THE DAUGAVET PROPERTY FOR SPACES OF LIPSCHITZ FUNCTIONS

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Abstract. For a compact metric space K the space $\mathrm{Lip}(K)$ has the Daugavet property if and only if the norm of every $f \in \mathrm{Lip}(K)$ is attained locally. If K is a subset of an L_p-space, $1 < p < \infty$, this is equivalent to the convexity of K.

1. Introduction

A Banach space X is said to have the *Daugavet property* if

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{1.1}$$

for every rank-1 operator $T\colon X \to X$; then (1.1) also holds for all weakly compact operators on X and even all operators that do not fix copies of ℓ_1. The Daugavet property was introduced in [5] and further studied in [10] and [6], but examples of spaces having the Daugavet property have long been known; e.g., $C[0,1]$, $L_1[0,1]$, $L_\infty[0,1]$, the disk algebra, H^∞, etc.

In this paper we shall investigate the Daugavet property for spaces of Lipschitz functions. Throughout, (K, ρ) stands for a complete metric space that is not reduced to a singleton. The space of all Lipschitz functions on K will be equipped with the seminorm

$$\|f\| = \sup\left\{ \frac{|f(t_1) - f(t_2)|}{\rho(t_1, t_2)} \colon t_1 \neq t_2 \in K \right\}.$$

If one quotients out the kernel of this seminorm, i.e., the constant functions, one obtains the Banach space $\mathrm{Lip}(K)$, whose norm will also be denoted by $\|\,.\,\|$. Equivalently, one can fix a point $t_0 \in K$ and consider the Banach space $\mathrm{Lip}_0(K)$ consisting of all Lipschitz functions on K that vanish at t_0, with the Lipschitz constant as an actual norm. It is easily seen that $\mathrm{Lip}(K)$ and $\mathrm{Lip}_0(K)$ are isometrically isomorphic. In this paper we prefer the first point of view, but will refer to the elements of $\mathrm{Lip}(K)$ as functions rather than equivalence classes, as is familiar with L_p-spaces.

Since $\mathrm{Lip}[0,1]$ is isometric to $L_\infty[0,1]$ via differentiation almost everywhere, it is clear that $\mathrm{Lip}[0,1]$ has the Daugavet property. On the other hand the Hölder space $H^\alpha[0,1]$, being the dual of a space with the RNP [13,

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p. 83], fails the Daugavet property by the results of [16]; $H^\alpha[0,1]$ is just the Lipschitz space for $K = [0,1]$ with the metric $\rho_\alpha(s,t) = |s-t|^\alpha$. But for the unit square $Q = [0,1] \times [0,1]$ with the Euclidean metric it is far from obvious whether the Daugavet property holds for $\mathrm{Lip}(Q)$; in fact, this will turn out to be true as a special case of Theorem 3.1 below. The validity of the Daugavet property of $\mathrm{Lip}(Q)$ was asked in [15].

Whereas for the "classical" function spaces the validity of the Daugavet property is equivalent to a nonatomicity condition ([3] for $C(K)$ and $L_1(\mu)$, [16] for function algebras, [14] for L_1-preduals and [8] for the noncommutative case), in the setting of Lipschitz spaces it is a locality condition that plays a similar role, for in Theorem 3.3 we will show for a compact metric space K that the Daugavet property of $\mathrm{Lip}(K)$ is equivalent to the fact that every Lipschitz function on K almost attains its norm at close-by points; see Definition 2.2(a) for precision. We also characterise compact "local" metric spaces by a condition that is reminiscent of metric convexity (Proposition 2.8) and is sometimes even equivalent to it, e.g., for compact subsets of L_p, $1 < p < \infty$ (Proposition 2.9). As a result, for a compact subset of L_p, $1 < p < \infty$, the Daugavet property of $\mathrm{Lip}(K)$ is equivalent to the convexity of K.

An important tool to construct Lipschitz functions is McShane's extension theorem saying that if $M \subset K$ and $f\colon M \to \mathbb{R}$ is a Lipschitz function, then there is an extension to a Lipschitz function $F\colon K \to \mathbb{R}$ with the same Lipschitz constant; see [1, p. 12/13]. This will be used several times.

We will also make use of the following geometric characterisations of the Daugavet property from [5] and [2]. Part (iii) is particularly useful when one doesn't have full access to the dual space. As for notation, we denote the closed unit ball (resp. sphere) of a Banach space X by B_X (resp. S_X) and the closed ball with centre t and radius r in a metric space K by $B_K(t,r)$.

