E * f)(y) = \int \chi_E(x) f(x^{-1}y)\, dx = \int_E f(x^{-1}y)\, dx.$$
For every $\varphi \in L^\infty(G) = L^1(G)^*$ the following holds by Fubini's theorem:
$$\begin{aligned}\langle \varphi, \chi_E * f\rangle &= \int_G \varphi(y)(\chi_E * f)(y)\, dy = \int_G\int_E \varphi(y) f(x^{-1}y)\, dx\, dy \\ &= \int_E\int_G \varphi(y) f(x^{-1}y)\, dy\, dx = \int_E \langle \varphi, f_{x^{-1}}\rangle\, dx = \left\langle \varphi, \int_E f_{x^{-1}}\, dx \right\rangle.\end{aligned}$$
By the above this is what was claimed in (2). □

We need the following lemmata for the M-structure characterisation of Shapiro sets. Recall that φ_γ denotes the map which assigns to $f \in L^1$ the γ-th Fourier coefficient $\widehat{f}(\gamma)$. Also, recall from Definition 3.8 that $X^\sharp$ denotes the collection of those functionals on a subspace X of L^1 whose restrictions to the unit ball of X are m-continuous.

Lemma 4.8 *Let Λ be a nicely placed subset of Γ. Then Λ is a Shapiro set if and only if $\varphi_\gamma \in (L^1_\Lambda)^\sharp$ for all $\gamma \in \Lambda$.*

Proof: For the "if" part we have to show by Theorem 3.5 that $B_{L^1_{\Lambda'}}$ is m-closed for all subsets Λ' of Λ. Now
$$B_{L^1_{\Lambda'}} = \{f \in B_{L^1_\Lambda} \mid \widehat{f}(\gamma) = 0 \text{ for all } \gamma \in \Lambda \setminus \Lambda'\} = \bigcap_{\gamma \in \Lambda\setminus\Lambda'} \varphi_\gamma|_{B_{L^1_\Lambda}}{}^{-1}\{0\}\,.$$
Hence $B_{L^1_{\Lambda'}}$ is m-closed in $B_{L^1_\Lambda}$, which is m-closed in L^1 since Λ is nicely placed. Therefore $B_{L^1_{\Lambda'}}$ is m-closed in L^1, too.

On the other hand, assuming that for some $\gamma \in \Lambda$ the restriction of φ_γ to the unit ball of L^1_Λ is not m-continuous, we find a bounded sequence (f_n) in L^1_Λ such that $f_n \stackrel{m}{\longrightarrow} 0$, $a := \lim \widehat{f}_n(\gamma)$ exists, but $a \neq 0$. Consider $\Lambda' := \Lambda \setminus \{\gamma\}$ and $g_n := f_n - f_n(\gamma)\gamma$. Then (g_n) is a bounded sequence in $L^1_{\Lambda'}$ with $g_n \stackrel{m}{\longrightarrow} -a\gamma$. Since by assumption Λ' is nicely placed, we get $-a\gamma \in L^1_{\Lambda'}$, a contradiction. ☐

The next lemma says "If there is an m-continuous functional in $(L^1_\Lambda)^*$ which doesn't vanish at $\gamma \in L^1_\Lambda$, then the natural functional not vanishing at γ, i.e. φ_γ, is m-continuous on the unit ball of L^1_Λ". This will be used to show that "If there is an M-embedded predual of L^1_Λ $(= M_\Lambda)$, then it is the natural one, i.e. $C/C_{\Gamma\setminus\Lambda^{-1}}$".

Lemma 4.9 *If $\psi \in (L^1_\Lambda)^\sharp$, $\gamma \in \Lambda(\subset L^1_\Lambda)$, and $\psi(\gamma) \neq 0$, then $\varphi_\gamma \in (L^1_\Lambda)^\sharp$.*

Proof: Without loss of generality we may assume $\psi(\gamma) = 1$. Define a new functional by

$$\begin{array}{rcccl} \widetilde{\psi} & : & L^1_\Lambda & \longrightarrow & \mathbb{C} \\ & & f & \longmapsto & \int_G \psi(f_x)\,\overline{\gamma(x)}\,dx. \end{array}$$

Note that the space L^1_Λ is translation invariant and $x \mapsto f_x$ is continuous (see [541, Th. 1.1.5]); so the integral exists, and because of $\|f\| = \|f_x\|$ we have that $\widetilde{\psi}$ is bounded. We will show that $\widetilde{\psi}$ is m-continuous on the unit ball of L^1_Λ and $\widetilde{\psi} = \varphi_\gamma|_{L^1_\Lambda}$. The first statement follows since translation is isometric with respect to d_m, which was defined at the beginning of the previous section. Indeed, the supposed m-continuity of ψ implies: For every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|h|_m < \delta,\ h \in 2B_{L^1_\Lambda} \qquad \Longrightarrow \qquad |\psi(h)| < \varepsilon.$$

For $f, g \in B_{L^1_\Lambda}$ with $d_m(f,g) = |f-g|_m < \delta$, hence also $|f_x - g_x|_m < \delta$ for all $x \in G$, this yields $|\psi(f_x - g_x)| < \varepsilon$ $(x \in G)$. This immediately gives the m-continuity of $\widetilde{\psi}$ at f. Observing $\lambda_x = \lambda(x)\lambda$ for $\lambda \in \Lambda$ we find

$$\widetilde{\psi}(\lambda) = \int \psi(\lambda_x)\,\overline{\gamma(x)}\,dx = \int \psi(\lambda(x)\lambda)\,\gamma^{-1}(x)\,dx = \psi(\lambda)\int(\lambda\gamma^{-1})(x)\,dx.$$

Now $\int \lambda\gamma^{-1}\,dm = 0$ for $\lambda \neq \gamma$ (see e.g. [308, Lemma 23.19]), hence $\widetilde{\psi}(\lambda) = \delta_{\lambda\gamma}$. This shows that $\widetilde{\psi}$ and φ_γ agree on $\operatorname{lin}\Lambda$, so also on L^1_Λ (see Proposition 4.1). ☐

Theorem 4.10 *For a subset Λ of Γ the following are equivalent:*

(i) *Λ is a Shapiro set.*
(ii) *L^1_Λ is isometrically isomorphic to the dual of an M-embedded space.*
(ii′) *M_Λ is isometrically isomorphic to the dual of an M-embedded space.*
(iii) *$C/C_{\Gamma\setminus\Lambda^{-1}}$ is an M-embedded space.*
(iv) *$B_{L^1_\Lambda}$ is m-closed and $\varphi_\gamma|_{B_{L^1_\Lambda}}$ is m-continuous for all $\gamma \in \Lambda$.*

In (ii) *or* (ii′) *the M-embedded predual is isometrically isomorphic to $C/C_{\Gamma\setminus\Lambda^{-1}}$.*

PROOF: (ii) $\Leftrightarrow$ (ii′): If either L^1_Λ or M_Λ is the dual of an M-embedded space, it has the Radon-Nikodým property by Theorem III.3.1. Hence Λ is a Riesz set by Theorem 4.7, which means that $L^1_\Lambda = M_\Lambda$.

(i) $\Leftrightarrow$ (iv): This is Lemma 4.8.

(ii) $\Rightarrow$ (iv): By Theorem 3.10 the assumption is equivalent to

$$B_{L^1_\Lambda} \textit{ is } m\textit{-closed and } (L^1_\Lambda)^\sharp \textit{ separates } L^1_\Lambda.$$

In particular, for every $\gamma \in \Lambda$ there is a $\psi \in (L^1_\Lambda)^\sharp$ such that $\psi(\gamma) \neq 0$. By Lemma 4.9 this implies (iv). (Note that the M-embedded predual is isometrically isomorphic to $(L^1_\Lambda)^\sharp$ by Theorem 3.10.)

(iv) $\Rightarrow$ (ii): The second statement in (iv) says that $\varphi_\gamma \in (L^1_\Lambda)^\sharp$ for all $\gamma \in \Lambda$, consequently $(L^1_\Lambda)^\sharp$ separates L^1_Λ. Hence, by Theorem 3.10, L^1_Λ is isometrically isomorphic to the dual of the M-embedded space $(L^1_\Lambda)^\sharp$.

(iv) $\Rightarrow$ (iii): We have, as in "(ii) $\Leftrightarrow$ (ii′)", that $L^1_\Lambda = M_\Lambda$. This shows that the "natural" predual $C/C_{\Gamma\setminus\Lambda^{-1}}$ of M_Λ is a predual of L^1_Λ. On the other hand, we know from "(iv) $\Rightarrow$ (ii)" that $(L^1_\Lambda)^\sharp$ is the M-embedded predual of L^1_Λ.
To prove $C/C_{\Gamma\setminus\Lambda^{-1}} \cong (L^1_\Lambda)^\sharp$ (and hence the statement (iii)) we use the general fact: *If Y_1 and Y_2 are two isometric preduals of X with the embedded copy of Y_1 in X^* contained in the one of Y_2, then the embedded copies coincide, hence $Y_1 \cong Y_2$.* Now for $\gamma \in \Lambda$ the equivalence class $\gamma^{-1} + C_{\Gamma\setminus\Lambda^{-1}}$ acts as an element of $(C/C_{\Gamma\setminus\Lambda^{-1}})^{**} = (L^1_\Lambda)^*$ as $f \mapsto \int f\gamma^{-1}\,dm = \varphi_\gamma(f)$ $(f \in L^1_\Lambda)$. Since by assumption φ_γ is m-continuous on the unit ball of L^1_Λ we find (as embedded copies)

$$\Lambda^{-1} + C_{\Gamma\setminus\Lambda^{-1}} \subset (L^1_\Lambda)^\sharp.$$

The linear span of the left hand side is dense in $C/C_{\Gamma\setminus\Lambda^{-1}}$ (cf. Proposition 4.1), so the above remark finishes the proof.

(iii) $\Rightarrow$ (ii′): Clear by Proposition 4.1(e). □

Having completed the Banach space description of the classes of subsets Λ of Γ introduced at the beginning of this section, one could now try to use techniques and methods from harmonic analysis to construct and further investigate nicely placed and Shapiro sets and thus special classes of L- and M-embedded spaces. However, we refrain from doing this here because it would lead us too far away; we refer to the Notes and Remarks section where some of the results in this connection are collected.
So far there doesn't seem to be much influence in the other direction, i.e. applications of Banach space properties of L- and M-embedded spaces to questions in harmonic analysis. However, we point out that the Shapiro sets which are defined by Banach space properties are, by means of convergence in measure, much easier to handle than the Riesz sets and that almost all known examples of Riesz sets are actually Shapiro sets – cf. [257, §3].
We conclude this section with two examples. The first one gives yet another proof of Example III.1.4(h), which is immediate after the above preparation. The second one applies the results proved in this section to the solution of the RNP-problem for L-embedded spaces mentioned on p. 176.

