QUOTIENTS OF BANACH SPACES WITH THE DAUGAVET PROPERTY

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Abstract. We consider a general concept of Daugavet property with respect to a norming subspace. This concept covers both the usual Daugavet property and its weak* analogue. We introduce and study analogues for narrow operators and rich subspaces in this general setting and apply the results to show that a quotient of $L_1[0,1]$ over an ℓ_1-subspace can fail the Daugavet property. The latter answers a question posed to us by A. Pełczyński in the negative.

1. Introduction

Throughout the paper X stands for a Banach space. Recall that X has the *Daugavet property* if the identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|, \tag{1.1}$$

called *the Daugavet equation*, holds true for every rank-one operator $T\colon X \to X$. (We shall find it convenient to abbreviate this by writing $X \in \mathrm{DPr}$.) It is known that in this case (1.1) holds for the much wider class of so-called *narrow operators*. This class includes all strong Radon-Nikodým operators (which map the unit ball into a set with the Radon-Nikodým property) and in particular compact and weakly compact operators, operators not fixing a copy of ℓ_1 and linear combinations of the above mentioned types of operators [10].

Among the spaces with the Daugavet property are $C(K)$-spaces and vector-valued $C(K)$-spaces for perfect compact Hausdorff spaces K, $L_1(\mu)$- and vector-valued $L_1(\mu)$-spaces for non-atomic μ, wide classes of Banach algebras, but also some exotic spaces like Talagrand's space from [14] (see [9]) or Bourgain-Rosenthal's space from [3] (see [11]). All the spaces with the Daugavet property are non-reflexive, moreover they cannot have the Radon-Nikodým property and necessarily contain "many" copies of ℓ_1 [9].

There are several results on the stability of the Daugavet property under passing to "big" subspaces or quotients over "small" subspaces. In particular, if $X \in \mathrm{DPr}$, then $X/E \in \mathrm{DPr}$ for every reflexive subspace $E \subset X$ [13]. A preliminary version of that theorem appeared in [9] for $X = L_1[0,1]$. In a private conversation after a talk by the third-named author on the results

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of [9], A. Pełczyński asked whether the last result can be generalized for $E \subset L_1[0,1]$ being not necessarily reflexive, but having the Radon-Nikodým property (RNP). The question appeared quite non-trivial for the authors of [9], maybe because the efforts were concentrated on attempts to prove that $L_1[0,1]/E \in \mathrm{DPr}$ if E has the RNP. This question was reiterated in [13] and [7]. In this paper we are now going to present a negative answer.

Our approach to Pełczyński's question will be indirect. If the answer was positive, this would mean that every subspace $E \subset L_1[0,1]$ with the RNP has the following "smallness" property, since the RNP is hereditary: $L_1[0,1]/F \in \mathrm{DPr}$ for every subspace $F \subset E$. We introduce this property formally in a general setting (we call it *poverty*) and characterise it geometrically. Then we give a description of poor subspaces of $L_1(\Omega,\Sigma,\mu)$, and using that description we present an ℓ_1-subspace E of $L_1[0,1]$ which is not poor. Since ℓ_1 has the RNP, this leads to a counterexample.

To do all this we use duality arguments, but in order to be able to apply these arguments we have to consider a generalisation of the Daugavet property. Let us recall that a subspace $Y \subset X^*$ is said to be *norming* (more precisely 1-norming) if

$$\|x\| = \sup\{|y^*(x)|\colon y^* \in Y,\ \|y^*\| \le 1\}$$

for every $x \in X$. This is equivalent to saying that the closed unit ball of Y is weak* dense in the closed unit ball of X^*. Throughout the paper Y denotes a norming subspace of X^*.

Definition 1.1. We say that X has the *Daugavet property with respect to* Y ($X \in \mathrm{DPr}(Y)$) if the Daugavet equation (1.1) holds true for every rank-one operator $T\colon X \to X$ of the form $T = y^* \otimes x$, where $x \in X$ and $y^* \in Y$.

This generalisation of the ordinary Daugavet property was introduced in an equivalent form in [1]. It was motivated by the fact that the Daugavet property is not stable under passing to ultraproducts (this was proved in [11] and an open problem at the time when [1] was written), but the ultraproduct of spaces with the Daugavet property has the Daugavet property with respect to the ultraproduct of the dual spaces. The basic motivation for us in the present paper is that the Daugavet property does not generally pass to the dual, but it is obvious that if the original space X has the Daugavet property, then $X^* \in \mathrm{DPr}(X)$.

The structure of the paper is as follows. Section 2 contains characterisations of the Daugavet property with respect to Y in terms of slices, similar to [9]. Our eventual aim is to study "small" subspaces $Z \subset X$ of spaces with the Daugavet property, called poor subspaces; this will be done in Section 5. It will turn out that $Z \subset X$ is poor if and only if $Z^\perp \subset X^*$ enjoys a variant of the properties called richness and wealth in [10]. Such spaces are defined by means of a smallness property of the corresponding quotient map called narrowness. Narrow operators are studied in our context in Section 4. In order to prove that weakly compact operators on dual spaces are X-narrow we have included the technical Section 3 about convex combinations of slices. Finally, in Section 6 we characterise poor subspaces of $C(K)$ and $L_1(\mu)$ and derive that both spaces contain copies of ℓ_1 that are not poor provided they are separable, K is perfect and μ is atomless. This will lead to a negative answer to Pełczyński's question mentioned above (see Theorem 6.10).

A reader who is interested in that result only might wish to skip Section 3 and Section 4 apart from Definition 4.3, and he or she might also wish to only consider the case $X = Y^*$ in Section 5.

Much of the paper follows the lines of [10], and we omit proofs if they don't differ much from those in that paper.

We use standard notation such as B_X and S_X for the unit ball and the unit sphere of a Banach space X, and we employ the notation

$$S_U(x^*, \varepsilon) = \Big\{x \in U\colon x^*(x) > \sup_{u\in U} x^*(u) - \varepsilon\Big\}$$

for the slice of a bounded convex subset $U \subset X$ determined by $x^* \in S_{X^*}$. In the case of $U = B_X$ we omit the index U in the notation above:

$$S(x^*, \varepsilon) = \{x \in B_X\colon x^*(x) > 1 - \varepsilon\}.$$

For $\varepsilon > 0$ and $x \in S_X$ we consider the weak* slice of the dual ball B_{X^*}, i.e.,

$$S(x, \varepsilon) = \{x^* \in B_{X^*}\colon x^*(x) > 1 - \varepsilon\},$$

as a particular case of a slice. When we have a need to stress in what space the slice is considered we use notation like $S(X, x^*, \varepsilon)$ or $S(X^*, x, \varepsilon)$. The symbol $\operatorname{ex} C$ stands for the set of extreme points of a set C. In this paper we deal with real Banach spaces although our results extend to the complex case with minor modifications.

