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THICKNESS OF THE UNIT SPHERE, ℓ_1-TYPES, AND THE ALMOST DAUGAVET PROPERTY

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Abstract. We study those Banach spaces X for which S_X does not admit a finite ε-net consisting of elements of S_X for any $\varepsilon < 2$. We give characterisations of this class of spaces in terms of ℓ_1-type sequences and in terms of the almost Daugavet property. The main result of the paper is: a separable Banach space X is isomorphic to a space from this class if and only if X contains an isomorphic copy of ℓ_1.

1. Introduction

For a Banach space X, R. Whitley [13] introduced the following parameter, called *thickness*, which is essentially the inner measure of non-compactness of the unit sphere S_X:

$$T(X) = \inf\{\varepsilon > 0\colon \text{there exists a finite } \varepsilon\text{-net for } S_X \text{ in } S_X\},$$

or equivalently, $T(X)$ is the infimum of those ε such that the unit sphere of X can be covered by a finite number of balls with radius ε and centres in S_X. He showed in the infinite dimensional case that $1 \leq T(X) \leq 2$, and in particular that $T(C(K)) = 1$ if K has isolated points and $T(C(K)) = 2$ if not.

In this paper we concentrate on the spaces with $T(X) = 2$. Our main results are the following; B_X denotes the closed unit ball of X.

Theorem 1.1. *For a separable Banach space X the following conditions are equivalent:*

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(a) $T(X) = 2$;

(b) *there is a sequence* $(e_n) \subset B_X$ *such that for every* $x \in X$

$$\lim_{n\to\infty} \|x + e_n\| = \|x\| + 1;$$

(c) *there is a norming subspace* $Y \subset X^*$ *such that the equation*

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{1.1}$$

holds true for every rank-one operator $T\colon X \to X$ *of the form* $T = y^* \otimes x$, *where* $x \in X$ *and* $y^* \in Y$.

Theorem 1.2. *A separable Banach space* X *can be equivalently renormed to have thickness* $T(X) = 2$ *if and only if* X *contains an isomorphic copy of* ℓ_1.

We mention that it has been proved in [1] that a space with thickness $T(X) = 2$ contains a copy of ℓ_1.

Recall that a subspace $Y \subset X^*$ is said to be norming (or 1-norming) if for every $x \in X$

$$\sup_{y^*\in S_Y} |y^*(x)| = \|x\|.$$

Y is norming if and only if S_Y is weak* dense in B_{X^*}.

Condition (b) of Theorem 1.1 links our investigations to the theory of types [10]. Recall that a type on a separable Banach space X is a function of the form

$$\tau(x) = \lim_{n\to\infty} \|x + e_n\|$$

for some bounded sequence (e_n). In [10] the notion of an ℓ_1-type is defined by means of convolution of types; a special instance of this is a type generated by a sequence (e_n) satisfying

$$\tau(x) = \lim_{n\to\infty} \|x + e_n\| = \|x\| + 1. \tag{1.2}$$

To simplify notation let us call a type like this a *canonical* ℓ_1*-type* and a sequence $(e_n) \subset B_X$ satisfying (1.2) a *canonical* ℓ_1*-type sequence*.

Condition (c) links our investigations to the theory of Banach spaces with the *Daugavet property* introduced in [8] and developed further for instance in the papers [2] [5], [6], [9]; see also the survey [12]. We will say that a Banach space X has the *Daugavet property with respect to* Y ($X \in \mathrm{DPr}(Y)$) if the *Daugavet equation* (1.1) holds true for every rank-one operator $T\colon X \to X$ of the form $T = y^* \otimes x$, where $x \in X$ and $y^* \in Y$, and it has the *almost Daugavet property* or is an *almost Daugavet space* if it has $\mathrm{DPr}(Y)$ for some norming subspace $Y \subset X^*$. This definition is a generalization (introduced in [7]) of the by now well-known Daugavet property of [8], which is $\mathrm{DPr}(Y)$ with $Y = X^*$.