Example 4.11 *$\mathbb{N}$ is a Shapiro subset of $\mathbb{Z}$, hence $C(\mathbb{T})/A$ is an M-embedded space.*

PROOF: In the proof of Example 3.6(a) we actually showed that every subset Λ of $\mathbb{N}$ is nicely placed. □

Example 4.12 *For $n \in \mathbb{N}_0$ put $D_n := \{l2^n \mid |l| \le 2^n\}$ and $\Lambda := \bigcup_{n\in\mathbb{N}_0} D_n$. Then Λ is a nicely placed Riesz set which is not Shapiro. Hence L^1_Λ is an L-embedded Banach space with the Radon-Nikodým property, which is a dual space but not the dual of an M-embedded space.*

PROOF: The statements about L^1_Λ follow from the properties of Λ by Definition 4.2 and Theorems 4.7 and 4.10. Note that by Proposition 4.1(e) L^1_Λ is a dual space if Λ is Riesz. We prepare the actual proof by the following: Let for $m \in \mathbb{N}_0$

$$P_m := 2^m + 2^{m+1}\mathbb{Z}.$$

Observing that for $n \neq 0$

$$n \in P_m \iff 2^m \text{ divides } n \text{ and } 2^{m+1} \text{ doesn't divide } n$$

we see that (P_m) is a partition of $\mathbb{Z} \setminus \{0\}$. Note that $n \in D_k$ implies that 2^k divides n, hence 2^m divides n for $m \le k$, consequently $n \notin P_m$ for $m < k$. This gives for $m < k$

$$n \in P_m \quad \Longrightarrow \quad n \notin D_k,$$

therefore

$$P_m \cap \Lambda \subset \bigcup_{k \le m} D_k,$$

so $P_m \cap \Lambda$ is finite for all $m \in \mathbb{N}_0$.
To prove that Λ is a Riesz set we have to show that $\mu = \mu_a + \mu_s \in M_\Lambda$ implies $\mu_s = 0$. The latter will be implied by $\widehat{\mu_s}(n) = 0$ for all $n \neq 0$, since the Fourier transform of a singular measure can't have finite support.
For $n \in \mathbb{Z} \setminus \{0\}$ we find $m \in \mathbb{N}_0$ such that $n \in P_m$. There is a discrete measure $\sigma \in M(\mathbb{T})$ with finite support such that $\widehat{\sigma} = \chi_{P_m}$ (cf. the proof of Proposition 4.4(e)). Since M_Λ and M_{P_m} are ideals, we get

$$\mu * \sigma \in M_{\Lambda \cap P_m}.$$

But $\Lambda \cap P_m$ is finite, consequently $\mu * \sigma$ is a trigonometric polynomial. In particular

$$(\mu * \sigma)_s = 0.$$

Now [441, Lemme 1]

$$(\mu * \sigma)_s = \mu_s * \sigma.$$

[Indeed, $(\eta * \delta_x)(E) = \int \eta(y^{-1}E)\, d\delta_x(y) = \eta(x^{-1}E)$ shows that $\eta \in M_s$ implies $\eta * \delta_x \in M_s$. Writing

$$\mu * \sigma = (\mu_a + \mu_s) * \sigma = \mu_a * \sigma + \mu_s * \sigma$$

we get $\mu_a * \sigma \in M_a = L^1$ (since L^1 is an ideal) and $\mu_s * \sigma \in M_s$. Hence the uniqueness of the Lebesgue decomposition gives the claim.]

Evaluating the Fourier transform of $(\mu * \sigma)_s$ at n we find

$$0 = \widehat{(\mu * \sigma)_s}(n) = \widehat{\mu_s * \sigma}(n) = \widehat{\mu_s}(n)\,\widehat{\sigma}(n) = \widehat{\mu_s}(n).$$

From this we deduce that $\mu_s = 0$, and Λ is a Riesz set.
To show that Λ is nicely placed, we have to prove by Theorem 3.5 that $B_{L^1_\Lambda}$ is m-closed. So let (f_k) be a sequence in the unit ball of L^1_Λ which converges in measure to some $g \in L^1$. We want to show that $\widehat{g}(n) = 0$ for $n \in \mathbb{Z} \setminus \Lambda$.
As above we find $m \in \mathbb{N}_0$ such that $n \in P_m$ and σ with $\widehat{\sigma} = \chi_{P_m}$. Since

$$(f * \delta_x)(y) = \int f(yz^{-1})\,d\delta_x(z) = f(yx^{-1}) = f_{x^{-1}}(y)$$

and σ is discrete with finite support, we conclude that

$$f_k * \sigma \;\longrightarrow\; g * \sigma$$

in measure. But $f_k * \sigma$ belongs to the finite-dimensional space $L^1_{\Lambda \cap P_m}$, so the convergence is also in norm. In particular

$$\lim_k \widehat{f_k * \sigma}(n) = \widehat{g * \sigma}(n) = \widehat{g}(n)$$

so that $\widehat{f_k * \sigma}(n) = \widehat{f_k}(n) = 0$ for all $k \in \mathbb{N}$ yields $\widehat{g}(n) = 0$.
Finally, to obtain that Λ is not a Shapiro set we will prove that φ_0 is not m-continuous on the unit ball of L^1_Λ (cf. Lemma 4.8).
It is easy to find $f_n \in L^1(\mathbb{T})$ such that

$$\|f_n\|_1 \le 2, \qquad \int f_n\,dm = \widehat{f_n}(0) = \varphi_0(f_n) = 0 \quad \text{and} \quad f_n \longrightarrow \chi_{\mathbb{T}} \;\; m\text{−a.e.}$$

Using the density of $T_{\mathbb{Z}\setminus\{0\}}$ in $L^1_{\mathbb{Z}\setminus\{0\}}$ (Proposition 4.1) we find an L^1-bounded sequence (t_n) in $T_{\mathbb{Z}\setminus\{0\}}$ with $t_n \stackrel{m}{\longrightarrow} \chi_{\mathbb{T}}$. Each t_n has the form

$$t_n(z) = \sum_{\substack{l=-m_n \\ l\neq 0}}^{m_n} a_{l,n} z^l.$$

So

$$t_n(z^{2^{m_n}}) = \sum a_{l,n} z^{l2^{m_n}} =: \widetilde{t_n}(z)$$

and $\widetilde{t_n} \in T_{D_{m_n}} \subset T_\Lambda$. Since z and $z^{2^{m_n}}$ have the same distribution it follows easily that $m\{|\chi_{\mathbb{T}} - t_n| \ge a\} = m\{|\chi_{\mathbb{T}} - \widetilde{t_n}| \ge a\}$ and $\|t_n\| = \|\widetilde{t_n}\|$. So $(\widetilde{t_n})$ is a bounded sequence in $T_\Lambda \subset L^1_\Lambda$ with $\widetilde{t_n} \stackrel{m}{\longrightarrow} \chi_{\mathbb{T}}$, but $\varphi_0(\widetilde{t_n}) = \int \widetilde{t_n}\,dm = 0$ and $\varphi_0(\chi_{\mathbb{T}}) = 1$. Therefore the restriction of φ_0 to the unit ball of L^1_Λ is not m-continuous. □

It is (at least implicitly) contained in the above proof that φ_n is m-continuous on the unit ball of L^1_Λ for all $n \in \Lambda \setminus \{0\}$ – so that all but one character are in $(L^1_\Lambda)^\sharp$. In this connection the following is open:

QUESTION. *Does there exist a nicely placed Riesz subset Λ of $\mathbb{Z}$ such that, for all $n \in \Lambda$, $\varphi_n|_{B_{L^1_\Lambda}}$ is* not *m-continuous?*

Let us indicate where the interest in this question comes from. We now know that for an L-embedded space X with X_s not w^*-closed it does not follow that X fails the RNP (cf. Remark 2.10). However it is unknown if w^*-density of X_s is sufficient for X to fail the RNP. It is not hard to show that a set Λ as in the above question would give a counterexample to that.
There is another offspring of the RNP-problem which remains open: Although the original question whether or not an L-embedded space with the RNP is the dual of an M-embedded space is settled in the negative by Example 4.12, the following is left unanswered.

QUESTION. *Is every L-embedded space with the RNP isometric to a dual space?*

Observe that for the L^1_Λ-spaces studied in this section the answer is *yes* by Theorem 4.7 – so harmonic analysis can't help here. This question is related to Talagrand's work on Banach lattices who proved in [593] that a separable Banach lattice with the RNP is a dual lattice. See [265, p. 260, Remark 6] for more comments on this.