2. Basic descriptions of the generalised Daugavet property

In this section we collect generalisations of characterisations of the standard Daugavet property to the setting of $\mathrm{DPr}(Y)$. Let us start with a simple lemma about slices; cf. [9, Lemma 2.1].

Lemma 2.1. *The following statement holds true in any Banach space X for any norming subspace $Y \subset X^*$:*

Let $y \in S_X$, $x_0^* \in S_Y$ and $\varepsilon \in (0, 1)$. Assume that there is some $x \in S(x_0^*, \varepsilon/8)$ such that $\|x + y\| > 2 - \varepsilon/8$. Then there is an $x_1^* \in S_Y$ such that $S(x_1^*, \varepsilon/8) \subset S(x_0^*, \varepsilon)$ and $\|e + y\| > 2 - \varepsilon$ for all $e \in S(x_1^*, \varepsilon/8)$.

Proof. Since $\|x + y\| > 2 - \varepsilon/8$ there is an $x^* \in S_Y$ such that $x^*(x + y) > 2 - \varepsilon/8$. Then

$$x^*(x) > 1 - \frac{\varepsilon}{8} \quad\text{and}\quad x^*(y) > 1 - \frac{\varepsilon}{8}. \tag{2.1}$$

Define $x_1^* \in S_{X^*}$ by

$$x_1^* = \frac{x_0^* + x^*}{\|x_0^* + x^*\|};$$

we remark that

$$\|x_0^* + x^*\| \geq (x_0^* + x^*)x > 2 - \frac{\varepsilon}{4}.$$

Then for every $e \in S(x_1^*, \varepsilon/8)$ we have

$$(x_0^* + x^*)(e) > \Big(1 - \frac{\varepsilon}{8}\Big)\Big(2 - \frac{\varepsilon}{4}\Big), \tag{2.2}$$

so

$$x_0^*(e) > 1 - 2\frac{\varepsilon}{8} - \frac{\varepsilon}{4} + \frac{\varepsilon}{8}\frac{\varepsilon}{4} > 1 - \frac{\varepsilon}{2},$$

i.e., $e \in S(x_0^*, \varepsilon)$, and the inclusion $S(x_1^*, \varepsilon/8) \subset S(x_0^*, \varepsilon)$ is proved. Further, (2.2) implies that $x^*(e) > 1 - \varepsilon/2$ which together with (2.1) means that $\|e + y\| \geq x^*(e + y) > 2 - \varepsilon$. □

The following result is the analogue of [9, Lemma 2.2].

Theorem 2.2. *If Y is a norming subspace of X^*, then the following assertions are equivalent.*

(i) *X has the Daugavet property with respect to Y.*

(ii) *For every $x \in S_X$, for every $\varepsilon > 0$, and for every $y^* \in S_Y$ there is some $y \in S(y^*, \varepsilon)$ such that*

$$\|x + y\| \geq 2 - \varepsilon. \tag{2.3}$$

(iii) *For every $x \in S_X$, for every $\varepsilon > 0$, and for every $y^* \in S_Y$ there is a slice $S(y_1^*, \varepsilon_1) \subset S(y^*, \varepsilon)$ with $y_1^* \in S_Y$ such that (2.3) holds for every $y \in S(y^*, \varepsilon_1)$.*

Proof. The implication (iii) ⇒ (ii) is obvious; the implication (ii) ⇒ (iii) follows from Lemma 2.1. What remains to prove is the equivalence (i) ⇔ (ii).

Let us start with (i) ⇒ (ii). Fix some $x \in S_X$, $x^* \in S_Y$ and $\varepsilon > 0$ and consider the operator $T\colon X \to X$, $T := x^* \otimes x$. According to (1.1), $\|\mathrm{Id} + T\| = 2$, so there is a $y \in S_X$ such that $\|y + Ty\| \geq 2 - \varepsilon/2$ and $x^*(y) \geq 0$. Substituting the value of Ty we obtain that

$$\|y + x^*(y)x\| \geq 2 - \varepsilon/2$$

which means that $x^*(y) \geq 1 - \varepsilon/2$ (i.e., $y \in S(x^*, \varepsilon)$) and

$$\|y + x\| \geq \|y + x^*(y)x\| - |x^*(y) - 1| \geq 2 - \varepsilon,$$

which proves the implication.

For the converse implication (ii) ⇒ (i) consider an operator $T\colon X \to X$, $T = x^* \otimes x$, where $x \in X$ and $x^* \in Y$. Since the validity of (1.1) for T implies (1.1) for all operators of the form aT with $a > 0$, it is sufficient to consider the case of $\|T\| = 1$, and the representation $T = x^* \otimes x$ can be taken in such a way that $x \in S_X$ and $x^* \in S_Y$. Due to (ii), for every $\varepsilon > 0$ there is a $y \in S(x^*, \varepsilon)$ satisfying (2.3). Then

$$\begin{aligned} \|\mathrm{Id} + T\| &\geq \|y + Ty\| = \|y + x^*(y)x\| \\ &\geq \|y + x\| - |x^*(y) - 1| \geq (2 - \varepsilon) - \varepsilon, \end{aligned}$$

which by arbitrariness of ε means that $\|\mathrm{Id} + T\| = 2$.

□

3. A useful tool: convex combinations of slices

This section deals with a technical device that will be useful in the proof of Theorem 4.8.

Definition 3.1. Let Y be a norming subspace of X^*, and let $U \subset X$ be convex and bounded. A subset $V \subset U$ is called a quasi-$\sigma_U(X,Y)$ neighbourhood if it is a finite convex combination of slices of U generated by elements of Y; i.e., there are $\lambda_k \geq 0$, $k = 1, \dots, n$, with $\sum_{k=1}^n \lambda_k = 1$ and slices $S_1, \dots, S_n \subset U$ generated by elements of Y such that $\lambda_1 S_1 + \dots + \lambda_n S_n = V$.

The following lemma is known for the ordinary weak topology (see [2, Lemme 5.3]; it was rediscovered in [13]). The $\sigma(X, Y)$-version proof coincides almost word-to-word with the original one.

Lemma 3.2. *Under the conditions of the above definition every relatively $\sigma(X, Y)$-open subset $A \subset U$ contains a quasi-$\sigma_U(X, Y)$ neighbourhood.*

The next theorem and its corollary were essentially proved by Shvidkoy [13]. He considered the ordinary Daugavet property, but the proof in the general case is virtually the same.