In this language Theorem 1.2 says, by Theorem 1.1, that a separable Banach space can be renormed to have the almost Daugavet property if and only if it contains a copy of ℓ_1.

In Section 2 we present a characterisation of almost Daugavet spaces in terms of ℓ_1-sequences of the dual. The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4.

The following lemma is the main technical prerequisite that we use; it is the analogue of [8, Lemma 2.2]. Up to part (v) it was proved in [7]; however, (v) follows along the same lines. By a slice of B_X we mean a set of the form

$$S(y^*, \varepsilon) = \{x \in B_X\colon \operatorname{Re} y^*(x) \geq 1 - \varepsilon\}$$

for some $y^* \in S_{X^*}$ and some $\varepsilon > 0$, and a weak* slice $S(y, \varepsilon)$ of the dual ball B_{X^*} is a particular case of slice, generated by element $y \in S_X \subset X^{**}$.

Lemma 1.3. *If Y is a norming subspace of X^*, then the following assertions are equivalent.*

(i) *X has the Daugavet property with respect to Y.*

(ii) *For every $x \in S_X$, for every $\varepsilon > 0$, and for every $y^* \in S_Y$ there is some $y \in S(y^*, \varepsilon)$ such that*

$$\|x + y\| \geq 2 - \varepsilon. \tag{1.3}$$

(iii) *For every $x \in S_X$, for every $\varepsilon > 0$, and for every $y^* \in S_Y$ there is a slice $S(y_1^*, \varepsilon_1) \subset S(y^*, \varepsilon)$ with $y_1^* \in S_Y$ such that* (1.3) *holds for every $y \in S(y^*, \varepsilon_1)$.*

(iv) *For every $x^* \in S_Y$, for every $\varepsilon > 0$, and for every weak* slice $S(x, \varepsilon)$ of the dual ball B_{X^*} there is some $y^* \in S(x, \varepsilon)$ such that $\|x^* + y^*\| \geq 2 - \varepsilon$.*

(v) *For every $x^* \in S_Y$, for every $\varepsilon > 0$, and for every weak* slice $S(x, \varepsilon)$ of the dual ball B_{X^*} there is another weak* slice $S(x_1, \varepsilon_1) \subset S(x, \varepsilon)$ such that $\|x^* + y^*\| \geq 2 - \varepsilon$ for every $y^* \in S(x_1, \varepsilon_1)$.*

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2. A characterisation of almost Daugavet spaces by means of ℓ_1-sequences in the dual

For the sake of easy notation we introduce two definitions.

Definition 2.1. Let E be subspace of a Banach space F and $\varepsilon > 0$. An element $e \in B_F$ is said to be $(\varepsilon, 1)$-*orthogonal* to E if for every $x \in E$ and $t \in \mathbb{R}$

$$\|x + te\| \geq (1 - \varepsilon)(\|x\| + |t|). \tag{2.1}$$

Definition 2.2. Let E be a Banach space. A sequence $\{e_n\}_{n\in\mathbb{N}} \subset B_E \setminus \{0\}$ is said to be an *asymptotic ℓ_1-sequence* if there is a sequence of numbers $\varepsilon_n > 0$ with $\prod_{n\in\mathbb{N}}(1 - \varepsilon_n) > 0$ such that e_{n+1} is $(\varepsilon_n, 1)$-orthogonal to $Y_n := \operatorname{lin}\{e_1, \dots, e_n\}$ for every $n \in \mathbb{N}$.

Evidently every asymptotic ℓ_1-sequence is $1/\prod_{n\in\mathbb{N}}(1 - \varepsilon_n)$-equivalent to the unit vector basis in ℓ_1, and moreover every element of the unit sphere of $E_m := \operatorname{lin}\{e_k\}_{k=m+1}^{\infty}$ is $\big(1 - \prod_{n\geq m}(1 - \varepsilon_n), 1\big)$-orthogonal to Y_m for every $m \in \mathbb{N}$.

The following lemma is completely analogous to [8, Lemma 2.8]; instead of [8, Lemma 2.1] it uses (v) of Lemma 1.3. So we state it without proof.