IV.5 Notes and remarks

GENERAL REMARKS. Parallel to the study of spaces which are M-ideals in their biduals, some of the first results on L-embedded spaces were obtained in [291] and [292]. However the main contributions to this latter class were obtained by G. Godefroy in [256], [257] and [265], and by D. Li in [394] and [395]. The Example 1.1(b) dates back to [591, Th. 3], whereas the L-embeddedness of L^1/H^1_0 was first explicitly proved in [21, Th. 2]. The results on subspaces and quotients in Theorem 1.2 and Corollary 1.3 are from [394], where however the general invariance principle Lemma I.1.15 was not used. The special case where the bigger space is $L^1(\mu)$ was previously established in [256]. Lemma 1.4 is due to Pfitzner [492], the Propositions and Examples 1.5 to 1.10 are essentially from [291]. The fact that ℓ^1 is isometric to a 1-complemented subspace of $(\oplus \sum \ell^1(n))_{\ell^\infty}$ which was used in Example 1.7(b) was observed by W. B. Johnson [353, p. 303].
Let us have a closer look at the question whether the bidual of an L-embedded space is again L-embedded. Not even for "nice" examples, i.e. for L-embedded subspaces of L^1-spaces, is this true: The real Banach space $Y = (\oplus \sum \ell^1(n))_{c_0}$ is isometric to a subspace of c_0, so $Y^\perp$ is an L-embedded subspace of ℓ^1 by Proposition 1.10. However, $(Y^\perp)^{**} \cong Y^{\perp\perp\perp} \subset \ell^{1**}$ fails to be L-embedded, because otherwise $\ell^{1**}/Y^{\perp\perp\perp} \cong (\ell^1/Y^\perp)^{**} \cong (Y^*)^{**}$ would be. It is open for which finite dimensional spaces E_n the bidual of $(\oplus \sum E_n)_{\ell^1}$ is L-embedded and also whether the even duals of L^1/H^1_0 have this property. Note that by a result of Bourgain ([89] or [90, Cor. 5.4]) $(L^1/H^1_0)^{(2n)}$ is weakly sequentially complete.
A natural question which seems to have not yet been thoroughly investigated is the L-embeddedness of the Bochner space $L^1(X) := L^1([0,1], X)$. For some classes of L-embedded spaces this is known to be the case, e.g., if X is the predual of a von Neumann

algebra, then so is $L^1(X)$ (cf. [592, p. 263]), and if $X \subset L^1[0,1]$ is nicely placed, then so is $L^1(X) \subset L^1([0,1]^2)$. Less obvious is the case of reflexive X which are trivially L-embedded. In this case $L^1(X)$ is in fact L-embedded as can be deduced from results of Levin [392], who has presented a decomposition of $(L^\infty(Y))^*$ for arbitrary Banach spaces Y, as follows. Let M_1 denote the space of absolutely continuous Y^*-valued σ-additive vector measures and M_2 the space of absolutely continuous Y^*-valued purely finitely additive vector measures. Note that $M_1, M_2 \subset (L^\infty(Y))^*$ in a canonical fashion. Further, M_3 is to denote the annihilator of $\{\varphi \otimes y \mid \varphi \in L^\infty, y \in Y\}$. Then Levin's theorem asserts that

$$(L^\infty(Y))^* = M_1 \oplus_1 (M_2 \oplus M_3).$$

If Y is the dual of a reflexive space X, we obtain from the RNP of X and X^*

$$(L^1(X))^{**} = L^1(X) \oplus_1 (M_2 \oplus M_3),$$

which proves that $L^1(X)$ is L-embedded. As for the case of general L-embedded spaces X, we would like to mention Talagrand's result that $L^1(X)$ is weakly sequentially complete whenever X is [596]. Thus a positive answer to the problem of L-embeddedness of $L^1(X)$ would provide a proof of a special case of Talagrand's theorem. Going one step further, one might wonder about the projective tensor product of L-embedded spaces; recall that $L^1(X) = L^1 \widehat{\otimes}_\pi X$. Here even less is known. For example, it is not clear whether $L^p \widehat{\otimes}_\pi L^p$ is L-embedded for $1 < p < \infty$, $p \neq 2$, or can at least be so renormed. (The tensor product $\ell^p \widehat{\otimes}_\pi \ell^p$ is L-embedded since it is the dual space of $K(\ell^{p^*}, \ell^p)$, cf. Example III.1.4(g).) On the other hand, Pisier has shown in [499, Th. 4.1, Rem. 4.4] that $L^1/H^1 \widehat{\otimes}_\pi L^1/L^1_\Lambda$, for $\Lambda = \mathbb{N} \setminus \{3^n \mid n \geq 0\}$, contains a copy of c_0 and hence fails to be weakly sequentially complete. Thus this space cannot be renormed to be L-embedded; note that Λ is nicely placed because $\mathbb{N}$ is a Shapiro subset of $\mathbb{Z}$, hence L^1/L^1_Λ is L-embedded by Corollary 1.3. The remark following Example 1.7 was shown to us by T.S.S.R.K. Rao. It is unknown whether there is a commutative space as in Example 1.8, more precisely: does there exist a nonreflexive L-embedded subspace Y of an L^1-space such that Y^* has no nontrivial M-ideals? Lemma 1.11 to Proposition 1.13 are from [394] – we only have removed the unnecessary restriction in Proposition 7 of this article. The extreme point result in Proposition 1.14 is due to Deutsch [153, Th. 1]. The smoothness of H^1 (Remark 1.17) is a "folklore result". We haven't been able to locate its first appearance in the literature; the strict convexity of $C(\mathbb{T})/A$, however, appears e.g. in [332, Cor. 4], but the reasoning there is different.

In the Notes and Remarks to Chapter I we discussed several "semi" notions and observed that a Banach space which is a semi M-ideal in its bidual is necessarily M-embedded. It remained open for quite a while whether an analogous statement holds for L-embedded spaces. Let us say that a Banach space X is semi L-embedded if X is a semi L-summand in X^{**}, i.e., if there exists a (nonlinear) projection π from X^{**} onto X satisfying

$$\begin{aligned} \|x^{**}\| &= \|\pi(x^{**})\| + \|x^{**} - \pi(x^{**})\| \\ \pi(\lambda x^{**} + x) &= \lambda \pi(x^{**}) + x \end{aligned}$$

for $x \in X$, $x^{**} \in X^{**}$ and $\lambda \in \mathbb{K}$. Recently Payá and Rodríguez [479] succeeded in constructing semi L-embedded real Banach spaces which are not L-embedded. Their construction involves sets of constant width, which are defined as follows. A closed convex bounded subset S of a Banach space E is a set of constant width if $\operatorname{diam}(x^*(S)) = \operatorname{diam}(y^*(S))$ whenever $\|x^*\| = \|y^*\|$, $x^*, y^* \in E^*$. Let now E be M-embedded, $S \subset E^{**}$ be a weak* compact set of constant width and form the compact convex set $K = S^{\mathbb{N}}$. Then the space $A(K)$ of affine continuous functions is semi L-embedded, and if E^{**} enjoys the IP (Definition II.4.1) and S is not a ball, then $A(K)$ is not L-embedded. Let us mention that the proof of this result relies on several concepts studied in Chapter II, notably the IP and the notion of a pseudoball. Thus, the easiest way to produce a properly semi L-embedded space is to start with a triangle $S \subset \mathbb{R}^2$, or more generally a compact convex set $S \subset \mathbb{R}^n$ without a centre of symmetry, and to consider $A(S^{\mathbb{N}})$. By the way, in this case the resulting space $A(S^{\mathbb{N}})$ turns out to be a renorming of ℓ^1. For the time being it is not known whether a semi L-embedded space can be renormed to be L-embedded; note that semi L-embedded spaces are weakly sequentially complete and hence contain ℓ^1 unless they are reflexive [479]. In this connection a result due to Godefroy is worth mentioning, which says that every Banach space containing ℓ^1 can be renormed to be an L-summand in some subspace of its bidual [259]. Luckily there is no need to investigate more general norm decompositions (cf. the Notes and Remarks to Chapter I) in the case of the inclusion $X \subset X^{**}$, since results in [103] and [254] show that M- resp. semi L-embedded spaces are the only possibilities within a natural class of Banach spaces, namely where X is a so-called absolute subspace of X^{**}. The results of [479] are also surveyed in [525].

Godefroy has given several proofs of the weak sequential completeness of L-embedded spaces. The first one in [253] uses, among other things, admissible sets (a generalisation of weakly compact sets defined by a geometric condition in the bidual) and Baire's results on points of continuity of functions of the first Baire class. The second one in [256, Lemme 4] avoids all this and is completely elementary. The proof in [257, p. 308] uses upper envelopes and again properties of Baire-1 functions. The extreme discontinuity of elements of X_s (Proposition 2.1) was pointed out in [54], and this result was further simplified in [454, Lemma 6.1]. We used the set-valued modulus of continuity of this last article in order to obtain also the converse in Proposition 2.1. We don't state the technical problem one encounters if one uses the standard definition of oscillation in the complex case. However, we mention the following question which shows which kind of difficulty one faces: if X is a complex Banach space and $\operatorname{Re} x^{**}|_{B_{X^*}}$ is w^*-continuous at $x^* \in S_{X^*}$, is then also $x^{**}|_{B_{X^*}}$ w^*-continuous at x^*? The ace of diamonds lemma, Proposition 2.5, is an unpublished argument of Godefroy he suggested in a talk in Oldenburg. It has the following generalization [266, Th. II.1]: if (x_n) is a weak Cauchy sequence of fixed points of an isometric bijection $S : X^* \to X^*$, then the w^*-limit x^* of (x_n^*) is a fixed point too. The proof in [266] is very easy; however it doesn't yield the ℓ^1-subsequence one obtains with the argument in the text. A special case of [266, Th. II.1], namely that a space X admitting an isometric symmetry $S : X^{**} \to X^{**}$ with $X = \ker(Id - S)$ is weakly sequentially complete, had already been obtained in [253, Prop. 10]. Proposition 11 of this last paper (which gives the converse of the last-mentioned result for Banach lattices) and the connection with u-ideals (see below) are worth noticing. The simple local reflexivity argument, Remark 2.4, yielding ℓ^1-subspaces in nonreflexive L-embedded spaces, was first observed in [291]. Property (V^*) for L-embedded spaces, i.e. Lemma 2.6 and Theorem 2.7, is a recent achievement due to Pfitzner [492].

The RNP-problem for L-embedded spaces was studied in [394] and solved in [265]. Between the assumptions that X_s is not w^*-closed (which is not sufficient for X to fail the RNP) and that B_{X_s} is w^*-dense in $B_{X^{**}}$ (which suffices by Remark 2.10(a)) lies the condition that X_s has a positive characteristic in X^{**}. In [394, Th. 10] it is claimed that this last hypothesis already gives that X can't have the RNP. D. Li has asked us to mention that the proof contains a gap and it is open whether the result is true or not. The factorization property Proposition 2.12 appears in [395, Th. 1]. The special case of this last result for operators $T : L^1 \to L^1/H_0^1$ was proved in [171, Th. 2] by different methods – it is an important tool for the characterisation of the analytic Radon-Nikodým property. Besides the Corollaries 2.13 and 2.14 the article [395] contains many more applications of the factorization property.