Theorem 3.3. *Let Y be a norming subspace of X^* and $X \in \mathrm{DPr}(Y)$. Then for every $\varepsilon > 0$, every $x \in S_X$ and every quasi-$\sigma_{B_X}(X, Y)$ neighbourhood V there exists an element $v \in V$ such that $\|v + x\| \geq 2 - \varepsilon$.*

Proof. Let $V = \sum_{k=1}^{n} \lambda_k S_k$ be a representation of V as a convex combination of slices. Using repeatedly (ii) of Theorem 2.2 one can construct $x_k \in S_k$ such that $\|(x + \sum_{j<k} \lambda_j x_j) + \lambda_k x_k\| \geq \|x + \sum_{j<k} \lambda_j x_j\| + \lambda_k - \varepsilon/n$. Then $v = \sum_{k=1}^{n} \lambda_k x_k$ will be the element of V we need. □

Corollary 3.4. *If Y is a norming subspace of X^* and X has the Daugavet property with respect to Y, then the following is true: For every $x \in S_X$, for every $\varepsilon > 0$, and for every $\sigma(X, Y)$-open subset $U \subset X$ intersecting B_X there is some $y \in U \cap B_X$ such that $\|x + y\| \geq 2 - \varepsilon$.*

4. Narrow operators with respect to a norming subspace

We will eventually study subspaces satisfying a certain smallness condition called "poverty"; this will be dual to the notion of a "rich" subspace from [10]. The latter class is defined by the requirement that the canonical quotient map is "narrow". This section deals with such operators.

First we will recall and modify some definitions from [10]. Let X, E be Banach spaces.

Definition 4.1. An operator $T \in L(X, E)$ is said to be a *strong Daugavet operator* if for every two elements $x, y \in S_X$ and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|z + x\| > 2 - \varepsilon$ and $\|Tz - Ty\| < \varepsilon$. We denote the class of all strong Daugavet operators on X by $\mathcal{SD}(X)$.

Corollary 3.4 shows that if $X \in \mathrm{DPr}(Y)$ then every $T \in L(X, E)$ of the form $T = f \otimes e$, where $e \in E$ and $f \in Y$, is a strong Daugavet operator, and conversely, thanks to Theorem 2.2, if every $f \in Y \subset X^* = L(X, \mathbb{R})$ is strongly Daugavet, then X has the Daugavet property with respect to Y.

There is an obvious connection between strong Daugavet operators and the Daugavet equation (cf. [10, Lemma 3.2]).

Lemma 4.2. *If T: $X \to X$ is a strong Daugavet operator, then T satisfies the Daugavet equation* (1.1).

Definition 4.3. Let $X \in \mathrm{DPr}(Y)$. An operator $T \in L(X, E)$ is said to be *narrow with respect to Y* (or *Y-narrow* for short) if for every $x, e \in S_X$, $\varepsilon > 0$ and every slice $S \subset B_X$ generated by an element of Y and containing e there is an element $v \in S$ such that $\|x + v\| > 2 - \varepsilon$ and $\|Tv - Te\| < \varepsilon$. We denote the class of all Y-narrow operators on X by $\mathcal{NAR}_Y(X)$.

The notations $\mathcal{SD}(X)$ and $\mathcal{NAR}_Y(X)$ do not mention the range space E because the corresponding definitions do not actually depend on the values of T, but only on the norms of those values, i.e., these are not properties of the operator T itself, but just of the seminorm $x \mapsto \|T(x)\|$ on X. For more about this ideology see [10].

The following statement is a complete analogue of [10, Lemma 3.10(a)], so we omit the proof.

Lemma 4.4. *Let $T \in \mathcal{NAR}_Y(X)$. Let $S_1, \dots, S_n \subset B_X$ be a finite collection of slices generated by elements of Y, and let $U \subset B_X$ be a convex combination of these slices, i.e., there are $\lambda_k \geq 0$ with $\sum_{k=1}^n \lambda_k = 1$ such that $\lambda_1 S_1 + \dots + \lambda_n S_n = U$. Then for every $\varepsilon > 0$, every $x_1 \in S_X$ and every $w \in U$ there exists an element $u \in U$ such that $\|u + x_1\| > 2 - \varepsilon$ and $\|T(w - u)\| < \varepsilon$.*

Let us recall an operation with operators that was introduced in [10]. For operators $T_1\colon X \to E_1$ and $T_2\colon X \to E_2$ define

$$T_1 \tilde{+} T_2\colon X \to E_1 \oplus_1 E_2, \quad x \mapsto (T_1 x, T_2 x);$$

i.e.,

$$\|(T_1 \tilde{+} T_2)x\| = \|T_1 x\| + \|T_2 x\|.$$

Remark 4.5. Let X, E be Banach spaces, $Y \subset X^*$ be a norming subspace, $T \in L(X, E)$. If $T \tilde{+} y^* \in \mathcal{SD}(X)$ for every $y^* \in Y$, then $T \in \mathcal{NAR}_Y(X)$. In the setting of $Y = X^*$ this was actually given as the definition of a narrow operator in [10], and our definition was given as an equivalent condition in Lemma 3.10 of [10].

We are now going to introduce a class of operators that turn out to be Y-narrow; they correspond to the strong Radon-Nikodým operators in the case $Y = X^*$, which contain the weakly compact operators. We need two technical definitions.

Definition 4.6. Let X, E be Banach spaces, $F \subset E^*$ be a norming subspace and $\varepsilon > 0$. A point e of a convex subset $A \subset E$ is said to be an (F, ε)-*denting point* if there is a functional $f \in S_F$ and a $\delta > 0$ such that $\|e - a\| < \varepsilon$ whenever $a \in A$ satisfies the condition $f(a) > f(e) - \delta$. We say that $A \subset E$ is F-*dentable* if for every $\varepsilon > 0$ the set A is contained in the closed convex hull of its (F, ε)-denting points. An operator $T \in L(X, E)$ is said to be F-*dentable* if $T(B_X)$ is F-dentable. An E^*-dentable operator is called *dentable*.

Definition 4.7. An operator $T \in L(X, E)$ is said to be *hereditarily F-dentable* if for every $x^* \in X^*$ the operator

$$T \tilde{+} x^*\colon X \to E \oplus_1 \mathbb{R}, \quad x \mapsto (Tx, x^* x)$$

is $\tilde{F}$-dentable, where $\tilde{F}$ consists of all functionals $(f, \beta)\colon E \oplus_1 \mathbb{R} \to \mathbb{R}$, with $f \in F$ and $\beta \in \mathbb{R}$, of the form $(f, \beta)\,((e, t)) = f(e) + \beta t$.

Remark that every strong Radon-Nikodým operator and in particular every weakly compact operator is hereditarily dentable, by well-known geometric characterisations of sets with the RNP [4, Chap. 3].

Theorem 4.8. *If $X \in \mathrm{DPr}(Y)$, $T\colon X \to E$ is a hereditarily F-dentable operator and $T^*(F) \subset Y$, then T is Y-narrow.*

Proof. According to Remark 4.5 it is sufficient to prove that $\tilde{T} = T\tilde{+}x^* \in \mathcal{SD}(X)$ for every $x^* \in Y$. Fix $x, z \in S_X$ and $\varepsilon > 0$. By Definition 4.1, to prove the theorem we have to find an element $v \in B_X$ such that $\|x + v\| > 2 - \varepsilon$ and $\|\tilde{T}(z - v)\| < \varepsilon$.