Lemma 2.3. *Let Y be a norming subspace of X^*, $X \in \mathrm{DPr}(Y)$, and let $Y_0 \subset Y$ be a finite-dimensional subspace. Then for every $\varepsilon_0 > 0$ and every weak* slice $S(x_0, \varepsilon_0)$ of B_{X^*} there is another weak* slice $S(x_1, \varepsilon_1) \subset S(x_0, \varepsilon_0)$ of B_{X^*} such that every element $e^* \in S(x_1, \varepsilon_1)$ is $(\varepsilon_0, 1)$-orthogonal to Y_0. In particular there is an element $e_1^* \in S(x_0, \varepsilon_0) \cap S_Y$ which is $(\varepsilon_0, 1)$-orthogonal to Y_0.*

We need one more definition.

Definition 2.4. A sequence $\{e_n^*\}_{n\in\mathbb{N}} \subset B_{X^*}$ is said to be *double-norming* if $\operatorname{lin}\{e_k^*\}_{k=n}^{\infty}$ is norming for every $n \in \mathbb{N}$.

Here is the main result of this section.

Theorem 2.5. *A separable Banach space X is an almost Daugavet space if and only if X^* contains a double-norming asymptotic ℓ_1-sequence.*

Proof. First we prove the "if" part. Let $\{e_n^*\}_{n\in\mathbb{N}} \subset B_{X^*}$ be a double-norming asymptotic ℓ_1-sequence, and let $\varepsilon_n > 0$ with $\prod_{n\in\mathbb{N}}(1 - \varepsilon_n) > 0$ be such that e_{n+1}^* is $(\varepsilon_n, 1)$-orthogonal to $Y_n := \operatorname{lin}\{e_1^*, \dots, e_n^*\}$ for every $n \in \mathbb{N}$. Let us prove that X has the Daugavet property with respect to $Y = \overline{\operatorname{lin}}\{e_n^*\}_{n\in\mathbb{N}}$ where the closure is meant in the norm topology. To do this let us apply (iv) of Lemma 1.3.

Fix an $x^* \in S_Y$, an $\varepsilon > 0$ and a weak* slice $S(x, \varepsilon)$ of the dual ball B_{X^*}. Denote in addition to $Y_m = \operatorname{lin}\{e_1^*, \dots, e_m^*\}$, $E_m := \operatorname{lin}\{e_k^*\}_{k=m+1}^{\infty}$. Using the definition of Y select an $m \in \mathbb{N}$ and an $x_m^* \in Y_m$ such that $\|x^* - x_m^*\| < \varepsilon/2$ and $\prod_{n\geq m}(1 - \varepsilon_n) > 1 - \varepsilon/2$. Since E_m is norming, there is a $y^* \in S(x, \varepsilon) \cap S_{E_m}$.

Taking into account that every element of the unit sphere of E_m is $(\varepsilon/2, 1)$-orthogonal to Y_m we obtain

$$\|x^* + y^*\| \geq \|x_m^* + y^*\| - \|x^* - x_m^*\| \geq 2 - \varepsilon.$$

For the "only if" part we proceed as follows. First we fix a sequence of numbers $\varepsilon_n > 0$ with $\prod_{n\in\mathbb{N}}(1 - \varepsilon_n) > 0$ and a dense sequence (x_n) in S_X. We can choose these x_n in such a way that each of them appears in the sequence (x_n) infinitely many times. Assume now that $X \in \mathrm{DPr}(Y)$ with respect to a norming subspace $Y \subset X^*$. Starting with $Y_0 = \{0\}$, $\varepsilon_0 = 1$ and applying Lemma 2.3 step-by-step we can construct a sequence $\{e_n^*\}_{n\in\mathbb{N}} \subset S_Y$ in such a way that each e_{n+1}^* belongs to $S(x_n, \varepsilon_n)$ and is $(\varepsilon_n, 1)$-orthogonal to Y_n, where $Y_n = \mathrm{lin}\{e_1^*, \dots, e_n^*\}$ as before. This inductive construction ensures that the e_n^*, $n \in \mathbb{N}$, form an asymptotic ℓ_1-sequence. On the other hand this sequence meets every slice $S(x_n, \varepsilon_n)$ infinitely many times, and this implies by density of (x_n) that (e_n^*) is double-norming. □

In Corollary 3.5 we shall observe a somewhat more pleasing version of the last result.