We have already said in the text that the main result of the first part of Section IV.3, Theorem 3.4, is due to Buhvalov and Lozanovskii [99]. Lemma 3.1 and Corollary 3.2 have simply been extracted from their proof – see also [256, proof of Lemme 1]. The second proof of Corollary 3.2 is from [265, Lemma I.1]. Proposition 3.3 combines [256, Lemme 1] and [99, Lemma 1.1]. Note that this result fails for unbounded sets C, e.g. $L^1[0,1]$ is not μ-closed in $L^0[0,1]$. We remark also that for convex bounded subsets C of $L^1(\mu)$ a fourth equivalent condition in Proposition 3.3 is that C is closed in $L^1(\mu)$ with respect to convergence in $\|\,.\,\|_{1,w}$, where $\|f\|_{1,w} = \sup_{t>0} t \cdot \mu\{|f| > t\}$ denotes the weak-L^1 "norm". This was proved by Godefroy in [257, Lemma 1.7]. The characterisation of L-embedded subspaces of L^1-spaces in Theorem 3.5 is from [256, Lemme 1, Lemme 23]. In our development it is a simple consequence of Theorem 1.2; however this latter result wasn't available at the time of [256]. In [99, Th. 2.4′] it was already shown that subspaces of $L^1(\mu)$ with μ-closed unit balls are 1-complemented in their biduals.

We would now like to digress a bit and comment on the topology τ_μ of convergence in measure. We have said in the text that it is the vector space topology best suited to describe almost everywhere convergence. However, much deeper is the insight gained from the Buhvalov-Lozanovskii theorem: μ-closedness is a substitute for compactness. From the examples in [99] which substantiate this, we mention only the most striking one [99, Th. 1.3]: if (C_α) is a decreasing net of nonempty convex bounded sets in $L^1(\mu)$ which are μ-closed, then $\bigcap C_\alpha$ is nonempty. [For a proof simply use the w^*-compactness of $\overline{C_\alpha}^{w^*}$ in $L^1(\mu)^{**}$ and Theorem 3.4.] Note that the τ_λ-closed set $B_{L^1[0,1]}$ is not τ_λ-compact, since e.g. the sequence (r_n) of Rademacher functions has no τ_λ-convergent subsequence. On the other hand, the above weak form of compactness is strong enough to imply ([99, Th. 2.3]):

THEOREM. *A subspace Y of $L^1(\mu)$ is reflexive iff $\tau_\mu = \tau_{\|\,\|_1}$ on B_Y.*

[PROOF: If Y is reflexive and (f_n) is a sequence in B_Y τ_μ-convergent to $f \in B_Y$, we show that every subsequence (g_n) of (f_n) has a subsequence (h_n) which is $\|\,.\,\|_1$-convergent to f. In fact, by weak compactness (g_n) has a weakly convergent subsequence (h_n). Then $h_n \xrightarrow{w} f$ and [178, Th. IV.8.12] gives $h_n \xrightarrow{\|\,\|} f$. Conversely, let us establish Smulian's condition for reflexivity: each decreasing sequence (C_n) of nonempty, convex, closed, and bounded sets has nonempty intersection ([151, p. 58], nowadays one deduces this easily from James' theorem: for $y^* \in S_{Y^*}$ consider $C_n = \{y^* \geq 1 - 1/n\} \cap B_Y$). By assumption the C_n are μ-closed, so the conclusion obtains by what was observed above.]

We finish this intermission with some remarks on the case of infinite measures μ. First of all in this situation the "right" definition of convergence in measure is $f_n \xrightarrow{\mu} f$ iff (f_n) converges to f in measure on every set of finite measure. (This is suggested by the σ-finite case if one transforms μ to finite $\nu = h\mu$ with an integrable and strictly positive h.) But note that without further restrictions this convergence doesn't behave well, e.g. if there are atoms of infinite measure, the limit is not uniquely determined. The necessary assumptions and results can be found in [99]. We finally mention that in ℓ^1, τ_μ-convergence is pointwise convergence, so on bounded sets $\tau_\mu = \sigma(\ell^1, c_0)$ – another indication for the "compactness" of τ_μ-closed sets. In this way one finds examples of subspaces X of $L^1(\mu)$ such that B_X is μ-closed, but X is not: take a subspace X of ℓ^1 which is w^*-closed, but pointwise dense.
The less artificial Example 3.6 of this phenomenon is due to Godefroy, part (a) is [257, Ex. 3.1] and part (b) a personal communication from him. The paper [256] contains further examples of L-embedded subspaces of $L^1(\mu)$, for instance $H^1(U)$ for special subsets U of $\mathbb{C}^n$ [256, Th. 20] and certain abstract Hardy spaces $H^1(\mu)$ [256, Lemme 10]. Lemma 3.7 is from [256, Lemme 7]. A partial result pointing to a characterisation of duals of M-embedded spaces among L-embedded subspaces of $L^1(\mu)$ was [256, Th. 6]. The full version given in Proposition 3.9 and Theorem 3.10 was established in [265].
It was G. Godefroy who first applied concepts of M-structure theory to harmonic analysis in [256] and then extensively in [257]; see also the papers [265] and [266]. Definition 4.2, Lemma 4.3, Proposition 4.5, the equivalence of (i) and (iii) in Theorem 4.10, and essentially Lemma 4.8 come from [257], which contains many other results about which we shall say more in the next subsection. Theorem 4.7 is from [431] – if its proof is too complicated the blame is on us (we just wanted to be precise on the measure theoretical technicalities). Lemma 4.9, Theorem 4.10, and Example 4.12 were obtained in [265]. The set Λ in Example 4.12 is attributed to A. B. Aleksandrov in [257, p. 322]. For more information on Riesz sets one should consult the references listed in [257, p. 302] and the more recent articles [98], [170], [228] and [498].
Finally, we mention the note [77] where fixed points of nonlinear contractions on nicely placed subspaces of L^1 are investigated.

Techniques for constructing nicely placed and Shapiro sets. Although Riesz, Shapiro, and nicely placed sets are not really "thin" sets, it is – in view of the fact that subsets of Riesz and Shapiro sets are again of this type – justified to consider them as in some sense small. So it is most interesting to find big examples and to investigate to what extent known examples can be enlarged. Note, however, the different behaviour of nicely placed sets: the set Λ from Example 4.12 is nicely placed, yet $\Lambda \setminus \{0\}$ is not. Let us begin by noting the stability under products:

> Theorem. *If Λ and Λ' are nicely placed, resp. Shapiro or Riesz subsets of countable discrete groups Γ and Γ', then $\Lambda \times \Lambda'$ is a nicely placed, resp. Shapiro or Riesz subset of $\Gamma \times \Gamma'$.*

For Riesz sets this was proved in [432, Th. 1.2], for the other cases see [257, Th. 2.7]. The theorem implies in particular that the quotient of $C(\mathbb{T}^2)$ by the bidisk algebra is M-embedded.

Next we will describe two techniques for constructing nicely placed and Shapiro sets. The first one uses ordered groups and the second one the localisation argument of Meyer from [441]. Recall that a subset Π of Γ with

$$\Pi\,\Pi \subset \Pi, \quad \Pi \cap \Pi^{-1} = \{e\}, \quad \text{and} \quad \Pi \cup \Pi^{-1} = \Gamma$$

induces a (total) ordering on Γ by $\gamma \geq \lambda$ iff $\gamma\lambda^{-1} \in \Pi$. Γ admits such an ordering if and only if $G = \widehat{\Gamma}$ is connected [541, 2.5.6(c) and 8.1.2(a)]. The main reference for ordered groups (always in the above sense) is Chapter 8 in [541]; but see also Chapter VII in [239]. Further recall that a subset Λ of Γ is said to be a $\Lambda(1)$-*set*, if there are $0 < q < 1$ and $C > 0$ such that

$$\|t\|_1 \leq C\|t\|_q \qquad \forall\, t \in T_\Lambda,$$

i.e. if the "norms" $\|\,.\,\|_1$ and $\|\,.\,\|_q$ are equivalent on T_Λ. It is known that Λ is a $\Lambda(1)$-set if and only if L^1_Λ is reflexive (see [40] together with [529], or [290] for an elementary proof). We refer to [540], [185], and [424] for more information on $\Lambda(1)$-sets. Building on similar results to those in [540] and [564] Godefroy proved in [257, Th. 2.1]:

THEOREM A. *A subset Λ of an ordered group Γ such that*

$$\Lambda \cap \{\gamma' \in \Gamma \mid \gamma' \leq \gamma\} \ \text{ is a } \Lambda(1)\text{-set for all } \gamma \in \Gamma$$

is a Shapiro set.

Note as a trivial application (finite sets are $\Lambda(1)$-sets) that this provides another proof of the fact that $\mathbb{N}$ is a Shapiro set, equivalently that $C(\mathbb{T})/A$ is an M-embedded space. We also remark (or recall) that proving that a subset Λ is a Shapiro set includes the proof of, or uses, an F. and M. Riesz theorem. In the demonstration of the above theorem G. Godefroy applies at this point the ideas of Helson and Lowdenslager to come to a function algebra setting where the abstract F. and M. Riesz theorem in the form of p. 109 can be used.

To exhibit at least a part of the argument leading to Theorem A let us prove that the positive cone Π of Γ is nicely placed. We use Lemma I.1.15 as in the first part of Example III.1.4(h) and show that $Q(L^1_\Pi)^{\perp\perp} \subset (L^1_\Pi)^{\perp\perp}$, where P is the L-projection from $L^1(G)^{**}$ onto $L^1(G)$. Write $L^1(G)^* = L^\infty(G) = C(K)$ with $K = \mathsf{M}_{L^\infty(G)}$ and carry m to a measure $\widetilde{m}$ on K by means of $\int_G f\,dm = \int_K \widetilde{f}\,d\widetilde{m}$, $f \in L^\infty(G)$, where $f \mapsto \widetilde{f}$ of $L^\infty(G)$ denotes the Gelfand transform. After the identifications $L^1(G) = L^1(\widetilde{m})$ and $L^1(G)^{**} = M(K)$, the L-projection P is simply the Radon-Nikodým projection $P\mu = \mu_a$ where μ_a is the $\widetilde{m}$-absolutely continuous part of $\mu \in M(K)$. From the properties of Π it follows easily that $A := C_\Pi(G)$ is a function algebra in $C(G)$, that $\varphi : A \to \mathbb{C}$, $\varphi(f) = \int_G f\,dm$ is multiplicative, and that m is the unique representing measure for φ. By [95, Th. 4.1.2, p. 212] $H^\infty(m) := \overline{A}^{w*} \subset L^\infty(G)$ is a logmodular algebra in $C(K) = L^\infty(G)$. In particular $H^\infty(m)$ separates K, and multiplicative functionals on $H^\infty(m)$ have unique representing measures. We use this for the canonical extension ϕ of φ to $H^\infty(m)$ and its representing measure $\widetilde{m}$. Hence the abstract F. and M. Riesz theorem is applicable and yields $PH^\infty_0(m)^\perp \subset H^\infty_0(m)^\perp$, where $H^\infty_0(m) = \ker\phi = \overline{C_{\Pi\setminus\{e\}}}^{w*}$.