Since $\tilde{T}(B_X)$ is $\tilde{F}$-dentable there are $\lambda_k \geq 0$ with $\sum_{k=1}^n \lambda_k = 1$ and $(\tilde{F}, \varepsilon/2)$-denting points $e_1, \dots, e_n \in T(B_X)$ such that

$$\left\| Tz - \sum_{k=1}^n \lambda_k e_k \right\| < \varepsilon/2.$$

By the definition of an $\tilde{F}$-denting point there are slices $S_k = S_{\tilde{T}(B_X)}(f_k, \varepsilon_k)$ of $T(B_X)$ generated by elements of $\tilde{F}$ such that $e_k \in S_k$ and the diameter of each of the S_k is less than $\varepsilon/2$. Denote $W := \sum_{k=1}^n \lambda_k S_k$. Since $\operatorname{dist}(Tz, W) < \varepsilon/2$ and $\operatorname{diam} W < \varepsilon/2$ we have

$$\|Tz - w\| < \varepsilon \qquad \text{for every } w \in W. \tag{4.1}$$

Denote $y_k^* := \tilde{T}^* f_k$. By assumption, $y_k^* \in Y$. Consider slices $V_k = \{v \in B_X\colon y_k^*(x) > \varepsilon_k\}$ and the quasi-$\sigma_{B_X}(X, Y)$ neighbourhood $V := \sum_{k=1}^n \lambda_k V_k$. Since $\tilde{T}(V_k) \subset S_k$ and consequently $\tilde{T}(V) \subset W$, (4.1) implies that

$$\|Tz - Tv\| < \varepsilon \qquad \text{for every } v \in V.$$

It remains to apply Theorem 3.3 to get a $v \in V$ with $\|x + v\| > 2 - \varepsilon$. □

Corollary 4.9. *A weak*-weakly continuous operator on a dual space $X^* \in \mathrm{DPr}(X)$ is X-narrow.*

Proof. It is clear that such an operator is weakly compact. The hereditary dentability of a weakly compact operator $T\colon X^* \to E$ has been mentioned in the above remark; it remains to observe that $T^*(E^*) \subset X$ if T is weak*-weakly continuous. □

Note that a weakly compact adjoint operator $S^*\colon X^* \to V^*$ is weak*-weakly continuous, i.e., $S^{**}(V^{**}) \subset X$; cf. [6, Section VI.4].

5. Rich and poor subspaces

In [10] we introduced rich subspaces $Z \subset X$, building on work in [8] and [12]. We showed that this condition is equivalent to saying that every superspace $Z \subset \tilde{Z} \subset X$ has the Daugavet property; the latter property was called wealth in [10]. We now extend and dualise these ideas.

5.1. **Richness.** The next proposition shows a kind of stability of the Daugavet property when one passes from the original space to a "big" subspace.

Lemma 5.1. *Let $X \in \mathrm{DPr}(Y)$. Then for every $x \in S_X$, every $\varepsilon > 0$ and for every separable subspace $V \subset Y$ there is an $x^* \in S_{X^*}$ such that $x^*(x) \geq 1 - \varepsilon$ and $\|x^* + f\| = 1 + \|f\|$ for all $f \in V$.*

Proof. Consider a dense sequence $(f_n)_{n=1}^\infty \subset V$ such that every element is repeated infinitely many times in the sequence. Applying (v) of Theorem 2.2 to the slice $S(x, \varepsilon)$ of B_{X^*} and to f_1 and then applying it step-by-step to f_n and to the slices obtained in the previous steps, we construct a sequence of closed slices $\overline{S}(x, \varepsilon) \supset \overline{S}(x_1, \varepsilon_1) \supset \overline{S}(x_2, \varepsilon_2) \supset \dots$ with $\varepsilon_n < 1/n$ such that $\|x^* + f_n\| \geq 2 - \varepsilon_{n-1}$ for all $x^* \in S(x_n, \varepsilon_n)$. By w^*-compactness of all $\overline{S}(x_n, \varepsilon_n)$, there is a point $x^* \in \bigcap_{n=1}^{\infty} \overline{S}(x_n, \varepsilon_n) \subset \overline{S}(x, \varepsilon)$. This is exactly the point we need. □

Proposition 5.2. *Let $X \in \mathrm{DPr}(Y)$, and let $Z \subset X$ be a subspace such that $Z^\perp$ is a separable subspace of Y. Then $Z \in \mathrm{DPr}(Y|_Z)$.*

Proof. Let $z \in S(Z)$, and let

$$S = \{z^* \in Z^* = X^*/Z^\perp \colon \|z^*\| \leq 1,\ z^*(z) \geq 1 - \varepsilon\}$$

be a slice of B_{Z^*}. Fix a $[z^*] \in S(Y/Z^\perp)$. We have to prove the existence of an $[x^*] \in S$ such that $\|x^* + z^*\| = 2$. Applying Lemma 5.1 with $x = z$ and $V = \mathrm{lin}(\{z^*\} \cup Z^\perp)$ we obtain an $x^* \in S_{X^*}$ such that $x^*(z) \geq 1 - \varepsilon$ and

$$\|x^* + f\| = 1 + \|f\| \qquad \text{for all } f \in V.$$

Then $[x^*] \in S$ and

$$\begin{aligned} \|[x^* + z^*]\| &= \inf_{f \in Z^\perp} \|x^* + z^* + f\| \\ &= \inf_{f \in Z^\perp} (1 + \|z^* + f\|) = 1 + \|[z^*]\| = 2. \end{aligned}$$

□

If a subspace $Z \subset X$ satisfies the conditions of the proposition above then so do all the subspaces of X containing Z. Hence Z has the property that $\tilde{Z} \in \mathrm{DPr}(Y|_{\tilde{Z}})$ for every subspace $\tilde{Z} \subset X$ containing Z. Let us formalise this property.

Definition 5.3. Let $X \in \mathrm{DPr}(Y)$. A subspace $Z \subset X$ is said to be *wealthy with respect to* Y if $\tilde{Z} \in \mathrm{DPr}(Y|_{\tilde{Z}})$ for every subspace $\tilde{Z} \subset X$ containing Z.

Thus Proposition 5.2 can be rephrased by saying that $Z \subset X$ is wealthy with respect to Y if $Z^\perp$ is separable and $X \in \mathrm{DPr}(Y)$.

The main result of this subsection is a characterisation of Y-wealthy subspaces through Y-narrow operators, analogous to [10, Theorem 5.12].

Definition 5.4. Let $X \in \mathrm{DPr}(Y)$. A subspace $Z \subset X$ is said to be *rich with respect to* Y if the quotient map $q \colon X \to X/Z$ is a Y-narrow operator.

It turns out that the following theorem holds.

Theorem 5.5. *Let X be a Banach space and Y be a norming subspace of X^* such that $X \in \mathrm{DPr}(Y)$. Then for a subspace $Z \subset X$ the following properties are equivalent:*

(i) *Z is wealthy with respect to Y.*
(ii) *Z is rich with respect to Y.*

The proof is very similar to the proof of [10, Theorem 5.12].