We conclude the section with two examples.

Proposition 2.6. *The real space ℓ_1 is an almost Daugavet space.*

Proof. To prove this statement we will construct a double-norming asymptotic ℓ_1-sequence $(f_n) \subset \ell_\infty = (\ell_1)^*$. At first consider a sequence $(g_n) \subset \ell_\infty$ of elements $g_n = (g_{n,j})_{j\in\mathbb{N}}$ with all $g_{n,j} = \pm 1$ satisfying the following independence condition: for arbitrary finite collections $\alpha_s = \pm 1$, $s = 1, \dots, n$, the set of those j that $g_{s,j} = \alpha_s$ for all $s = 1, \dots, n$ is infinite (for instance, put $g_{s,j} := r_s(t_j)$, where the r_s are the Rademacher functions and $(t_j)_{j\in\mathbb{N}}$ is a fixed sequence of irrationals that is dense in $[0, 1]$). These g_n, $n \in \mathbb{N}$, form an isometric ℓ_1-sequence, and moreover if one changes a finite number of coordinates in each of the g_n to some other ± 1, the independence condition will survive, so the modified sequence will still be an isometric ℓ_1-sequence.

Now let us define the vectors $f_n = (f_{n,j})_{j\in\mathbb{N}}$, $f_{n,j} = \pm 1$, in such a way that for $k = 1, 2, \dots$ and $n = 2^k + 1, 2^k + 2, \dots, 2^{k+1}$ the vectors $(f_{n,j})_{j=1}^k \in \ell_\infty^{(k)}$ run over all extreme points of the unit ball of $\ell_\infty^{(k)}$, i.e., over all possible k-tuples of ± 1; for the remaining values of indices we put $f_{n,j} = g_{n,j}$. As we have already remarked, the f_n form an isometric ℓ_1-sequence. Moreover, for every $k \in \mathbb{N}$ the restrictions of the f_n to the first k coordinates form a double-norming sequence over $\ell_1^{(k)}$, so $(f_n)_{n\in\mathbb{N}}$ is a double-norming sequence over ℓ_1. □

Some ideas of the previous proof will enter into the proof of Theorem 1.1. As a consequence of that theorem, the complex space ℓ_1 is almost Daugavet as well. It is worth noting that ℓ_1 fails the Daugavet property and cannot even be renormed to have it (see e.g. [8, Cor. 2.7]).

Since ℓ_∞ is isomorphic to $L_\infty[0,1]$, which has the Daugavet property, ℓ_∞ can be equivalently renormed to possess the Daugavet property. Let us show that in the original norm it is not even an almost Daugavet space. This is a special case of the following proposition in which $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$.

Proposition 2.7. *No Banach space of the form $Z = X \oplus_\infty \mathbb{K}$ is an almost Daugavet space.*

Proof. Let us call a functional $z_0^* \in Z^*$ a *Daugavet functional* if

$$\|\mathrm{Id} + z_0^* \otimes z_0\| = 1 + \|z_0^* \otimes z_0\| \qquad \text{for every } z_0 \in Z.$$

We shall show that $z_0^* = (x_0^*, b_0)$ is not a Daugavet functional if $b_0 \neq 0$. Hence all the Daugavet functionals lie in the weak* closed subspace $(\{0\} \oplus X)^\perp$ of Z^*.