Since by definition and Proposition IV.4.1 $L^1_\Pi = \overline{A}^{\|\ \|_1}$ and $(L^1_\Pi)^\perp = H^\infty_0(m)$, we obtain the desired invariance $P(L^1_\Pi)^{\perp\perp} \subset (L^1_\Pi)^{\perp\perp}$.

We will now describe in more detail a second technique which actually achieves a local characterisation using the Bohr compactification of Γ. Let us recall the relevant facts: if we write G_d for $G = \widehat{\Gamma}$ equipped with the discrete topology, then $b\Gamma := \widehat{G_d}$ is a compact group which contains Γ as a dense subgroup [541, 1.8]. We will write τ for the topology induced on Γ by $b\Gamma$. A τ-neighbourhood of $\gamma_0 \in \Gamma$ is given by $U(\gamma_0, K, \varepsilon) = \{\gamma \in \Gamma \mid |\gamma(x) - \gamma_0(x)| < \varepsilon,\ \ \forall x \in K\}$ with $\varepsilon > 0$ and $K \subset G_d$ compact, i.e. finite. Hence τ is the topology of pointwise convergence on G. We write $\mathfrak{U}_\tau(\gamma_0)$ for the set of τ-neighbourhoods of γ_0. We will say a family $\mathcal{C}$ of subsets of Γ is *localizable* if for $\Lambda \subset \Gamma$

$$\Lambda \in \mathcal{C} \quad \Longleftrightarrow \quad \forall \gamma \in \Gamma\ \exists U \in \mathfrak{U}_\tau(\gamma)\ \ U \cap \Lambda \in \mathcal{C}$$

It is easy to check that if $\mathcal{C}$ is localizable then its hereditary family $\mathcal{C}_{\text{her}} := \{\Lambda \subset \Gamma \mid \Lambda' \in \mathcal{C},\ \ \forall \Lambda' \subset \Lambda\}$ is localizable. Note that the families of Riesz and Shapiro sets are just the hereditary families of the Haar invariant and the nicely placed sets (see Lemma 4.3).

Theorem B. ([441, Th. 1], [257, Th. 2.3]) *The families of Haar invariant, nicely placed, Riesz, and Shapiro sets are localizable. We even have, for* $\Lambda \subset \Gamma$*, that the condition*

$$\forall \gamma \notin \Lambda\ \ \exists U \in \mathfrak{U}_\tau(\gamma)\ \ \exists E \quad \gamma \notin E,\, U \cap \Lambda \subset E,\, E \text{ nicely placed} \qquad (*)$$

is sufficient for Λ *to be nicely placed.*

Proof: The Riesz and the Shapiro parts follow from what was remarked above if we prove the assertion for the other two classes. In fact, for Haar invariant sets we will prove a stronger statement, namely: $\Lambda \subset \Gamma$ is Haar invariant if for all $\gamma \notin \Lambda$ there exists $U \in \mathfrak{U}_\tau(\gamma)$ with $U \cap \Lambda$ Haar invariant. We have to show that $\mu \in M_\Lambda$ implies $\mu_s \in M_\Lambda$, i.e. $\widehat{\mu_s}(\gamma) = 0$ for $\gamma \notin \Lambda$. Choose $U \in \mathfrak{U}_\tau(\gamma)$ as in the assumption and take $\widetilde{U} \in \mathfrak{U}_{b\Gamma}(\gamma)$ with $\widetilde{U} \cap \Gamma = U$. Using the regularity of the algebra $A(b\Gamma) = \{\widehat{f} \mid f \in L^1(G_d)\}$ we find some $f \in L^1(G_d)$ such that $\widehat{f}(\gamma) = 1$ and $\widehat{f}|_{b\Gamma \setminus \widetilde{U}} = 0$ [541, Th. 2.6.2]. Since $L^1(G_d) = \ell^1(G)$ we can view f as a discrete measure $\nu \in M(G)$ and we get

$$\widehat{\nu}(\gamma) = 1 \text{ and } \widehat{\nu}|_{\Gamma \setminus U} = 0.$$

Then $\mu * \nu \in M_{U \cap \Lambda}$ and

$$(\mu * \nu)_s = \mu_s * \nu.$$

[Proof: Because $(\eta * \delta_x)(E) = \int \eta(y^{-1}E)\ d\delta_x(y) = \eta(x^{-1}E)$ we have that $\eta \in M_s$ implies that $\eta * \delta_x \in M_s$. Since M_s is a closed subspace of M and every discrete measure is a limit of measures of finite support we conclude from the above: $\eta \in M_s \Rightarrow \eta * \nu \in M_s$. Writing $\mu * \nu = (\mu_a + \mu_s) * \nu = \mu_a * \nu + \mu_s * \nu$, we get $\mu_a * \nu \in M_a = L^1$ and $\mu_s * \nu \in M_s$. The uniqueness of the Lebesgue decomposition now gives the claim.]

Using that $U \cap \Lambda$ is Haar invariant for $\mu * \nu$ we get $0 = \widehat{(\mu * \nu)}_s(\gamma) = \widehat{\mu_s * \nu}(\gamma) = \widehat{\mu_s}(\gamma)\widehat{\nu}(\gamma) = \widehat{\mu_s}(\gamma)$. This is what we wanted to show.

We prepare the proof of the case of nicely placed sets by showing the following

CLAIM: *For $f_n \in B_{L^1}$, $f \in L^1$, and a discrete measure ν the convergence $f_n \xrightarrow{m} f$ implies $f_n * \nu \xrightarrow{m} f * \nu$.*

[PROOF: It is clearly enough to show that, for a bounded sequence (f_n) in L^1, $f_n \xrightarrow{m} 0$ implies $f_n * \nu \xrightarrow{m} 0$. Assume $\|f_n\| \le 1$ for all $n \in \mathbb{N}$. Observing $g * \delta_x = g_{x^{-1}}$, hence $\|g\|_m = \|g * \delta_x\|_m$, we get $f_n * \nu_0 \xrightarrow{m} 0$ for all measures ν_0 of finite support. Let $\varepsilon > 0$. Choose a finitely supported measure ν_0 with $\|\nu - \nu_0\| \le \varepsilon^2$. Then

$$m\{|f_n * (\nu - \nu_0)| \ge \varepsilon\} \le \frac{1}{\varepsilon}\|f_n * (\nu - \nu_0)\| \le \frac{1}{\varepsilon}\|f_n\|\|\nu - \nu_0\| \le \varepsilon$$

hence $\|f_n * (\nu - \nu_0)\|_m \le \varepsilon$. Pick n_0 such that $\|f_n * \nu_0\|_m \le \varepsilon$ for $n \ge n_0$. Then

$$\|f_n * \nu\|_m \le \|f_n * \nu_0\|_m + \|f_n * (\nu - \nu_0)\|_m \quad \text{for} \quad n \ge n_0,$$

i.e. $f_n * \nu \xrightarrow{m} 0$.]
To prove the theorem for nicely placed sets assume $\Lambda \subset \Gamma$ is given as in assumption $(*)$ of the theorem. We have to prove that $f_n \in B_{L^1_\Lambda}$, $f \in L^1$, $f_n \xrightarrow{m} f$ imply $f \in L^1_\Lambda$. For $\gamma \notin \Lambda$ choose U and E according to the assumption and a discrete measure ν as in proof of the first part. By the above claim we get $f_n * \nu \xrightarrow{m} f * \nu$. We have $f_n * \nu \in L^1_{U \cap \Lambda} \subset L^1_E$, and because E is nicely placed we deduce that $f * \nu \in L^1_E$. Finally $\gamma \notin E$ yields $0 = \widehat{f * \nu}(\gamma) = \widehat{f}(\gamma)\widehat{\nu}(\gamma) = \widehat{f}(\gamma)$. ◻

Before we come to concrete applications let us note the following general consequence.

COROLLARY.

(a) *Let Λ_1 be a nicely placed subset of Γ and Λ_2 a τ-closed subset of Γ. Then $\Lambda_1 \cup \Lambda_2$ is nicely placed. In particular, every τ-closed subset is nicely placed.*

(b) *Let Λ_1 be a τ-closed Riesz subset of Γ and Λ_2 a Shapiro subset of Γ. Then every set Λ such that $\Lambda_1 \subset \Lambda \subset \Lambda_1 \cup \Lambda_2$ is Riesz and nicely placed.*

[PROOF: (a) By Theorem B it is enough to show that for $\gamma \notin \Lambda_1 \cup \Lambda_2$ there is $U \in \mathfrak{U}_\tau(\gamma)$ such that $U \cap (\Lambda_1 \cup \Lambda_2)$ is nicely placed. Clearly $U := (\Gamma \setminus \Lambda_2) \cup \Lambda_1$ has these properties.
(b) We can write $\Lambda = \Lambda_1 \cup \Lambda'$ for some $\Lambda' \subset \Lambda_2$, which by assumption is nicely placed. Hence Λ is nicely placed by part (a). Also, Λ' is a Riesz set by Proposition 4.5(b); [441, Th. 2] shows that Λ is Riesz.]

The following examples come from [257].

EXAMPLES.

(a) Let P be the set of prime numbers including 1. Then $\Lambda = \{-1\} \cup P$ is τ-closed and $-\mathbb{N} \cup P$ is Riesz and nicely placed.

(b) $Q = \{n^2 \mid n \in \mathbb{Z}\}$ is τ-closed and $-\mathbb{N} \cup Q$ is Riesz and nicely placed.

(c) If $A \subset \mathbb{Z}^2$ is such that (1) $\{n \in \mathbb{Z} \mid (n \times \mathbb{Z}) \cap A \neq \emptyset\}$ is bounded from below and (2) $(n \times \mathbb{Z}) \cap A$ is bounded from below or from above for all $n \in \mathbb{Z}$, then A is a Shapiro set.

(d) If B is the set of all $(n,m) \in \mathbb{Z}^2$ which are contained in a plane sector whose opening is less than π, then B is a Shapiro set.

We remark that the set P of prime numbers is not a $\Lambda(1)$-set [446], so part (a) can't be deduced from Theorem A. Bochner proved in 1944 that the sets B in part (d) are Riesz sets (see also [239, Th. VII.3.1] or [541, Th. 8.2.5]). That sets A as in (c) are Riesz sets was shown by Aleksandrov in [5].