5.2. **Poverty as a dual property to richness.**

Definition 5.6. Let $X \in \mathrm{DPr}$. A subspace $Z \subset X$ is said to be *poor* if $X/\tilde{Z} \in \mathrm{DPr}$ for every subspace $\tilde{Z} \subset Z$.

Our study of poor subspaces uses duality, so let us start with a very simple observation that we state as a proposition for easy reference.

Proposition 5.7. *A Banach space X has the Daugavet property if and only if $X^* \in \mathrm{DPr}(X)$. Hence, a subspace Z of a space X with the Daugavet property is poor if and only if for every subspace $\tilde{Z} \subset Z$ its dual $(X/\tilde{Z})^* = \tilde{Z}^\perp$ has the Daugavet property with respect to $X/\tilde{Z}$.*

Now we are ready to give the basic characterisations of poverty.

Theorem 5.8. *Let $X \in \mathrm{DPr}$. For a subspace $Z \subset X$ the following conditions are equivalent.*

(i) *Z is poor.*
(ii) *$X/\tilde{Z} \in \mathrm{DPr}$ for every subspace $\tilde{Z} \subset Z$ of codimension $\operatorname{codim}_Z \tilde{Z} \le 2$.*
(iii) *$Z^\perp$ is a subspace of X^* that is rich with respect to X.*
(iv) *For every $x^*, e^* \in S_{X^*}$, $\varepsilon > 0$ and for every $x \in S_X$ such that $e^*(x) > 1 - \varepsilon$ there is an element $v^* \in B_{X^*}$ with the following properties: $v^*(x) > 1-\varepsilon$, $\|x^*+v^*\| > 2-\varepsilon$ and $\|(e^*-v^*)|_Z\| < \varepsilon$; that is, the quotient map from X^* onto $X^*/Z^\perp$ is narrow with respect to X.*

Proof. (i) $\Rightarrow$ (ii) follows immediately from the definition of poor subspaces.

Let us prove (ii) $\Rightarrow$ (i). According to Proposition 5.7, we have to prove that for every subspace $Z_1 \subset Z$, its dual $Z_1^\perp$ has the Daugavet property with respect to X/Z_1. Fix $Z_1 \subset Z$. Applying (ii) of Theorem 2.2 we see that for every $x^* \in S_{Z_1^\perp}$, $\varepsilon > 0$ and every $[x] \in S_{X/Z_1}$ we have to find $y^* \in S_{Z_1^\perp}$ such that $y^*([x]) \ge 1 - \varepsilon$ and $\|x^* + y^*\| \ge 2 - \varepsilon$. Since $[x] \in S_{X/Z_1}$, there exists $z^* \in S_{Z_1^\perp}$ such that $z^*([x]) = 1$. Denote $\tilde{Z} = Z \cap \ker x^* \cap \ker z^*$. Evidently, $\tilde{Z}$ is a subspace of Z of $\operatorname{codim}_Z \tilde{Z} \le 2$ and $Z_1 \subset \tilde{Z}$. Also remark that

$$1 = \|[x]_{X/Z_1}\| \ge \|[x]\|_{X/\tilde{Z}} \ge z^*([x]) = 1,$$

which implies $[x] \in S_{X/\tilde{Z}}$. By our assumption $\tilde{Z}^\perp$ has the Daugavet property with respect to $X/\tilde{Z}$, and hence for $x^* \in S_{\tilde{Z}^\perp}$ and $[x] \in S_{X/\tilde{Z}}$ there is $y^* \in S_{\tilde{Z}^\perp}$ such that $y^*([x]) \ge 1-\varepsilon$ and $\|x^*+y^*\| \ge 2-\varepsilon$. Then $y^* \in S_{\tilde{Z}^\perp} \subset S_{Z_1^\perp}$, and it meets all the requirements.

Now we will prove that (ii) $\Leftrightarrow$ (iii). Theorem 5.5 implies that (iii) holds if and only if $Z^\perp$ is a subspace of X^* that is wealthy with respect to X; and this is equivalent to the claim that for every $x^*, y^* \in S_{X^*}$ the space $W = \operatorname{lin}(Z^\perp \cup \{x^*, y^*\})$ has the Daugavet property with respect to $X/W_\perp$ (cf. [10, Lemma 5.6(iii)]. But for a space $\hat{Z} \supset Z^\perp$ the existence of $x^*, y^* \in S_{X^*}$ such that $W = \operatorname{lin}(Z^\perp \cup \{x^*, y^*\})$ is equivalent to the existence of a space $\tilde{Z} \subset Z$ such that $W = \tilde{Z}^\perp$ and $\operatorname{codim}_Z \tilde{Z} \le 2$. Thus we get that (iii) is equivalent to the claim that $\tilde{Z}^\perp \in \mathrm{DPr}(X/\tilde{Z})$ for every subspace $\tilde{Z} \subset Z$ of $\operatorname{codim}_Z \tilde{Z} \le 2$, which is equivalent to (ii) according to Proposition 5.7.

The remaining equivalence (iii) $\Leftrightarrow$ (iv) is just a reformulation of the definition of a rich subspace. □

As a corollary we can give a proof of the following theorem of Shvidkoy [13].

Corollary 5.9. *Let $X \in \mathrm{DPr}$ and let Z be a reflexive subspace of X. Then the quotient space X/Z also has the Daugavet property.*

Proof. Since every subspace of a reflexive space is also reflexive, the statement of this corollary is equivalent to the claim that every reflexive subspace Z of $X \in \mathrm{DPr}$ is poor. According to Theorem 5.8, it is sufficient to prove that $Z^\perp$ is a subspace of X^* that is rich with respect to X, i.e., that the quotient map $q\colon X^* \to X^*/Z^\perp$ is an X-narrow operator. As $X^*/Z^\perp$ is isometric to Z^*, which is reflexive, this follows from Corollary 4.9. □

6. Applications to the geometry of $C(K)$ and L_1

For a compact Hausdorff space K denote by $M(K)$ the dual space of $C(K)$, i.e., $M(K)$ is the Banach space of all (not necessarily positive) finite regular Borel signed measures on K. (In the sequel, all measures on K will be tacitly assumed to be finite regular Borel measures.) We are going to prove a theorem which gives a characterisation of operators on $M(K)$ that are narrow with respect to $C(K)$. For this theorem we will need the following lemma in which ∂A denotes the boundary of a set $A \subset K$.