Lemma 1.1. *The following assertions are equivalent:*

(i) *X has the Daugavet property.*

(ii) *For every $y \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$ there exists some $x \in S_X$ such that $x^*(x) \ge 1-\varepsilon$ and $\|x+y\| \ge 2-\varepsilon$.*

(iii) *For every $\varepsilon > 0$ and for every $y \in S_X$ the closed convex hull of the set $\{u \in (1+\varepsilon)B_X\colon \|y+u\| \ge 2-\varepsilon\}$ contains S_X.*

2. Local metric spaces

Let us recall that a metric space K is called *metrically convex* if for any two points $t_1, t_2 \in K$ two closed balls $B_K(t_1, r_1)$ and $B_K(t_2, r_2)$ intersect if and only if $\rho(t_1,t_2) \le r_1 + r_2$.

Clearly, convex subsets of normed spaces are metrically convex, and $S^1 = \{(x,y) \in \mathbb{R}^2\colon x^2+y^2 = 1\}$ is metrically convex for the geodesic metric, but not for the Euclidean metric.

We shall need the following lemma.

Lemma 2.1. *A complete metric space K is metrically convex if and only if for every two distinct points $t, \tau \in K$ there is an isometric embedding ϕ: $[0, a] \to K$ (where $a = \rho(t, \tau)$) such that $\phi(0) = t$, $\phi(a) = \tau$. In other words, K is metrically convex if and only if every two points of K can be connected by an isometric copy of a linear segment.*

Proof. The property displayed in the lemma clearly implies the metric convexity of K. To prove the converse, let K be metrically convex and let t and τ be two points at a distance a; we shall label them t_0 and t_a. Then there is a point $t_{a/2} \in B_K(t_0, a/2) \cap B_K(t_a, a/2)$. It follows that $\rho(t_0, t_{a/2}) = \rho(t_{a/2}, t_a) = a/2$. Likewise, pick points $t_{a/4} \in B_K(t_0, a/4) \cap B_K(t_{a/2}, a/4)$ and $t_{3/4 \cdot a} \in B_K(t_{a/2}, a/4) \cap B_K(t_a, a/4)$. Continuing in this manner, one obtains for each dyadic rational $d \in [0, 1]$ a point $t_{da} \in K$ such that $\rho(t_{da}, t_{d'a}) = |d - d'|a$. The mapping $da \mapsto t_{da}$ can now be extended to an isometric mapping ϕ: $[0, a] \to K$, as requested. $\square$

The following definition is crucial for this paper.

Definition 2.2. Let K be a metric space.

(a) The space K is called *local* if for every $\varepsilon > 0$ and for every function $f \in \mathrm{Lip}(K)$ there are two distinct points $\tau_1, \tau_2 \in K$ such that $\rho(\tau_1, \tau_2) < \varepsilon$ and

$$\frac{f(\tau_2) - f(\tau_1)}{\rho(\tau_1, \tau_2)} > \|f\| - \varepsilon. \tag{2.1}$$

(b) Let $f \in \mathrm{Lip}(K)$ and $\varepsilon > 0$. A point $t \in K$ is said to be an *ε-point* of f if in every neighbourhood $U \subset K$ of t there are two points $\tau_1, \tau_2 \in U$ for which (2.1) holds true.

(c) The space K is called *spreadingly local* if for every $\varepsilon > 0$ and for every function $f \in \mathrm{Lip}(K)$ there are infinitely many ε-points of f.

The next proposition provides a large class of examples.

Proposition 2.3. *A metrically convex complete metric space K is spreadingly local.*

Proof. Fix an $\varepsilon > 0$ and a function $f \in \mathrm{Lip}(K)$ with $\|f\| = 1$. Select $t, \tau \in K$ with $\rho(t, \tau) > 0$ such that

$$f(\tau) - f(t) > (1 - \varepsilon)\rho(t, \tau).$$

Denote $a = \rho(t, \tau)$ and apply Lemma 2.1 to this pair of points. The function $F = f \circ \phi$: $[0, a] \to \mathbb{R}$, where ϕ is from Lemma 2.1, is 1-Lipschitz. Hence $|F'| \le 1$ a.e. on $[0, a]$ and

$$\int_0^a F'(r)\, dr = f(\tau) - f(t) > (1 - \varepsilon)a.$$

Therefore there are infinitely many points $r_i \in [0, a]$ with $F'(r_i) > 1 - \varepsilon$. Let us show that every point of the form $t_i = \phi(r_i)$ is an ε-point of f. By the definition of the derivative we have

$$\frac{F(r_i+\delta_i)-F(r_i)}{\delta_i} > 1-\varepsilon.$$

for sufficiently small $\delta_i \in (0,\varepsilon)$. Denote $\tau_i = \phi(r_i+\delta_i)$. Then $\rho(t_i,\tau_i) = \delta_i$ and $f(\tau_i) - f(t_i) > (1-\varepsilon)\delta_i$. □