PROOF: Observe first that any arithmetical progression $n+k\mathbb{Z}$ is τ-clopen. [For $z = e^{i\frac{2\pi}{k}}$ and $0 < \varepsilon < |z-1|$ we obtain that $U(0,\{z\},\varepsilon) = \{m \in \mathbb{Z} \mid |z^m - 1| < \varepsilon\} = k\mathbb{Z}$ is τ-open. An open subgroup of a topological group is also closed, and the map $m \mapsto m+n$ is a homeomorphism.] For $n \notin \Lambda$ and $|n|$ not prime we get $\Lambda \cap n\mathbb{Z} = \emptyset$. If $|n|$ is prime for $n \notin \Lambda$ then $n\mathbb{Z} \setminus \{-n\}$ is an open neighbourhood not meeting Λ. Hence $\mathbb{Z} \setminus \Lambda$ is τ-open. For the set Q of square numbers one has $(n + 3n^2\mathbb{Z}) \cap Q = \emptyset$ if $n < 0$, and $(n + p^{2k+1}\mathbb{Z}) \cap Q = \emptyset$ if $n \geq 0$ and $n \notin Q$ (with p prime and $k \geq 0$ such that $n = p^{2k+1}n'$ and $p \nmid n'$). We leave the elementary number theory to the reader.
As in the one-dimensional case one shows that $k\mathbb{Z} \times \mathbb{Z}$ is τ-closed. Thus $\{0\} \times \mathbb{Z}$ and $F \times \mathbb{Z}$ are τ-closed for $F \subset \mathbb{Z}$ finite. Since the assumptions (1) and (2) on the set A are hereditary, it suffices to prove that sets A as in (c) are nicely placed. We will show the weak localization property $(*)$ of Theorem B. Let $\gamma = (n,m) \notin A$ and assume that $(n \times \mathbb{Z}) \cap A$ is bounded below. Then $\{(n,l) \in A \mid l < m\}$ and $F = \{k \in \mathbb{Z} \mid k < n,\ (k \times \mathbb{Z}) \cap A \neq \emptyset\}$ are finite. So $(F \times \mathbb{Z}) \cup \{(n,l) \in A \mid l < m\}$ is τ-closed and its complement U is a τ-neighbourhood of (n,m). We have $U \cap A \subset E := \{(k,l) \mid (k,l) \geq (n,m+1)\} = (n,m+1) + \mathbb{Z}^2_+$ for the lexicographical order on $\mathbb{Z}^2$. By Proposition 4.4(c) and the fact, proved after Theorem A, that $\mathbb{Z}^2_+$ is nicely placed we deduce that E is nicely placed – this is what we wanted to show. Part (d) is most easily shown using Theorem A. We omit the details and refer to [257]. □

In [228] C. Finet generalized not only the results of Section IV.4, but also the techniques for constructing nicely placed and Shapiro sets described above, to the setting of compact groups and compact commutative hypergroups. The latter are compact spaces which have enough structure so that an abstract convolution on $M(K)$ can be defined. An extension to transformation groups was established in [229] and [230].

WEAK* CONTINUITY PROPERTIES OF FUNCTIONALS ON THE DUAL UNIT BALL.
In the first half of this subsection we consider the continuity of individual elements of X^{**} as functions on (B_{X^*}, w^*) in order to put Lemma III.2.14 and Proposition IV.2.1 in a general perspective. The second half relates properties of X with certain subsets of X^{**} defined in terms of w^*-continuity properties. This will give us an alternative approach to property (u) for M-embedded spaces, Theorem III.3.8.
An element $x^{**} \in X^{**}$ has a point of w^*-continuity iff x^{**} belongs to X. It is clearly a different story to ask whether $x^{**}|_K$ has a point of w^*-continuity for a subset $K \subset X^*$. In this case one has to intersect w^*-neighbourhoods in X^* with K, and geometric properties of K may enter to help establishing points of w^*-continuity relative to K. We exclusively consider $K = B_{X^*}$, and all topological concepts refer to the relative w^*-topology on K. By the Krein-Smulian theorem, $x^{**}|_{B_{X^*}}$ has a point of w^*-continuity in the open unit ball iff x^{**} belongs to X. Hence, apart from the trivial case, w^*-continuity can only

occur at points $x^* \in S_{X^*}$. A simple but important sufficient condition for this to happen is that the w^*- and the w-topology agree at x^* (which clearly is the case if and only if *all* $x^{**} \in X^{**}$ are w^*-continuous at x^*). A geometric characterisation of these $x^* \in S_{X^*}$ was given in Lemma III.2.14. That $w^* = w$ at x^* is certainly implied by the stronger condition that $w^* = \|\,.\,\|$ at x^*, which in turn is the case if $x^* \in w^*\text{-sexp}\, B_{X^*}$. Note, however, that $x^* = (0,1) \in \ell^1 \oplus_1 \mathbb{K} = C(\alpha\mathbb{N})^* = c^*$ is strongly exposed in the unit ball, yet all $x^{**} \in c^{**} \setminus c$ are w^*-discontinuous at x^*. In particular, there is no relation between extreme points of B_{X^*} and points of w^*-w-continuity. Nevertheless, by the above observation we have an important isomorphic condition which guarantees many points of w^*-continuity: if X is an Asplund space then $\overline{\text{co}}^{w^*}\mathcal{C}_{w^*,w} = B_{X^*}$, where $\mathcal{C}_{w^*,w}$ denotes the set of points of continuity of $id : (B_{X^*}, w^*) \to (B_{X^*}, w)$. In [251] Godefroy showed that also the assumption that X^{**} is smooth or that the norm of X is Fréchet differentiable on a dense set imply $\overline{\text{co}}^{w^*}\mathcal{C}_{w^*,w} = B_{X^*}$. By Corollary III.2.15 $\mathcal{C}_{w^*,w} = S_{X^*}$ holds for M-embedded spaces X; this can be understood as a maximal continuity relation. In the proof of Corollary III.2.16 we remarked that an $x^* \in \mathcal{C}_{w^*,w}$ belongs to every norming subspace V of X^*. This is easily seen to give that if $\overline{\text{co}}^{w^*}\mathcal{C}_{w^*,w} = B_{X^*}$, then the norm closed linear span of $\mathcal{C}_{w^*,w}$ is the smallest norming subspace of X^*. Hence the existence of a smallest norming subspace of X^*, called property (P) in [251], is necessary for the existence of "sufficiently many" points of w^*-continuity. For the relation of property (P) with strongly unique preduals we refer to [251] and [258, Section I]. A necessary geometric condition for the existence of many w^*-continuity points of a single $x^{**} \in X^{**}$ is provided by [253, Lemme 1.2] and [258, Lemma I.4]: if the points of w^*-continuity of x^{**} separate X then $\bigcap_{x\in X} B_{X^{**}}(x, \|x - x^{**}\|)$ contains at most one point. Of course points of w^*-continuity of x^{**} depend on the norm, yet the following renorming result is quite remarkable [259, Prop. II.1]:

For a separable Banach space X the following assertions hold:

(a) *X^* is separable iff there is an equivalent norm $|\,.\,|$ on X such that all $x^{**} \in X^{**}$ are w^*-continuous at all $x^* \in S_{(X^*,|\,.\,|)}$.*
(b) *X does not contain a copy of ℓ^1 iff for all $x^{**} \in X^{**}$ there is an equivalent norm $|\,.\,|$ on X such that x^{**} is w^*-continuous at all $x^* \in S_{(X^*,|\,.\,|)}$.*

Let us now turn to the points of w^*-discontinuity. Behrends proved in [54] that if $X = A(K)$, the space of affine continuous functions on a compact convex set K, or if X is the range of a contractive projection in X^{**}, then every $x^{**} \in X^{**} \setminus X$ has a point of w^*-discontinuity in S_{X^*}. He introduced the space $\text{cont}(X) = \{x^{**} \in X^{**} \mid x^{**}$ is w^*-continuous at every $x^* \in S_{X^*}\}$, which is a closed subspace of X^{**} containing X, and showed that the centers y_B of pseudoballs B in X (cf. Th. II.1.6) belong to $\text{cont}(X)$, hence spaces with $\text{cont}(X) = X$ don't contain proper pseudoballs and thus can't be proper M-ideals (Th. II.3.10). The equivalence, established in Proposition IV.2.1, between extreme discontinuity of $x^{**} \in X^{**}$ and x^{**} being ℓ^1-orthogonal to X, deserves special comments. Maurey showed in [435] that a separable real Banach space X contains a copy of ℓ^1 iff there is an $x^{**} \in X^{**}\setminus\{0\}$ satisfying $\|x^{**}+x\| = \|x^{**}-x\|$ for all $x \in X$. This was largely simplified and extended to the nonseparable case by Godefroy in the following form: X contains a copy of ℓ^1 iff there is an equivalent norm $|\,.\,|$ on X and an $x^{**} \in X^{**}\setminus\{0\}$ such

that $|x^{**} + x| = |x^{**}| + |x|$ for all $x \in X$, i.e., such that X becomes an L-summand in some subspace of X^{**}, [259, Th. II.4]. Incidentally, these results show that one has to use more than one ℓ^1-orthogonal direction in X^{**} to establish property (V^*) for L-embedded spaces. Note that the Maurey-Godefroy theorem provides an isomorphic characterisation of the existence of ℓ^1-orthogonal directions; and there is even a local one for separable X [261, Lemma 9.1]: there is an $x^{**} \in X^{**} \setminus \{0\}$ such that $\|x^{**} + x\| = \|x^{**}\| + \|x\|$ for all $x \in X$ iff for all finite dimensional subspaces E of X and all $\varepsilon > 0$ there is an $x_{E,\varepsilon} \in X$ with $\|x_{E,\varepsilon}\| = 1$ such that $\|x + x_{E,\varepsilon}\| \ge (1-\varepsilon)(\|x\| + \|x_{E,\varepsilon}\|)$ for all $x \in E$. Norms satisfying this last condition were called octahedral in [259]. The proof of the quoted result in [261] is quite indirect and uses the so-called ball topology of X. All the above brings up the question: which $x^{**} \in X^{**}$ can be turned into ℓ^1-orthogonal directions by renorming? By Proposition IV.2.1 such an x^{**} is necessarily not of the first Baire class. In [259, p. 12] Godefroy remarked that if $x^{**} \in X^{**} \setminus X$ is such that there exists $Y \subset X$ with $Y \simeq \ell^1$ and $x^{**} \in Y^{\perp\perp}$, then there is an equivalent norm $|\,.\,|$ on X such that x^{**} is ℓ^1-orthogonal to X with respect to this new norm. However, not every ℓ^1-orthogonal element x^{**} is in the w^*-closure of a subspace Y isomorphic to ℓ^1, cf. [259, p. 12] and [626]. E. Werner obtained the following partial answer to the above question [626, Th. 2]: $x^{**} \in X^{**} \setminus \{0\}$ does not satisfy property (4) below iff for all $0 < \alpha < 1$ there is an equivalent norm $|\,.\,|$ on X such that $|x^{**} + x| \ge |x^{**}| + \alpha|x|$ for all $x \in X$.
In the following we want to relate some properties of a Banach space X with certain subsets $S \subset X^{**}$ defined by the continuity behaviour of their elements. The most important subset in this respect is the set functions of the first Baire class. To clarify this concept consider the following properties for a topological space K and $f : K \to \mathbb{R}$:

(1) f is the pointwise limit of a sequence of continuous functions on K.
(2) $f^{-1}(F)$ is a G_δ-subset of K for every closed set F.
(3) For all closed subsets $A \subset K$ the restriction $f_{|A}$ is continuous on a dense subset of A.
(4) For all closed subsets $A \subset K$ the restriction $f_{|A}$ is continuous at some point in A.