Lemma 6.1. *Let K be compact, $f \in C(K)$, and μ be some positive measure on K. Then for every $\varepsilon > 0$ there exists a step function $\tilde f = \sum_{k=1}^n \beta_k \chi_{A_k}$ on K such that $\mu(\partial A_k) = 0$ for $k = 1, \dots, n$, $A_1 \cup \dots \cup A_n = K$ and $\|f - \tilde f\|_\infty < \varepsilon$.*

Proof. Since the image measure $\nu = \mu \circ f^{-1}$ on $\mathbb{R}$ has at most countably many atoms, it is possible to cover $f(K)$ by finitely many half-open intervals $I_k = (\beta_{k-1}, \beta_k]$ of length $< \varepsilon$ such that $\nu(\{\beta_0, \dots, \beta_n\}) = 0$. Let $A_k = f^{-1}(I_k)$; then $\tilde f = \sum_{k=1}^n \beta_k \chi_{A_k}$ works. □

Theorem 6.2. *Let K be a perfect compact Hausdorff space. An operator T on $M(K)$ is narrow with respect to $C(K)$ if and only if for every open subset $U \subset K$, for every two probability measures π_1, π_2 on U and for every $\varepsilon > 0$ there is a probability measure ν on U such that $\|T(\nu - \pi_1)\| < \varepsilon$ and $\|\pi_2 - \nu\| > 2 - \varepsilon$.*

Proof. We first prove the "only if" part. By the definition of a narrow operator (Definition 4.3), for every $x, e \in S_{M(K)}$, $\varepsilon > 0$ and every weak* slice S of $B_{M(K)}$ containing e, there exists $v \in S$ such that $\|x + v\| > 2 - \varepsilon$ and $\|T(e - v)\| < \varepsilon$. Fix $\varepsilon_1 > 0$ and let $x = -\pi_2$ and $e = \pi_1$. Since U is open and $\pi_1(U) = 1$, we can find $f \in C(K)$ taking values in $[0, 1]$ with $\operatorname{supp} f \subset U$ and $\int f\, d\pi_1 > 1 - \varepsilon_1$. By Definition 4.3 there exists $\tilde\nu \in S_{M(K)}$ such that the following inequalities hold:

$$\int f\, d\tilde\nu > 1 - \varepsilon_1, \quad \|T(\tilde\nu - \pi_1)\| < \varepsilon_1, \quad \|\pi_2 - \tilde\nu\| > 2 - \varepsilon_1.$$

Let $\hat\nu = \tilde\nu^+|_U$. Using the properties of f we have $\|\tilde\nu - \hat\nu\| < 2\varepsilon_1$ and thus

$$1 - 3\varepsilon_1 < \|\hat\nu\| \le 1 + 2\varepsilon_1, \quad \|T(\hat\nu - \pi_1)\| < \varepsilon_1(1 + 2\|T\|), \quad \|\pi_2 - \hat\nu\| > 2 - 3\varepsilon_1.$$

Hence for $\nu = \hat\nu / \|\hat\nu\|$ we have $\|\nu - \hat\nu\| = \big|1 - \|\hat\nu\|\big| < 3\varepsilon_1$ and consequently

$$\|\pi_2 - \nu\| \ge \|\pi_2 - \hat\nu\| - \|\nu - \hat\nu\| > 2 - 3\varepsilon_1 - 3\varepsilon_1 = 2 - 6\varepsilon_1,$$

and

$$\|T(\nu - \pi_1)\| \le \|T(\hat\nu - \pi_1)\| + \|T(\hat\nu - \nu)\| < (1 + 5\|T\|)\varepsilon_1.$$

Then taking $\varepsilon_1 = \min\{\frac{\varepsilon}{6}, \frac{\varepsilon}{1+5\|T\|}\}$ completes the proof of the "only if" part.

Now consider the "if" part. Given $\mu_1, \mu_2 \in S_{M(K)}$, $\varepsilon > 0$ and a weak* slice S of $B_{M(K)}$ containing μ_1, we have to find $\nu \in S$ such that $\|\mu_2 + \nu\| > 2 - \varepsilon$ and $\|T(\mu_1 - \nu)\| < \varepsilon$. Since one can wiggle the slice S a bit, there is, by Lemma 6.1, no loss of generality in replacing S by a slice generated by a function of the form $f = \sum_{k=1}^{n} \beta_k \chi_{A_k}$, where $A_1, \dots, A_n$ are measurable sets with $(|\mu_1| + |\mu_2|)(\bigcup_{k=1}^{n} \partial A_k) = 0$. (Note that in general this new slice will not be relatively weak* open.) On the other hand, using the Hahn decomposition theorem we have $K = \bigcup_{i=1}^{4} B_i$, where B_1 is a set on which μ_1 is positive and μ_2 is negative, B_2 is a set on which μ_2 is positive and μ_1 is negative, and B_3 (resp. B_4) is a set where both μ_1 and μ_2 are positive (resp. negative).

Fix $\varepsilon_1 > 0$ and let G_1 be an open set such that $G_1 \supset B_1$ and $|\mu_i|(G_1 \setminus B_1) < \varepsilon_1$ $(i = 1, 2)$. Define $C_k = G_1 \cap A_k$ and let $U_k = \operatorname{int} C_k$, $k = 1, \dots, n$. Clearly $C_k \setminus U_k \subset \partial A_k$, so the U_k are open sets with the following properties: $U_k \subset C_k$ and $(|\mu_1| + |\mu_2|)(C_k \setminus U_k) = 0$.

Consider those U_k for which $\mu_1(U_k \cap B_1) \neq 0$, $\mu_2(U_k \cap B_1) \neq 0$ and define two probability measures on U_k by

$$\mu_{i,k} = \frac{\mu_i|_{U_k \cap B_1}}{\mu_i(U_k \cap B_1)} \quad (i = 1, 2).$$

By assumption there exists a probability measure $\hat{\nu}_k$ on U_k such that

$$\|T(\hat{\nu}_k - \mu_{1,k})\| < \varepsilon_1 \quad \text{and} \quad \|\mu_{2,k} - \hat{\nu}_k\| > 2 - \varepsilon_1.$$

Define $\nu_k = \mu_1(U_k \cap B_1) \cdot \hat{\nu}_k$. Then we have

$$\|\nu_k\| = \nu_k(U_k) = \mu_1(U_k \cap B_1), \quad \|\mu_1|_{U_k}\| - \varepsilon_1 \leq \|\nu_k\| \leq \|\mu_1|_{U_k}\| + \varepsilon_1 \tag{6.1}$$

and

$$\begin{aligned} \|\mu_2|_{U_k} + \nu_k\| &= \|\mu_2(U_k \cap B_1) \cdot \mu_{2,k} + \mu_2|_{U_k \setminus B_1} + \mu_1(U_k \cap B_1) \cdot \hat{\nu}_k\| \\ &\geq \||\mu_2(U_k \cap B_1)| \cdot \mu_{2,k} - |\mu_1(U_k \cap B_1)| \cdot \hat{\nu}_k\| - \|\mu_2|_{U_k \setminus B_1}\| \\ &\geq |\mu_2|(U_k) + |\mu_1|(U_k) - 4\varepsilon_1 \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \|T(\nu_k - \mu_1|_{U_k})\| &\leq \|T(\mu_1(U_k \cap B_1) \cdot (\hat{\nu}_k - \mu_{1,k}))\| + \|T(\mu_1|_{U_k \setminus B_1})\| \\ &\leq \varepsilon_1(1 + \|T\|). \end{aligned} \tag{6.3}$$

For U_k with $\mu_1(U_k \cap B_1) = 0$ or $\mu_2(U_k \cap B_1) = 0$, the inequalities (6.1)–(6.3) hold with $\nu_k = \mu_1|_{U_k \cap B_1}$.