So let $x_0^* \in X^*$ and $b_0 \neq 0$ with $\|x_0^*\| + |b_0| = 1$, $z_0^* = (x_0^*, b_0)$ and let $z_0 = (0, -|b_0|/b_0)$. If $z = (x, a) \in B_Z$, i.e., $\|x\| \le 1$ and $|a| \le 1$, then

$$\begin{aligned} \|z + z_0^*(z) z_0\| &= \max\{\|x\|, |\, a - z_0^*(z)|b_0|/b_0 \,|\} \\ &\le \max\{1, |\, a - (x_0^*(x_0) + b_0 a)|b_0|/b_0 \,|\} \\ &\le \max\{1, \|x_0^*\| + (1 - |b_0|)\} < 2. \end{aligned}$$

This shows that z_0^* is not a Daugavet functional. □

If K is a compact Hausdorff space with an isolated point, then $C(K)$ is of the form $X \oplus_\infty \mathbb{K}$, hence it fails the almost Daugavet property. But if K is an uncountable metric space, then $C(K)$ is isomorphic to $C[0,1]$ by Milutin's theorem [14, Th. III.D.19], hence it can be renormed to have the Daugavet property.

3. Proof of Theorem 1.1

Since the three properties considered in Theorem 1.1 hold for a complex Banach space X if and only if they hold for the underlying real space $X_\mathbb{R}$, we will tacitly assume in this section that we are dealing with real spaces.

We will accomplish the proof of Theorem 1.1 by means of the following propositions.

The following fact applied for separable spaces is equivalent to implication (c) $\Rightarrow$ (a) of Theorem 1.1.

Proposition 3.1. *Every almost Daugavet space X has thickness $T(X) = 2$.*

PROOF. Let $Y \subset X^*$ be a norming subspace with respect to which $X \in \mathrm{DPr}(Y)$. According to the definition of $T(X)$ we have to show that for every $\varepsilon_0 > 0$ there is no finite $(2-\varepsilon_0)$-net of S_X consisting of elements of S_X. In other words we must demonstrate that for every collection $\{x_1, \dots, x_n\} \subset S_X$ there is a $y_0 \in S_X$ with $\|x_k - y_0\| > 2 - \varepsilon_0$ for all $k = 1, \dots, n$. But this is an evident corollary of Lemma 1.3(iii): starting with an arbitrary $y_0^* \in S_{Y^*}$ and applying (iii) we can construct recursively elements $y_k^* \in S_{Y^*}$ and reals $\varepsilon_k \in (0, \varepsilon)$, $k = 1, \dots, n$, in such a way that $S(y_k^*, \varepsilon_k) \subset S(y_{k-1}^*, \varepsilon_{k-1})$ and

$$\|(-x_k) + y\| > 2 - \varepsilon_0$$

for every $y \in S(y_k^*, \varepsilon_k)$. Since $S(y_n^*, \varepsilon_n)$ is the smallest of the slices constructed, every norm-one element of $S(y_n^*, \varepsilon_n)$ can be taken as the y_0 we need. □

For spaces with the Daugavet property the previous proposition has been proved in [11, Prop. 4.1.6].

Let us now turn to the implication (a) ⇒ (b) of Theorem 1.1.

Proposition 3.2. *If $T(X) = 2$ and X is separable, then X contains a canonical ℓ_1-type sequence.*

PROOF. Fix a dense countable set $A = \{a_n \colon n \in \mathbb{N}\} \subset S_X$ and a null-sequence (ε_n) of positive reals. Since for every $n \in \mathbb{N}$ the n-point set $\{-a_1, \dots, -a_n\}$ is not a $(2-\varepsilon_n)$-net of S_X there is an $e_n \in S_X$ with $\|e_n - (-a_k)\| > 2 - \varepsilon_n$ for all $k = 1, \dots, n$. The constructed sequence (e_n) satisfies for every $k \in \mathbb{N}$ the condition

$$\lim_{n\to\infty} \|a_k + e_n\| = \|a_k\| + 1 = 2.$$

By the density of A in S_X and a standard convexity argument (cf. e.g. [12, page 78]) this yields that (e_n) is a canonical ℓ_1-type sequence. □

By the result in [1] mentioned in the introduction we obtain:

Corollary 3.3. *Every almost Daugavet space contains ℓ_1.*

It remains to prove the implication (b) ⇒ (c) of Theorem 1.1.