Actually this proposition applies to a slightly more general class of spaces K, defined by the requirement that for each pair of points $t,\tau \in K$ and each $\eta > 0$ there exists a curve of length $\le \rho(t,\tau)+\eta =: a_\eta$ joining t and τ. In other words, there exists a 1-Lipschitz mapping (having arclength as parameter) $\phi\colon [0,a_\eta] \to K$ with $\phi(0)=t$, $\phi(a_\eta)=\tau$. Such spaces could be termed *almost metrically convex*. A variant of the above proof then shows that almost metrically convex spaces are spreadingly local.

Example 2.4. There is a (noncompact) almost metrically convex space that is not metrically convex. Indeed, let

$$M = \{f \in L_1[0,1]\colon |f| = 1 \text{ a.e.}\};$$

this is a closed subset of L_1. Instead of the L_1-norm we shall use the following equivalent norm on L_1. Pick a total sequence of functionals $x_n^* \in S_{L_\infty}$ and put, for $f \in L_1$,

$$|||f||| = \|f\|_{L_1} + \left(\sum_{n=1}^\infty 2^{-n}|x_n^*(f)|^2\right)^{1/2}.$$

This norm is strictly convex. It follows that M, equipped with the metric $\rho(f,g) = |||f-g|||$, is not metrically convex since it is not convex; indeed, if $f,g \in M$, then *no* nontrivial convex combination belongs to M (unless $f=g$).

On the other hand, (M,ρ) is almost metrically convex. To see this let $f \ne g$ be two functions in M. For a Borel set $A \subset [0,1]$ define $h_A \in M$ by

$$h_A = f\chi_A + g\chi_{[0,1]\setminus A}.$$

Given $\varepsilon > 0$, pick $\varepsilon' \le \varepsilon|||f-g|||$ and $N \in \mathbb{N}$ such that $2(\sum_{n>N} 2^{-n})^{1/2} \le \varepsilon'$. Define a nonatomic vector measure taking values in $\mathbb{R}^{N+1}$ by

$$\mu(A) = \bigl(\textstyle\int_A |f-g|, x_1^*((f-g)\chi_A), \dots, x_N^*((f-g)\chi_A)\bigr).$$

By the Lyapunov convexity theorem [9, Th. 5.5] there exists a Borel set Δ such that $\mu(\Delta) = \frac{1}{2}\mu([0,1])$. We then have, since $g - h_\Delta = (g-f)\chi_\Delta$

$$\begin{aligned} |||g-h_\Delta||| &= \|g-h_\Delta\|_{L_1} + \left(\sum_{n=1}^\infty 2^{-n}|x_n^*(g-h_\Delta)|^2\right)^{1/2} \\ &\le \int_\Delta |f-g| + \left(\sum_{n=1}^N 2^{-n}|x_n^*((f-g)\chi_\Delta)|^2\right)^{1/2} + \varepsilon' \end{aligned}$$

$$= \frac{1}{2}\int_0^1 |f-g| + \frac{1}{2}\Bigl(\sum_{n=1}^{N} 2^{-n}|x_n^*(f-g)|^2\Bigr)^{1/2} + \varepsilon'$$
$$\le \frac{1}{2}|||f-g||| + \varepsilon' \le \Bigl(\frac{1}{2}+\varepsilon\Bigr)|||f-g|||$$

and likewise

$$|||f-h_\Delta||| \le \Bigl(\frac{1}{2}+\varepsilon\Bigr)|||f-g|||.$$

Let $F_0 = f$, $F_1 = g$, $F_{1/2} = h_\Delta$. Now we reiterate the above construction, first applying it to F_0, $F_{1/2}$ and $\varepsilon/2$ and then to $F_{1/2}$, F_1 and $\varepsilon/2$ to obtain functions $F_{1/4}, F_{3/4} \in M$ such that

$$\max\{|||F_0 - F_{1/4}|||, |||F_{1/2} - F_{1/4}|||\} \le \Bigl(\frac{1}{2}+\frac{\varepsilon}{2}\Bigr)|||F_0 - F_{1/2}|||,$$
$$\max\{|||F_{1/2} - F_{3/4}|||, |||F_1 - F_{1/4}|||\} \le \Bigl(\frac{1}{2}+\frac{\varepsilon}{2}\Bigr)|||F_1 - F_{1/2}|||.$$