Now define the measure μ_1^1 by

$$\mu_1^1|_{U_k} = \nu_k, \quad \mu_1^1|_{K \setminus \bigcup_{k=1}^{n} U_k} = \mu_1|_{K \setminus \bigcup_{k=1}^{n} U_k}.$$

From (6.1), (6.2), and (6.3) we obtain the following properties of μ_1^1:

$$\|\mu_1\| - n\varepsilon_1 \leq \|\mu_1^1\| \leq \|\mu_1\| + n\varepsilon_1, \quad \left| \int f \, d\mu_1^1 - \int f \, d\mu_1 \right| \leq n\varepsilon_1 \tag{6.4}$$

and

$$\begin{aligned}
\|\mu_2|_{G_1} + \mu_1^1|_{G_1}\| &\geq \left\|\sum_{k=1}^{n}(\mu_2|_{U_k} + \nu_k)\right\| - (|\mu_2| + |\mu_1|)\Big(\bigcup_{k=1}^{n} C_k \setminus U_k\Big) \\
&\geq \sum_{k=1}^{n}(|\mu_2|(U_k) + |\mu_1|(U_k)) - 4n\varepsilon_1 \\
&\geq |\mu_1|(G_1) + |\mu_2|(G_1) - (4n+2)\varepsilon_1, \qquad (6.5)\\
\|T(\mu_1^1 - \mu_1)\| &= \left\|\sum_{k=1}^{n} T(\nu_k - \mu_1|_{U_k})\right\| \leq (n + n\|T\|)\varepsilon_1. \qquad (6.6)
\end{aligned}$$

Now define $\tilde{B}_2 = B_2 \setminus G_1$. Notice that $\tilde{B}_2$ is a set of negativity for μ_1^1 and a set of positivity for μ_2. Following the same lines as above we define $G_2 \supset \tilde{B}_2$ and construct $\mu_1^2 \in M(K)$ such that

$$|\mu_2|(G_2 \setminus \tilde{B}_2) < \varepsilon_1, \quad |\mu_1^1|(G_2 \setminus \tilde{B}_2) < \varepsilon_1, \quad \|\mu_1^1\| - n\varepsilon_1 \leq \|\mu_1^2\| \leq \|\mu_1^1\| + n\varepsilon_1$$

and

$$\begin{aligned}
\left|\int f\,d\mu_1^2 - \int f\,d\mu_1^1\right| &\leq n\varepsilon_1, \\
\|T(\mu_1^2 - \mu_1^1)\| &\leq (n + n\|T\|)\varepsilon_1, \\
\|\mu_2|_{G_2} + \mu_1^2|_{G_2}\| &\geq |\mu_1|(G_2) + |\mu_2|(G_2) - (4n+2)\varepsilon_1.
\end{aligned}$$

From (6.4), (6.5), (6.6) and the above inequalities we obtain the estimates

$$1 - 2n\varepsilon_1 \leq \|\mu_1^2\| \leq 1 - 2n\varepsilon_1, \qquad \left|\int f\,d\mu_1^2 - \int f\,d\mu_1\right| \leq 2n\varepsilon_1$$

and

$$\begin{aligned}
\|T(\mu_1^2 - \mu_1)\| &\leq (2n + 2n\|T\|)\varepsilon_1, \qquad (6.7)\\
\|\mu_2|_{G_1\cup G_2} + \mu_1^2|_{G_1\cup G_2}\| &\geq |\mu_1|(G_1 \cup G_2) + |\mu_2|(G_1 \cup G_2) - (8n+10)\varepsilon_1.
\end{aligned}$$

Finally, the definition of the sets B_3 and B_4 implies that

$$\|\mu_2 + \mu_1^2\| \geq \|\mu_1\| + \|\mu_2\| - (8n+10)\varepsilon_1 = 2 - (8n+10)\varepsilon_1. \qquad (6.8)$$

Hence for ε_1 small enough, the normalized signed measure $\nu = \mu_1^2/\|\mu_1^2\|$ satisfies all the required conditions, which completes the proof of the theorem. □

Applying this theorem to the operator $\mu \mapsto \mu|_Z$ yields by Theorem 5.8:

Corollary 6.3. *Let K be a perfect compact. A subspace $Z \subset C(K)$ is poor if and only if for every open subset $U \subset K$, for every two probability measures π_1, π_2 on U and for every $\varepsilon > 0$ there is a probability measure ν on U such that $\|\nu - \pi_1\|_{Z^*} < \varepsilon$ and $\|\pi_2 - \nu\| > 2 - \varepsilon$.*

For a closed subset K_1 of K denote by R_{K_1} the natural restriction operator $R_{K_1}\colon C(K) \to C(K_1)$. Note that for an operator $S\colon E \to F$ between Banach spaces the following assertions are equivalent, by the (proof of) the open mapping theorem: (i) S is onto; (ii) $S(B_E)$ is not nowhere dense; (iii) 0 is an interior point of $S(B_E)$.

Corollary 6.4. *Let K be a perfect compact Hausdorff space, $K_1 \subset K$ be a closed subset with non-empty interior, and let Z be a poor subspace of $C(K)$. Then $R_{K_1}(B_Z)$ is nowhere dense in $B_{C(K_1)}$.*

Proof. Apply Corollary 6.3 with $U = \operatorname{int} K_1$, $\pi_1 = \pi_2$ and a sufficiently small $\varepsilon > 0$ to see that $R_{K_1}(B_Z)$ cannot contain a ball $rB_{C(K_1)}$ of radius $r > 0$. □

We now deal with poor subspaces of L_1. Let (Ω, Σ, μ) be a finite measure space. Denote by Σ^+ the collection of all $A \in \Sigma$ with $\mu(A) > 0$.