Proposition 3.4. *A separable Banach space X containing a canonical ℓ_1-type sequence is an almost Daugavet space.*

PROOF. We will use Theorem 2.5. Fix an increasing sequence of finite-dimensional subspaces $E_1 \subset E_2 \subset E_3 \subset \dots$ whose union is dense in X. Also, fix sequences $\varepsilon_n \searrow 0$ and $\delta_n > 0$ such that for all n

$$\prod_{k=n}^{\infty}(1-\delta_k) \geq 1-\varepsilon_n. \tag{3.1}$$

Passing to a subsequence if necessary we can find a canonical ℓ_1-type sequence (e_n) satisfying the following additional condition: For every $x \in \operatorname{lin}(E_n \cup \{e_1, \dots, e_n\})$ and every $\alpha \in \mathbb{R}$ we have

$$\|x+\alpha e_{n+1}\| \geq (1-\delta_n)(\|x\|+|\alpha|). \tag{3.2}$$

Then we have for every $x \in E_n$ and every $y = \sum_{k=n+1}^{M} a_k e_k$ by (3.1) and (3.2)

$$\|x+y\| \geq (1-\varepsilon_n)\|x\| + \sum_{k=n+1}^{M}(1-\varepsilon_{k-1})|a_k|. \tag{3.3}$$

Fix a dense sequence (x_n) in S_X such that $x_n \in E_n$ and every element of the range of the sequence is attained infinitely often, that is for each $m \in \mathbb{N}$ the set $\{n\colon x_n = x_m\}$ is infinite. Finally, fix an "independent" sequence $(g_n) \subset \ell_\infty$, $g_{n,j} = \pm 1$, as in the proof of Proposition 2.6.

Now we are ready to construct a double-norming asymptotic ℓ_1-sequence $(f_n^*) \subset X^*$. First we define $\tilde{f}_n^*$ on $F_n := \operatorname{lin}\{x_n, e_{n+1}, e_{n+2}, \dots\}$ by

$$\tilde{f}_n^*(x_n) = 1-\varepsilon_n, \tag{3.4}$$

$$\tilde{f}_n^*(e_k) = (1-\varepsilon_{k-1})g_{n,k} \qquad (\text{if } k>n). \tag{3.5}$$

By (3.3), $\|\tilde{f}_n^*\| \leq 1$, and indeed $\|\tilde{f}_n^*\| = 1$ by (3.5). Define $f_n^* \in X^*$ to be a Hahn-Banach extension of $\tilde{f}_n^*$. Condition (3.4) and the choice of (x_n) ensure that (f_n^*) is double-norming. Let us show that it is an isometric ℓ_1-basis. Indeed, due to our definition of an "independent" sequence, for an arbitrary finite collection $A = \{a_1, \dots, a_n\}$ of non-zero coefficients the set J_A of those $j > n$ that $g_{s,j} = \operatorname{sign} a_s$, $s = 1, \dots, n$, is infinite. So by (3.5)

$$\left\|\sum_{s=1}^{n} a_s f_s^*\right\| \geq \sup_{j\in J_A}\left(\sum_{s=1}^{n} a_s f_s^*\right)e_j = \sup_{j\in J_A}(1-\varepsilon_{j-1})\sum_{s=1}^{n}|a_s| = \sum_{s=1}^{n}|a_s|.$$

□

Since we have constructed an isometric ℓ_1-basis (over the reals) in the last proof, we have obtained the following version of Theorem 2.5.

Corollary 3.5. *A real separable Banach space X is an almost Daugavet space if and only if X^* contains a double-norming isometric ℓ_1-sequence.*

4. Proof of Theorem 1.2

We start with two lemmas.