Continuing in this manner, we can assign to each dyadic rational $d \in [0,1]$ a function $F_d \in M$ such that the curve $[0,1] \to M$, $t \mapsto F_t$, obtained from this by continuous extension, has a length that can be estimated from above by

$$\sup_n \Bigl(\frac{1}{2}+\frac{\varepsilon}{2^{n-1}}\Bigr)\Bigl(\frac{1}{2}+\frac{\varepsilon}{2^{n-2}}\Bigr)\cdots\Bigl(\frac{1}{2}+\varepsilon\Bigr)2^n \le \exp(2^{2-n}\varepsilon + \cdots + 2\varepsilon) \le e^{4\varepsilon}.$$

Therefore M is almost metrically convex.

We will need a lemma in order to control the Lipschitz constant of a function by the Lipschitz constant of some restriction under highly technical assumptions that we shall meet later. In the following, $\sqcup$ is used to indicate a disjoint union.

Lemma 2.5. *Let $A = B \sqcup C$ be a metric space, $r \in (0, 1/4]$, $\delta < r^2/16$, $\rho(B,C) > r$. Suppose $\tilde{C} \subset C$ is a δ-net of C such that every two points of $\tilde{C}$ are at least r-distant, and let $f\colon A \to \mathbb{R}$ be a function that is 1-Lipschitz on $B \sqcup \tilde{C}$ and also 1-Lipschitz on every ball $B_A(t,\delta)$ for $t \in \tilde{C}$. Then f is $(1 + r/2)$-Lipschitz on the whole space A.*

Proof. Consider arbitrary points $s_1 \ne s_2 \in A$. We have to prove that

$$\left|\frac{f(s_2)-f(s_1)}{\rho(s_1,s_2)}\right| \le 1 + r/2. \tag{2.2}$$

We have to distinguish three cases: firstly, when $s_1, s_2 \in B$; secondly, when $s_1, s_2 \in C$; and thirdly, when one of the points (say, s_1) belongs to B and the other belongs to C.

In the first case (2.2) holds true even with 1 on the right hand side by assumption on f. Consider the second case. If s_1, s_2 belong to the same ball of the form $B_A(t,\delta)$ for $t \in \tilde{C}$, then the job is likewise done. If not, let $t_1 \ne t_2 \in \tilde{C}$ be points such that $\rho(t_1, s_1) \le \delta$ and $\rho(t_2, s_2) \le \delta$. Then

$$\left|\frac{f(s_2)-f(s_1)}{\rho(s_1,s_2)}\right| \le \left|\frac{f(s_2)-f(t_2)}{\rho(s_1,s_2)}\right| + \left|\frac{f(t_2)-f(t_1)}{\rho(s_1,s_2)}\right| + \left|\frac{f(t_1)-f(s_1)}{\rho(s_1,s_2)}\right|$$

$$
\begin{aligned}
&\le \frac{\delta}{\rho(s_1,s_2)} + \frac{\rho(t_2,t_1)}{\rho(s_1,s_2)} + \frac{\delta}{\rho(s_1,s_2)} \\
&\le \frac{2\delta}{r-2\delta} + \frac{\rho(t_2,t_1)}{\rho(t_2,t_1)-2\delta} \\
&\le \frac{2\delta}{r-2\delta} + 1 + \frac{2\delta}{\rho(t_2,t_1)-2\delta} \\
&\le 1 + \frac{4\delta}{r-2\delta} \le 1 + r/2.
\end{aligned}
$$

In the last case find $t_2 \in \tilde{C}$ such that $\rho(t_2, s_2) \le \delta$. Then

$$
\begin{aligned}
\left|\frac{f(s_2)-f(s_1)}{\rho(s_1,s_2)}\right| &\le \left|\frac{f(s_2)-f(t_2)}{\rho(s_1,s_2)}\right| + \left|\frac{f(t_2)-f(s_1)}{\rho(s_1,s_2)}\right| \\
&\le \frac{\delta}{\rho(s_1,s_2)} + \frac{\rho(t_2,s_1)}{\rho(s_1,s_2)} \\
&\le \frac{\delta}{r} + \frac{\rho(t_2,s_1)}{\rho(t_2,s_1)-\delta} \\
&\le \frac{\delta}{r} + \frac{r}{r-\delta} = 1 + \frac{\delta}{r} + \frac{\delta}{r-\delta} \le 1 + r/2.
\end{aligned}
$$

This completes the proof of the lemma. □

Obviously, a spreadingly local space is local. In the compact case the converse is valid, too, as will be pointed out now.