Theorem 6.5. *Let (Ω, Σ, μ) be a non-atomic finite measure space. An operator T on $L_\infty := L_\infty(\Omega, \Sigma, \mu)$ is narrow with respect to $L_1 := L_1(\Omega, \Sigma, \mu)$ if and only if for every $\Delta \in \Sigma^+$ and for every $\varepsilon > 0$ there is $g \in S_{L_\infty}$ such that $g = 0$ off Δ and $\|Tg\| < \varepsilon$. Moreover, in the statement above g can be selected non-negative.*

Proof. First we prove the "if" part. By the definition of a narrow operator (Definition 4.3) for every $x, y \in S_{L_\infty}$, every $f \in S_{L_1}$ such that $\int f \cdot y \, d\lambda > 1 - \delta$ (i.e., $y \in S(f, \delta)$) and every $\varepsilon > 0$ we have to find $z \in S(f, \delta)$ such that $\|x + z\| > 2 - \varepsilon$ and $\|T(y - z)\| < \varepsilon$. By density of step functions we may assume without loss of generality that there is a partition $A_1, \dots, A_n$ of Ω such that the restrictions of x, y and f to A_k are constants, say a_k, b_k and c_k respectively. Fix some $\varepsilon_1 > 0$. Since $\|x\| = 1$, there exists k such that $|a_k| > 1 - \varepsilon_1$. Let $B \in \Sigma^+$ be a subset of A_k with $\mu(B) \le \varepsilon_1$ and $A_k \setminus B \in \Sigma^+$. By our assumption there exists $\hat{z} \in S_{L_\infty}$ such that $z \ge 0$, z is supported on B and $\|T(z)\| \le \varepsilon_1$. Denote $\tilde{z} = y + (\operatorname{sign}(a_1) - b_1)\hat{z}$. It is easy to see that $\|\tilde{z}\| = 1$, $\|x + \tilde{z}\| = 2 - \varepsilon_1$, $\|T(y - \tilde{z})\| < \varepsilon_1$ and $\tilde{z} \in S(f, \delta - \varepsilon_1)$. To finish the proof of this part it is sufficient to repeat the reasoning from the end of the proof of Theorem 6.2.

Now we consider the "only if" part. Since T is narrow with respect to L_1, T is also a strong Daugavet operator. Hence as in [10, Theorem 3.5] we can get a function $\tilde{g} \in S_{L_\infty}$ which satisfies all the requirements, except being non-negative. To fix this we argue as in [8, Lemma 1.4] and finally get some non-negative g possessing all the properties listed above. □

Again, specialising to the restriction operator $g \in L_\infty = (L_1)^* \mapsto g|_Z \in Z^*$ we obtain the following characterisation of poor subspaces.

Corollary 6.6. *Let (Ω, Σ, μ) be a non-atomic finite measure space. A subspace $Z \subset L_1(\Omega, \Sigma, \mu)$ is poor if and only if for every $\Delta \in \Sigma^+$ and for every $\varepsilon > 0$ there is $g \in S_{L_\infty}$ such that $g = 0$ off Δ and $\|g\|_{Z^*} < \varepsilon$. Moreover, in the statement above g can be selected non-negative.*

For a subset $A \in \Sigma^+$ denote by Q_A the natural restriction operator Q_A: $L_1(\Omega, \Sigma, \mu) \to L_1(A, \Sigma|_A, \mu|_A)$.

Corollary 6.7. *Let (Ω, Σ, μ) be a non-atomic finite measure space, $A \in \Sigma^+$ and let Z be a poor subspace of $L_1(\Omega, \Sigma, \mu)$. Then $Q_A(B_Z)$ is nowhere dense in $B_{L_1(A, \Sigma|_A, \mu)}$.*

Proof. Apply Corollary 6.6 with $\Delta = A$ and a sufficiently small $\varepsilon > 0$ to see that $Q_A(B_Z)$ cannot contain a ball $rB_{L_1(A)}$ of radius $r > 0$. □

The Corollaries 6.4 and 6.7 look very similar. The next definition extracts the significant common feature.

Definition 6.8. Let $X \in \mathrm{DPr}$. A subspace $E \subset X$ is said to be a *bank* if E contains an isomorphic copy of ℓ_1 and for every poor subspace Z of X, $q_E(B_Z)$ is nowhere dense in $B_{X/E}$ (here q_E denotes the natural quotient map $q_E \colon X \to X/E$). If $E \subset X$ is a bank, then $B_{X/E}$ will be called the *asset* of E.

In this terminology a poor subspace cannot cover a "significant part" of a bank's asset.

Theorem 6.9. *Let $X \in \mathrm{DPr}$ and $E \subset X$ be a bank with separable asset. Then X contains a copy of ℓ_1 which is not poor in X.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}} \subset \frac{1}{2}B_E$ be equivalent to the canonical basis of ℓ_1 and let $\{x_n\}_{n\in\mathbb{N}} \subset B_E$ be a sequence such that $\{q_E(x_n)\}_{n\in\mathbb{N}}$ is dense in $B_{X/E}$. Then, if one selects a sufficiently small $\varepsilon > 0$, the sequence of $u_n = e_n + \varepsilon x_n \in B_E$ is still equivalent to the canonical basis of ℓ_1, and the image of this sequence under q_E equals $\{\varepsilon q_E(x_n)\}_{n\in\mathbb{N}}$, which is dense in $\varepsilon B_{X/E}$. This means that the closed linear span of $\{u_n\}_{n\in\mathbb{N}}$ is the copy of ℓ_1 we need. □

The next theorem is an immediate corollary of Theorem 6.9.

Theorem 6.10. *In every $C(K)$-space with perfect metric compact K and in every separable $L_1(\Omega, \Sigma, \mu)$-space with non-atomic μ there is a subspace isomorphic to ℓ_1 that is not poor.*

Proof. Corollary 6.4 implies that if K is a perfect compact and $K_1 \subset K$ is a proper closed subset with non-empty interior, then $C_0(K \setminus K_1) := \{f \in C(K) \colon f(t) = 0 \ \forall t \in K_1\}$ is a bank with $B_{C(K_1)}$ being its asset. Corollary 6.7 implies that if (Ω, Σ, μ) is a non-atomic finite measure space and $A \in \Sigma^+$, then $L_1(\Omega \setminus A)$ is a bank with $B_{L_1(A)}$ being its asset. Separability of these assets follows from the separability of the spaces $C(K)$ and $L_1(\Omega, \Sigma, \mu)$ considered. It is left to apply Theorem 6.9. □

Theorem 6.10 answers Pełczyński's question mentioned in the introduction in the negative since it provides a non-poor ℓ_1-subspace $Z \subset L_1[0, 1]$. By definition this means that for some subspace $\tilde{Z} \subset Z$, $L_1[0, 1]/\tilde{Z}$ fails the Daugavet property; but Z has the RNP and so does its subspace $\tilde{Z}$. In fact, by Theorem 5.8 one can choose $\tilde{Z}$ of codimension ≤ 2, hence $\tilde{Z}$ is isomorphic to ℓ_1 as well. Let us some up these considerations.

Corollary 6.11. *There is a subspace $E \subset L_1[0, 1]$ that is isomorphic to ℓ_1 and hence has the RNP, but $L_1[0, 1]/E$ fails the Daugavet property.*

7. Some open questiions

1. Is it true that every separable space with the Daugavet property has an ℓ_1-subspace which is not poor?
2. Can the separability condition in Theorem 6.9 be omitted?
3. Is it true that every subspace without copies of ℓ_1 of a space with the Daugavet property is poor? We don't even know the answer in the case of $C[0, 1]$.

4. Is it true that if $X \in \mathrm{DPr}$ and $Y \subset X$ is a subspace with a separable dual, then the quotient space X/Y also has the Daugavet property? This question also appears in [13].

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