Lemma 4.1. *Let X be a linear space, $(e_n) \subset X$, and let p be a seminorm on X. Assume that (e_n) is an isometric ℓ_1-basis with respect to p, i.e., $p(\sum_{k=1}^n a_k e_k) = \sum_{k=1}^n |a_k|$ for all $a_1, a_2, \ldots \in \mathbb{K}$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define*

$$p_r(x) = \mathcal{U}\text{-}\lim_n p(x + re_n) - r$$

for $x \in X$ and $r > 0$. Then:

(a) $0 \le p_r(x) \le p(x)$ *for all* $x \in X$,
(b) $p_r(x) = p(x)$ *for all* $x \in \operatorname{lin}\{e_1, e_2, \ldots\}$,
(c) *the map* $x \mapsto p_r(x)$ *is convex for each* r,
(d) *the map* $r \mapsto p_r(x)$ *is convex for each* x,
(e) $p_r(tx) = tp_{r/t}(x)$ *for each* $t > 0$,
(f) $|p_r(x) - p_r(y)| \le p(x - y)$ *for all* $x, y \in X$.

Proof. The only thing that is not obvious is that p_r is positive; note that (b) is a well-known property of the unit vector basis of ℓ_1. Now, given $\varepsilon > 0$ pick n_ε such that

$$p(x + re_{n_\varepsilon}) \le \mathcal{U}\text{-}\lim_n p(x + re_n) + \varepsilon.$$

Then for each $n \neq n_\varepsilon$

$$\begin{aligned} p(x + re_n) &= p(x + re_{n_\varepsilon} + r(e_n - e_{n_\varepsilon})) \\ &\ge 2r - p(x + re_{n_\varepsilon}) \\ &\ge 2r - \mathcal{U}\text{-}\lim_n p(x + re_n) - \varepsilon; \end{aligned}$$

hence $2\mathcal{U}\text{-}\lim_n p(x + re_n) \ge 2r - 2\varepsilon$ and $p_r \ge 0$. □

Lemma 4.2. *Assume the conditions of Lemma 4.1. Then the function $r \mapsto p_r(x)$ is decreasing for each x. The quantity*

$$\bar{p}(x) := \lim_{r\to\infty} p_r(x) = \inf_r p_r(x)$$

satisfies (a)–(c) of Lemma 4.1 and moreover

$$\bar{p}(tx) = t\bar{p}(x) \qquad \text{for } t > 0,\ x \in X. \tag{4.1}$$

Proof. By Lemma 4.1(a) and (d), $r \mapsto p_r(x)$ is bounded and convex, hence decreasing. Therefore, $\bar{p}$ is well defined. Clearly, (4.1) follows from (e) above. □

Since for separable spaces the condition $T(X) = 2$ is equivalent to the presence of a canonical ℓ_1-type sequence and a canonical ℓ_1-type sequence evidently contains a subsequence equivalent to the canonical basis of ℓ_1, to prove Theorem 1.2 it is sufficient to demonstrate the following:

Theorem 4.3. *Let X be a Banach space containing a copy of ℓ_1. Then X can be renormed to admit a canonical ℓ_1-type sequence. Moreover if $(e_n) \subset X$ is an arbitrary sequence equivalent to the canonical basis of ℓ_1 in the original norm, then one can construct an equivalent norm on X in such a way that (e_n) is isometrically equivalent to the canonical basis of ℓ_1 and (e_n) forms a canonical ℓ_1-type sequence in the new norm.*

Proof. Let Y be a subspace of X isomorphic to ℓ_1, and let (e_n) be its canonical basis. To begin with, we can renorm X in such a way that Y is isometric to ℓ_1 and (e_n) is an isometric ℓ_1-basis.

Let $\mathcal{P}$ be the family of all seminorms $\tilde{p}$ on X that are dominated by the norm of X and for which $\tilde{p}(y) = \|y\|$ for $y \in Y$. By Zorn's lemma, $\mathcal{P}$ contains a minimal element, say p. We shall argue that

$$\lim_{n\to\infty} p(x + e_n) = p(x) + 1 \qquad \forall x \in X. \tag{4.2}$$

To show this it is sufficient to prove that for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$

$$\mathcal{U}\text{-}\lim_{n} p(x + e_n) = p(x) + 1 \qquad \forall x \in X. \tag{4.3}$$

To this end associate to p and $\mathcal{U}$ the functional $\bar{p}$ from Lemma 4.2. Note that $0 \le \bar{p} \le p$, but a priori $\bar{p}$ need not be a seminorm. However,

$$q(x) = \frac{\bar{p}(x) + \bar{p}(-x)}{2}$$

in the real case, resp.