Lemma 2.6. *If K is compact and local, then it is spreadingly local.*

Proof. We will prove by induction on n that for every $f \in \mathrm{Lip}(K)$ and for every $\varepsilon > 0$ there are n ε-points of f.

Thanks to the compactness of K every function $f \in \mathrm{Lip}(K)$ has a "0-point", i.e., a point that is an ε-point for every $\varepsilon > 0$. Indeed, take a sequence of pairs $t_n, \tau_n \in K$ satisfying Definition 2.2 with $\varepsilon = 1/n$, $n = 1, 2, \dots$, and take an arbitrary limit point of (t_n). So the start of the induction holds true. Now assume the statement for a fixed n and let us prove it for $n+1$.

Take an $f \in \mathrm{Lip}(K)$ with $\|f\| = 1$ and $\varepsilon \in (0, 1/4]$. Due to our hypothesis there are ε-points $t_1, \dots, t_n$ of f. Also, select two points $\tau_1, \tau_2 \in K$ distinct from all the t_i and such that

$$
\frac{f(\tau_2)-f(\tau_1)}{\rho(\tau_1,\tau_2)} > 1 - \varepsilon/4.
$$

Let $r \in (0, \varepsilon/4]$ be a number so small that the balls $U_i = B_K(t_i, r)$, $i = 1, \dots, n$, are disjoint and contain neither τ_1 nor τ_2. Fix a $\delta < r^2/16$, denote the interior of $B_K(t_i, \delta)$ by V_i and consider $\tilde{K} = (K \setminus \bigcup_{i=1}^n U_i) \sqcup \bigcup_{i=1}^n V_i$ as a subspace of the metric space K. Define $\tilde{f}\colon \tilde{K} \to \mathbb{R}$ as follows: $\tilde{f}(t) = f(t)$ for $t \in K \setminus \bigcup_{i=1}^n U_i$ and $\tilde{f}(t) = f(t_i)$ on the corresponding V_i. Lemma 2.5 implies that $\tilde{f}$ satisfies a Lipschitz condition on $\tilde{K}$ with the constant $1 + \varepsilon/2$.

Extend $\tilde{f}$ to a function on K preserving the Lipschitz constant, still denoted by $\tilde{f}$.

Take as t_{n+1} an arbitrary 0-point of the function $g = f + \tilde{f}$. Since

$$\|g\| \geq \frac{g(\tau_2) - g(\tau_1)}{\rho(\tau_1, \tau_2)} = 2\frac{f(\tau_2) - f(\tau_1)}{\rho(\tau_1, \tau_2)} > 2 - \varepsilon/2,$$

in every neighbourhood of t_{n+1} there are points s_1, s_2 with

$$\frac{f(s_2) - f(s_1)}{\rho(s_1, s_2)} + \frac{\tilde{f}(s_2) - \tilde{f}(s_1)}{\rho(s_1, s_2)} > 2 - \varepsilon/2. \tag{2.3}$$

This implies that t_{n+1} cannot belong to any V_i since in V_i the second fraction of (2.3) is zero, but the first one is not greater than 1; hence t_{n+1} differs from all the other t_i. On the other hand, by our construction $\|\tilde{f}\| \leq 1+\varepsilon/2$, so the second fraction of (2.3) is $\leq 1 + \varepsilon/2$. Hence there is an estimate for the first fraction, namely

$$\frac{f(s_2) - f(s_1)}{\rho(s_1, s_2)} > 1 - \varepsilon,$$

which means that t_{n+1} is an ε-point of f. □

Next we are going to characterise local metric spaces intrinsically, at least in the compact case, using the following geometric property that we have chosen to give an ad-hoc name.

Definition 2.7. A metric space K has *property* (Z) if the following condition is met: Given $t, \tau \in K$ and $\varepsilon > 0$, there is some $z \in K \setminus \{t, \tau\}$ satisfying

$$\rho(t, z) + \rho(z, \tau) \leq \rho(t, \tau) + \varepsilon \min\{\rho(z, t), \rho(z, \tau)\}. \tag{2.4}$$

A compact space satisfying (2.4) with $\varepsilon = 0$ is easily seen to be metrically convex. Thus, property (Z) is "ε-close" to metric convexity, and there are instances when (Z) actually implies metric convexity; see Corollary 2.10 and Remark 2.11 below.