Then trivially (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4), and by the results of Baire (3) $\Rightarrow$ (2) $\Rightarrow$ (1) if K is metrizable, and (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) if K is completely metrizable ([384, §31] and [294] – see also [92, Prop. 1E] and [428, p. 54f.]). Since we are mainly interested in $K = (B_{X^*}, w^*)$, let us record what is known for compact spaces: (4) $\Rightarrow$ (3) follows by an easy Baire category argument (cf. [443, Th. 3.1]), and (2) $\Rightarrow$ (3) is contained in [92, Lemma 1.C and Cor. 1.D]. That (3) $\not\Rightarrow$ (2) can be seen with $K = (B_{\ell^1(\Gamma)}, w^*)$, Γ uncountable and $f(x) = \sum x_\gamma$, where $x = (x_\gamma)$. (The RNP of $\ell^1(\Gamma)$ helps to show (4) for f.) The proof of (2) $\Rightarrow$ (1) in the metric case uses that closed sets are G_δ-sets, and for quite a while Baire's results were only extended to the setting of functions $f : X \to Y$ with metric X and suitable Y (cf. [287], [527]). Only recently Hansell proved the implication (2) $\Rightarrow$ (1) for normal K and real-valued f [288].
Today it seems to be a common convention – which we will follow – to call functions with property (1) functions of the first Baire class and those with property (2) functions of the first Borel class: [287], [288], [342], [527]. Note, however, that the important article [92] uses a different notation – also the class $B_r^{\S}(K)$ from [92] is in general not the set

of functions satisfying (3). The name *barely continuous* function introduced in [443] for those f fulfilling (4) was not accepted in the literature. For applications in Banach space theory the first Baire class functions seem to be most appropriate – see below. We define for a Banach space X

$$B_1(X) := \left\{ x^{**} \in X^{**} \;\middle|\; \text{there is a sequence } (x_n) \text{ in } X \text{ such that } x_n \overset{w*}{\longrightarrow} x^{**} \right\}$$

$$LWUC(X) := \left\{ x^{**} \in X^{**} \;\middle|\; \text{there is a wuC-series } \sum x_n \text{ in } X \text{ such that } x^{**} = \sum\nolimits^* x_n \right\}$$

and for a topological space K

$$\textit{Baire-1}(K) := \left\{ F : K \longrightarrow \mathbb{R} \;\middle|\; \text{there is a sequence } (f_n) \text{ in } C(K) \text{ which converges pointwise to } F \right\}$$

$$DBSC(K) := \left\{ F : K \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \text{there is a sequence } (f_n) \text{ in } C(K) \text{ which} \\ \text{converges pointwise to } F \text{ and there is} \\ C > 0 \text{ such that, putting } f_0 = 0, \\ \sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| \le C \text{ for all } k \in K \end{array} \right\}.$$

$LWUC(X)$ stands for "limits of weakly unconditionally Cauchy series" and $DBSC(K)$ for "differences of bounded semicontinuous functions". A justification has to be given for our definition of $DBSC$: of course one expects $DBSC$ to be $\{f_1 - f_2 \mid f_1, f_2$ semicontinuous and bounded on $K\}$ and this agrees indeed with the above if K is metrizable (see [87, Chap. IX, § 2, no. 7], [295, p. 3] or [374, p. 220]). We have introduced the above definitions in order to be able to formulate the following theorem succinctly.

THEOREM. *For a Banach space X the following holds:*

(a) $B_1(X) = X^{**} \cap \textit{Baire-1}(B_{X^*})$,
(b) $LWUC(X) = X^{**} \cap DBSC(B_{X^*})$.

[PROOF: Part (a) is well-known ([457, Lemma 1], [157, Basic Lemma, p. 235]) and part (b) is only a disguised form of Pełczyński's result that property (u) is inherited by subspaces ([484]; see also [422, Prop. 1.e.3] and Lemma I.2.9). To see the interesting inclusion "$\supset$" in (b) let (f_n) be a sequence in $C(B_{X^*})$ as in the definition of $DBSC(B_{X^*})$ which converges pointwise to $x^{**}|_{B_{X^*}}$. By part (a) there is a sequence (x_n) in X with $x_n \overset{w*}{\longrightarrow} x^{**}$. Putting $g_n := f_n - f_{n-1}$, we obtain that the sequence $(\sum_{k=1}^n g_k)_n$ satisfies $\sum_{n=1}^\infty |g_n(x^*)| \le C$ for all $x^* \in B_{X^*}$ and converges pointwise to $x^{**}|_{B_{X^*}}$. So we are in the situation of Lemma I.2.9 with $E = C(B_{X^*})$, $F = X$, and $x_n = g_n$.]
Clearly

$$X \subset LWUC(X) \subset B_1(X) \subset X^{**}.$$

We are now in a position to formulate properties of X in terms of these spaces:

- $X = B_1(X) \iff X$ is weakly sequentially complete

- For separable spaces X: $B_1(X) = X^{**} \iff \ell^1 \not\hookrightarrow X$
- $X = LWUC(X) \iff c_0 \not\hookrightarrow X$
- $LWUC(X) = B_1(X) \iff X$ has property (u)

[The second statement is the Odell-Rosenthal theorem (see e.g. [421, Th. 2.e.7]), the third is a result of Bessaga and Pełczyński (see e.g. [157, Th. 8, p. 45]).] The last statement is only is trivial reformulation of property (u). To obtain a more serious characterisation we can use the equivalent description of *DBSC* for metric spaces mentioned above and recall that property (u) is separably determined.

> Proposition. *A Banach space X has property (u) if and only if for every separable subspace Y of X and every $y^{**} \in Y^{**}$ the following implication holds:*
>
> *If $y^{**}|_{B_{Y^*}}$ is Baire-1 then it is the difference of two bounded lower semicontinuous functions.*

This together with Lemma I.2.8 and Lemma I.2.5 shows again that M-embedded spaces have property (u). It even gives $LWUC(X) = X^{**}$ for a separable M-embedded space X.
If one tries to pursue the idea of relating properties of a space X to subset of X^{**} further, it seems most promising to try to relate the existence of particular discontinuous functions in $B_1(X)$ to the existence of special subspaces of X. This was done by Haydon, Odell, and Rosenthal in [295]; also Rosenthal's recent paper [530] is relevant in this connection. A thorough study of subclasses of Baire-1 functions from a topological point of view can be found in [374]. We remark that it doesn't seem very appropriate to consider bigger sets than $B_1(X)$, e.g. $B_2(X) = \{x^{**} \in X^{**} \mid \text{there is a sequence } (x_n^{**}) \text{ in } B_1(X) \text{ such that } x_n^{**} \xrightarrow{w*} x^{**}\}$ is in general not norm closed in X^{**}, whereas $B_1(X)$ always is [436], [437]. Also $B_2(X) \neq X^{**} \cap \mathit{Baire\text{-}2}(X)$, cf. [175, p. 79]. However, there are important relations between properties of X and measure theoretic features of (the elements of) X^{**}, notably Haydon's nonseparable extension of the Odell-Rosenthal theorem using universally measurable functions. We refer to [181], [182], and [595] for these questions. In this regard [175] should also be mentioned, although this memoir focuses on the containment of ℓ^1.

u-ideals. Besides the more general norm decompositions extending M-ideals and L-summands which we discussed in the Notes and Remarks to Chapter I, there is a recent development which focuses only on the symmetry of the splitting $Y = X \oplus Z$. This concept, which leads to the notions of u-summands and u-ideals, was first introduced in [114] and later worked out in detail by Godefroy, Kalton, and Saphar in [263], see also [262]. Here the letter u stands for "unconditional", which turns out to be justified by the main results of [263]. In the following we report on those central results of the important paper [263] which are pertinent to M- and L-embedded spaces. Aspects relevant to M-ideals of compact operators and various approximation properties will be discussed in Notes and Remarks to Chapter VI.

A subspace X of a Banach space Y is called a *u-summand* if there is a subspace Z of Y such that $Y = X \oplus Z$ and

$$\|x + z\| = \|x - z\| \qquad \text{for all } x \in X,\, z \in Z\,.$$

The associated u-projection P is characterised by $\|Id - 2P\| = 1$. More precisely, X is a u-summand in Y iff X is the range of a u-projection iff X is the kernel of a u-projection. Since there is at most one u-projection onto a subspace X [263, Lemma 3.1], the complementary space Z is uniquely determined. Adopting another point of view we mention that u-projections P are in one-to-one correspondence with the isometric symmetries (involutions) S of the space Y, i.e. $S^2 = Id$, via $S = Id - 2P$. For example, this allows a complete description of the u-projections on a $C(K)$-space by the Banach-Stone theorem (cf. p. 341): An isometric symmetry S on $C(K)$ has the form $Sf = \theta(f \circ \varphi)$ with $\theta : K \to S_{\mathbb{K}}$ and $\varphi : K \to K$ continuous and satisfying $\theta^2 = \mathbf{1}_K$ and $\varphi^2 = Id_K$. Taking e.g. $K = [-1, 1]$ and $\varphi(x) = -x$ one obtains the u-summands of even and odd continuous functions in $C([0, 1])$.