$$q(x) = \int_0^1 \bar{p}(e^{2\pi i t}x)\,dt$$

in the complex case, defines a seminorm, and $q \le p$. (Lemma 4.1(f) implies that the integrand is a continuous function of t.) By Lemma 4.1(b) and by minimality of p we get that

$$q(x) = p(x) \qquad \forall x \in X. \tag{4.4}$$

Now, since $p(x) = p(\lambda x) \ge \bar{p}(\lambda x)$ whenever λ is a scalar of modulus 1, (4.4) implies that $p(x) = \bar{p}(x)$. Finally, by Lemma 4.1(a) and the definition of $\bar{p}$ we have $p(x) = p_r(x)$ for all $r > 0$; in particular $p(x) = p_1(x)$, which is our claim (4.3).

To complete the proof of the theorem, consider

$$|||x||| := p(x) + \|[x]\|_{X/Y};$$

this is the equivalent norm that we need. Indeed, clearly $|||x||| \le 2\|x\|$. On the other hand, $|||x||| \ge \frac{1}{3}\|x\|$. To see this assume $\|x\| = 1$. If $\|[x]\|_{X/Y} \ge \frac{1}{3}$, there is nothing to prove. If not, pick $y \in Y$ such that $\|x-y\| < \frac{1}{3}$. Then $p(y) = \|y\| > \frac{2}{3}$, and

$$|||x||| \ge p(x) \ge p(y) - p(x-y) > \frac{2}{3} - \|x-y\| > \frac{1}{3}.$$

Therefore, $\| \, . \, \|$ and $||| \, . \, |||$ are equivalent norms, and

$$\lim_{n\to\infty} |||x+e_n||| = \lim_{n\to\infty} p(x+e_n) + \|[x]\|_{X/Y} = p(x) + 1 + \|[x]\|_{X/Y} = |||x||| + 1$$

shows that (e_n) is a canonical ℓ_1-type sequence for the new norm. □

We would like to mention another proof of Theorem 4.3 that was suggested to us by W.B. Johnson. In this proof X is a real Banach space. Let again $Y \subset X$ be a subspace isometric to ℓ_1 with canonical basis (e_n). We denote by (r_n) the sequence of Rademacher functions in $L_\infty[0,1]$. Then there is a norm-1 operator from Y to $L_\infty[0,1]$ mapping e_n to r_n, for each n. Since $L_\infty[0,1]$ is 1-injective, the operator can be extended to a norm-1 operator $T\colon X \to L_\infty[0,1]$. If we let

$$|||x||| = \|Tx\| + \|[x]\|_{X/Y},$$

then this equivalent norm works; the details of the computation are the same as above.

(Added September 21, 2009.) In his diploma thesis, Simon Lücking has recently extended Proposition 2.6 and Corollary 3.5 to the complex case.

(Added May 31, 2010.) Barry Turett has kindly informed us that there are links between Theorem 4.3 and Gilles Godefroy's work. In [4] Godefroy defined a norm to be octahedral if, for some $u \neq 0$ in X^{**},

$$(*) \qquad \|x+u\| = \|x\| + \|u\| \quad \text{for all } x \in X.$$

He proved that a Banach space contains a copy of ℓ_1 (if and) only if X admits an equivalent octahedral norm. He also pointed out that $(*)$ implies the following condition $(**)$: For all finite-dimensional subspaces F of X and $\varepsilon > 0$ there exists $z \neq 0$ in X such that $\|x+z\| = (1-\varepsilon)(\|x\|+\|z\|)$ for all $x \in F$; and he noted that $(**)$ is sufficient for $(*)$ if X is separable. In this connection let us remark that $(**)$ is equivalent to $T(X) = 2$, hence Theorem 4.3 is equivalent to Godefroy's result in the separable case. In the non-separable case the interrelation remains open.

We would also like to take the opportunity to mention that the original method of proof of Theorem 4.3 has meanwhile proved useful in [3].

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