Here is the connection between locality and property (Z).

Proposition 2.8. *Let K be a metric space.*

(a) *If K is local, then K has property (Z).*

(b) *If K is compact and has property (Z), then K is local.*

Proof. (a) Assume that K fails property (Z), i.e., for some $t_0, \tau_0 \in K$ and $\varepsilon_0 > 0$ there are no points $z \in K \setminus \{t_0, \tau_0\}$ as in (2.4). For a point $z \in K$ let $r(z) = \rho(z, t_0)$, $s(z) = \rho(z, \tau_0)$ and $d = \rho(t_0, \tau_0)$. Pick $\varepsilon > 0$ with

$$\frac{\varepsilon}{1 - \varepsilon} < \frac{\varepsilon_0}{4}.$$

Now define $f\colon K \to \mathbb{R}$ by

$$f(z) = \begin{cases} \max\{d/2 - (1-\varepsilon)s(z), 0\} \\ \qquad \text{if } r(z) \ge s(z),\ r(z) + (1-2\varepsilon)s(z) \ge d, \\ -\max\{d/2 - (1-\varepsilon)r(z), 0\} \\ \qquad \text{if } r(z) \le s(z),\ (1-2\varepsilon)r(z) + s(z) \ge d. \end{cases}$$

This function is well defined, since for $r(z) = s(z)$ both parts of the definition yield 0, and all points of K are covered in the two "if" cases by our assumption on K; note that $2\varepsilon < \varepsilon_0$.

Let us show that f is a Lipschitz function with $\|f\| = 1$. Indeed, the only critical case is to estimate $f(z_2) - f(z_1)$ when $f(z_2) > 0$ and $f(z_1) < 0$; in this case

$$\begin{aligned} f(z_2) - f(z_1) &= \Big(\frac{d}{2} - (1-\varepsilon)s(z_2)\Big) + \Big(\frac{d}{2} - (1-\varepsilon)r(z_1)\Big) \\ &\le \Big(\frac{r(z_2) + (1-2\varepsilon)s(z_2)}{2} - (1-\varepsilon)s(z_2)\Big) \\ &\qquad + \Big(\frac{(1-2\varepsilon)r(z_1) + s(z_1)}{2} - (1-\varepsilon)r(z_1)\Big) \\ &= \frac{1}{2}(r(z_2) - s(z_2)) + \frac{1}{2}(s(z_1) - r(z_1)) \\ &\le \rho(z_1, z_2); \end{aligned}$$

also, the norm is attained at τ_0, t_0, i.e., $f(\tau_0) - f(t_0) = \rho(\tau_0, t_0)$.

Consider now points $z_1, z_2 \in K$ where

$$\frac{f(z_2) - f(z_1)}{\rho(z_2, z_1)} > 1 - \varepsilon; \tag{2.5}$$

we shall show that then z_1 is close to t_0 and z_2 is close to τ_0 so that their distance is necessarily big. Obviously, we must have $f(z_2) > 0$ and $f(z_1) < 0$ for (2.5) to subsist. In particular, we have

$$\rho(z_1, t_0) < \rho(z_1, \tau_0); \quad \rho(z_2, \tau_0) < \rho(z_2, t_0). \tag{2.6}$$

Hence

$$\begin{aligned} (1-\varepsilon)\rho(z_1, z_2) &< f(z_2) - f(z_1) \\ &= \Big(\frac{d}{2} - (1-\varepsilon)\rho(z_2, \tau_0)\Big) - \Big(\frac{d}{2} - (1-\varepsilon)\rho(z_1, t_0)\Big) \\ &= d - (1-\varepsilon)(\rho(z_2, \tau_0) + \rho(z_1, t_0)); \end{aligned}$$

in other words

$$(1-\varepsilon)(\rho(z_1, z_2) + \rho(z_2, \tau_0) + \rho(z_1, t_0)) < d$$

so that

$$\rho(z_k, t_0) + \rho(z_k, \tau_0) < \frac{d}{1-\varepsilon}, \quad k = 1, 2. \tag{2.7}$$

By our choice of ε_0, t_0, τ_0 and (2.6)

$$\rho(z_1, t_0) + \rho(z_1, \tau_0) \ge d + \varepsilon_0 \rho(z_1, t_0)$$

so that by (2.7)

$$d + \varepsilon_0 \rho(z_1, t_0) < \frac{d}{1-\varepsilon}$$

and hence $\rho(z_1, t_0) < d/4$ by our choice of ε. Likewise $\rho(z_2, \tau_0) < d/4$ and consequently $\rho(z_1, z_2) > d/2$. Therefore, K cannot be local.