The extension of the concept of a u-summand to that of a *u-ideal* proceeds in the obvious way: a subspace X of a Banach space Y is called a u-ideal if $X^\perp$ is a u-summand in Y^*. Since $X^\perp$ is then in particular the kernel of a contractive projection, there is a Hahn-Banach extension operator $L : X^* \to Y^*$ (see p. 135) which in turn induces a mapping $T : Y \to X^{**}$ with $Tx = x$ for $x \in X$. In this way the $\sigma(X^{**}, X^*)$-topology (in connection with the norm condition) is opened up as a tool for the study of u-ideals. If X is a u-ideal in Y, say $Y^* = V \oplus X^\perp$, the topological properties of V can be used to distinguish u-summands and "extreme" u-ideals. Namely, a u-ideal X in Y is a u-summand if and only if V is w^*-closed (compare the text preceding Proposition I.1.2), and a u-ideal X in Y is called *strict* if V is norming. Since this is equivalent to the condition $r(V, X^*) = 1$ (compare Definition II.3.7), strict u-ideals can be regarded as counterparts of the extreme M-ideals introduced in Definition II.3.7(b).

The main result on general u-ideals asserts that a u-ideal which is not a u-summand contains a subspace isomorphic to c_0 [263, Theorem 3.5]; this is an extension of Theorem II.4.7. Its proof uses a fundamental lemma from the theory of u-ideals which says in abridged form: if X is a u-ideal in Y, $T : Y \to X^{**}$ the operator defined above, and Ty is the $\sigma(X^{**}, X^*)$-limit of a sequence in X, then Ty is already the $\sigma(X^{**}, X^*)$-limit of a wuC-series $\sum x_k$ in X satisfying in addition $\|\sum_{k=1}^n \varepsilon_k x_k\| \le \|y\| + \varepsilon$ for all $n \in \mathbb{N}$ and $|\varepsilon_k| = 1$. A more elaborate version of this lemma together with the equivalence "X doesn't contain c_0 iff every wuC-series in X is unconditionally convergent" shows that if X is a u-ideal in Y and X contains no copy of c_0, then $T(Y) \subset X$, which easily gives the w^*-continuity of the u-projection in Y^*.

A much richer theory evolves in the special case when X is a u-ideal in X^{**}. Examples of this situation are M- and L-embedded spaces (the latter even being u-summands in X^{**}) and spaces for which the natural projection π_{X^*} in X^{***} with kernel $X^\perp$ is a u-projection (in which case X is a called a strict u-ideal). However the most remarkable class of examples are the order continuous Banach lattices. In this setting the extreme cases are easily characterised: an order continuous Banach lattice X is a u-summand in X^{**} iff X doesn't contain c_0, and X is a strict u-ideal in X^{**} iff X doesn't contain ℓ^1. It can be deduced from [253, Prop. 9] that X is weakly sequentially complete provided X is a u-summand in X^{**}.

By the above it only makes sense to compare M-embedded spaces with spaces which are strict u-ideals in their biduals. Clearly, they are easy to distinguish isometrically. The space c_0 with the norm $|x| = \|x\|_\infty + \|(x_n/2^n)\|_2$ is a strict u-ideal in its bidual which is not proximinal, and $X = c_0 \oplus_1 \mathbb{K}$ is one which lacks unique Hahn-Banach extension to X^{**}; thus they are not M-embedded. More interesting is that $\ell^2(c_0)$ is a space which is a strict u-ideal in its bidual and which can't be renormed to become an M-embedded space. The latter follows from

> PROPOSITION [263, Prop. 4.4] *If a separable M-embedded space has a boundedly complete Schauder decomposition $\oplus \sum E_n$, then all but finitely many E_n are reflexive.*

A crucial distinction between M- and L-embedded spaces is that in the first case we deal with the natural projection π_{X^*}. In certain cases this carries over to u-ideals: if X is separable or doesn't contain ℓ^1, then X is a strict u-ideal in X^{**} iff π_{X^*} is a u-projection. In this case every subspace and every quotient of X is a strict u-ideal in its bidual, every isometric isomorphism of X^{**} is the bitranspose of one in X, X is Asplund and X is the only predual of X^* which is a strict u-ideal in its bidual ([263, Prop. 5.2, Prop. 2.8, Theorem 5.7], for the case of M-ideals compare Th. III.1.6, Prop. III.2.2, Th. III.3.1, Prop. IV.1.9). The main result on strict u-ideals is a characterisation in terms of a quantitative version of Pełczyński's property (u), introduced in Section III.3. Denote by $\varkappa_u(X)$ the least constant $C > 0$ satisfying: for all x^{**} in $B_1(X)$ (i.e., there is a sequence (x_n) in X with $w^*\text{-}\lim x_n = x^{**}$) there is a wuC-series $\sum y_k$ in X such that $x^{**} = \sum^* y_k$ and $\|\sum_{k=1}^n \theta_k y_k\| \le C\|x^{**}\|$ for all $n \in \mathbb{N}$ and all $|\theta_k| = 1$. [If complex scalars θ_k are allowed, we write $\varkappa_h(X)$.] Note that with this notation X has property (u) iff $\varkappa_u(X) < \infty$, and the proof of Theorem III.3.8 actually yields $\varkappa_u(X) = 1$ for M-embedded spaces. The following theorem provides in particular another approach to that result.

> THEOREM [263, Theorem 5.4] *A Banach space X containing no copy of ℓ^1 is a strict u-ideal in its bidual iff $\varkappa_u(X) = 1$.*

There is a kind of quantitative version of this result which singles out the strict u-ideals among the u-ideals. It is shown in [263, Theorem 7.4] that for a Banach space X which is a u-ideal in its bidual and contains no copy of ℓ^1 the following conditions are equivalent: (1) X is a strict (u)-ideal in X^{**}, (2) $\varkappa_u(X) < 2$, (3) X^* contains no proper norming subspace. This can be used in the study of isomorphic (!) and isometric preduals. As an example we mention a special case of [263, Theorem 7.7]: If X is a separable M-embedded space enjoying the MAP and Y is a space which is a strict u-ideal in Y^{**} such that $d(X^*, Y^*) < 2$, then X is isomorphic to Y. We remark that the proof of this result relies on Theorem II.2.1.

For complex scalars there is a very natural subclass of u-projections and u-ideals. Let us note that a projection P on a complex Banach space Y is hermitian in the sense of p. 17 iff $\|Id_Y - (1+\lambda)P\| = 1$ whenever $|\lambda| = 1$. [This follows easily from the power series representation of e^{itP}.] This norm condition can be regarded as the natural complex analogue of the condition $\|Id - 2P\| = 1$ defining u-projections. Now, hermitian projections are termed *h-projections*, and *h-summands*, *h-ideals*, and *strict h-ideals* are defined in the obvious way. To see that M- and L-summands in complex spaces are h-summands we note that a decomposition $Y = X \oplus Z$ gives rise to h-summands if and only if

$$\|x + z\| = \|x + \lambda z\| \qquad \text{for all } x \in X,\ z \in Z,\ |\lambda| = 1.$$

The hermitian operators on $Y = C_{\mathbb{C}}(K)$ are multiplications by real valued functions [86, p. 91], so in this case h-summands and M-summands coincide. However, for noncommutative C^*-algebras $\mathfrak{A}$ they can differ since by a result of Sinclair [86, p. 95] a hermitian operator T on $\mathfrak{A}$ is of the form $Tx = hx + Dx$, where $h^* = h$ and D is a star derivation, i.e. $D(xy) = (Dx)y + xDy$ and $Dx^* = -(Dx)^*$. So taking e.g. $\mathfrak{A} = L(H)$, $D = 0$, and h a selfadjoint projection on H, one obtains nontrivial u-summands in $L(H)$; but $L(H)$ fails to have nontrivial M-summands as will be shown in Corollary VI.1.12.

Exploiting the theory of hermitian operators Godefroy, Kalton, and Saphar obtain even more satisfactory results for complex scalars than in the u-ideal setting. For example it can be shown that if X is an h-ideal in X^{**} then the induced mapping $T : X^{**} \to X^{**}$ satisfies $Tx^{**} = x^{**}$ not only for $x^{**} \in X$, but also for $x^{**} \in B_1(X) := \{x^{**} \in X^{**} \mid \text{there is a sequence } (x_n) \text{ in } X \text{ such that } w^*\text{-}\lim x_n = x^{**}\}$. This gives

Theorem [263, Theorem 6.4] *If X is an h-ideal in X^{**}, then $\varkappa_h(X) = 1$.*

Note that there is no strictness assumption in the above. The mapping T is not only the identity on $B_1(X)$, but a projection onto this space. Even more is true. A separable Banach space X is an h-ideal in X^{**} iff there is a hermitian projection T of X^{**} onto $B_1(X)$ such that for every $x^{**} \in S_{X^{**}}$ there is a net (x_d) in X with $w^*\text{-}\lim x_d = x^{**}$ and $\limsup \|x^{**} - (1 + \lambda)x_d\| \le 1$ for $|\lambda| = 1$ [263, Theorem 6.5]. Without this extra norm condition the result is false; one can even show that there is a separable Banach space Y such that $\varkappa_h(Y) = 1$ and $B_1(Y)$ is complemented in Y^{**} by a hermitian projection, yet Y is not an h-ideal in Y^{**}; in fact one can take $Y = K(c_0 \oplus_1 \mathbb{C})$. The characterisation of strict h-ideals among h-ideals is particularly pleasing. It is shown in [263, Theorem 6.6] that the following conditions are equivalent for a Banach space X which is an h-ideal in X^{**}: (1) X is a strict h-ideal in X^{**}, (2) X^* is an h-ideal in X^{***}, (3) X is an Asplund space, (4) X contains no copy of ℓ^1.

We close this summary by mentioning some situations where h-ideals coincide with M- or L-embedded spaces. For example, if X is a separable (complex) M-embedded space, then a subspace Z of X^* is an h-ideal in Z^{**} iff Z is L-embedded (cf. Prop. IV.1.10). The same conclusion obtains for subspaces Z of $L^1[0,1]$ (see [263, Cor. 6.10, Cor. 6.13]). There is also a dual result stating that, for a separable (complex) Banach space X such that X^* is L-embedded, a quotient Y of X is M-embedded iff Y is a strict h-ideal in Y^{**}.