(b) Assume that K is not local. Then there is a Lipschitz function f with $\|f\| = 1$ for which (2.1) is impossible for τ_1, τ_2 at small distance, viz. for $\rho(\tau_1, \tau_2) < \varepsilon$. By a compactness argument one hence deduces the existence of points $t, \tau \in K$ such that

$$\frac{f(\tau) - f(t)}{\rho(\tau, t)} = 1 \tag{2.8}$$

and $\rho(t, \tau)$ is minimal among all points as in (2.8). Now let $\varepsilon_n \searrow 0$ and apply condition (Z) to t, τ and ε_n. This yields a sequence of points $z_n \in K \setminus \{t, \tau\}$ such that

$$\rho(t, z_n) + \rho(z_n, \tau) \leq \rho(t, \tau) + \varepsilon_n \min\{\rho(z_n, t), \rho(z_n, \tau)\}. \tag{2.9}$$

Passing to a subsequence we may assume that (z_n) converges, say $z_n \to z_0$, and that without loss of generality

$$\rho(t, z_n) \leq \rho(\tau, z_n) \qquad \forall n \geq 1. \tag{2.10}$$

Note that

$$\rho(t, z_0) + \rho(z_0, \tau) = \rho(t, \tau). \tag{2.11}$$

If $z_0 \neq t$, then

$$\begin{aligned} 1 \geq \frac{f(z_0) - f(t)}{\rho(z_0, t)} &= \frac{f(\tau) - f(t)}{\rho(\tau, t)} \frac{\rho(\tau, t)}{\rho(z_0, t)} - \frac{f(\tau) - f(z_0)}{\rho(\tau, z_0)} \frac{\rho(z_0, \tau)}{\rho(z_0, t)} \\ &\geq \frac{\rho(\tau, t)}{\rho(z_0, t)} - \frac{\rho(z_0, \tau)}{\rho(z_0, t)} = 1 \end{aligned}$$

by (2.11), and thus f attains its norm at the pair z_0, t. But by (2.10)

$$\rho(t, z_0) \leq \frac{1}{2}(\rho(t, z_0) + \rho(\tau, z_0)) = \frac{1}{2}\rho(t, \tau),$$

which contradicts the minimality condition imposed on the pair t, τ.

Therefore, $z_n \to t$, and for sufficiently large n we have $\rho(t, z_n) < \varepsilon$ along with (2.9). But then

$$\begin{aligned} \frac{f(z_n) - f(t)}{\rho(z_n, t)} &= \frac{f(\tau) - f(t)}{\rho(\tau, t)} \frac{\rho(\tau, t)}{\rho(t, z_n)} - \frac{f(\tau) - f(z_n)}{\rho(\tau, z_n)} \frac{\rho(\tau, z_n)}{\rho(t, z_n)} \\ &\geq \frac{\rho(\tau, t) - \rho(\tau, z_n)}{\rho(t, z_n)} \geq 1 - \varepsilon \end{aligned}$$

by (2.9), which contradicts our choice of f, since $\rho(t, z_n) < \varepsilon$. □

The definition of locality immediately implies that a compact local space is connected; one just has to apply the definition with the indicator function of a set that is both open and closed. We will now present a class of compact metric spaces for which property (Z) and hence locality implies (metric) convexity. Recall that a Banach space $(E, \|\ .\ \|_E)$ is called *locally uniformly rotund* if for each $x \in S_E$ and $\eta > 0$ there is some $\delta = \delta_x(\eta) > 0$ such that $\|x - y\|_E \le \eta$ whenever $y \in B_E$ and $\|\frac{1}{2}(x + y)\|_E \ge 1 - \delta$.

Proposition 2.9. *Let $(E, \|\ .\ \|_E)$ be a smooth locally uniformly rotund Banach space and let $K \subset E$ be a compact subset with property (Z). Then K is convex.*

Proof. By a result of Vlasov ([12], [11, Th. 2.2, p. 368]) a compact Chebyshev subset of a smooth Banach space is convex. If we assume that K is not convex, this means that there are two points $P, Q \in K$ and a ball B whose interior does not intersect K with $P, Q \in \partial B$; we may assume that B is centred at the origin, $B = B_E(0, \alpha)$, and by scaling that $\|P - Q\|_E = 1$. Applying condition (Z) to P, Q and an arbitrary $\varepsilon > 0$ yields some $z = z(\varepsilon) \in K \setminus \{P, Q\}$ as in (2.4). We may as well assume that $z_0 = \lim_{\varepsilon\to 0} z(\varepsilon)$ exists; z_0 lies on the line segment $[P, Q]$ by strict convexity of E. Thus $z_0 = P$ or $z_0 = Q$; without loss of generality let us assume the latter. Fix, for the time being, ε and $z = z(\varepsilon)$ and put $r = \|z - Q\|_E$ $(< 1/2)$.

Now consider $Q(\lambda) = \lambda P + (1 - \lambda)Q$, $0 \le \lambda \le 1$. Let us estimate $\|z - Q(\lambda)\|_E$ in order to derive a contradiction. On the one hand we have, since $z \in K$ and thus $\|z\|_E \ge \alpha$,

$$\|z - Q(\lambda)\|_E \ge \|z\|_E - \|Q(\lambda)\|_E \ge \alpha - \|Q(\lambda)\|_E =: \varphi(\lambda).$$

Now φ is a concave function of λ with $\varphi(0) = 0$ and

$$\varphi(1/2) = \alpha - \left\|\frac{1}{2}(P + Q)\right\| > 0$$

by strict convexity. Hence with $\sigma = 2\varphi(1/2)$

$$\|z - Q(r)\|_E \ge \varphi(r) \ge \sigma r. \tag{2.12}$$

On the other hand, (2.4) means that $z \in B_E(P, 1 - r + \varepsilon r)$; therefore the point $w = \frac{1}{2}(z + Q(r))$ also belongs to this ball, but $w \notin \operatorname{int} B_E(Q, r - \varepsilon r)$. In other words,

$$\left\|\frac{(Q - z) + (Q - Q(r))}{2}\right\|_E = \left\|Q - \frac{z + Q(r)}{2}\right\|_E \ge r - \varepsilon r. \tag{2.13}$$

Specifically, let $\eta = \sigma/2$ and $0 < \varepsilon < \delta_{P-Q}(\eta)$. Then (2.13) and local uniform rotundity (note that $(Q - z)/r, (Q - Q(r))/r \in B_E$) imply that

$$\|z - Q(r)\|_E \le r\eta < r\sigma$$

contradicting (2.12). □

Proposition 2.9 applies in particular to L_p-spaces for $1 < p < \infty$ and most particularly to Hilbert spaces.

We can sum up the previous results as follows.

Corollary 2.10. *Let K be a compact metric space. Then the following are equivalent:*

(1) *K is local;*

(2) *K is spreadingly local;*
(3) *K has property (Z).*

If K is a subset of a smooth locally uniformly rotund Banach space, then a further equivalent condition is:

(4) *K is convex.*

Another link between locality and metric convexity is provided by the following technical remark.

Remark 2.11. Let us say that K satisfies (Z') if in addition to (2.4) in Definition 2.7 we require that

$$\rho(z,\tau) \le \rho(z,t).$$

Since one can exchange the roles of t and τ here, this means that there is one point as in (2.4) that is closer to τ than to t and another one that is closer to t than to τ. It is then possible to show that (Z') implies metric convexity for compact spaces; see below. Hence locality implies metric convexity for those compact metric spaces that are symmetric in the sense that for any two points in K there is an isometry on K swapping these two points.

To prove this remark, we rephrase property (Z') by saying that for every $\varepsilon > 0$ and every $t, \tau \in K$ there exists some $z \in K \setminus \{\tau\}$ such that

$$(1-\varepsilon)\rho(\tau,z) + \rho(t,z) \le \rho(t,\tau), \tag{2.14}$$

$$\rho(\tau,z) \le \rho(t,z). \tag{2.15}$$

The strategy of the proof will be to infer from this in the compact case that for every $\varepsilon > 0$ and every $t, \tau \in K$ there exists some $z \in K$ for which (2.14) holds and

$$\frac{1}{10}\rho(t,\tau) \le \rho(\tau,z) \le \frac{9}{10}\rho(t,\tau). \tag{2.16}$$

If we let $\varepsilon \to 0$ and consider a limit point z_0 of the $z = z(\varepsilon)$ satisfying (2.14) and (2.16), then we can be certain that $z_0 \ne t$ and $z_0 \ne \tau$, but

$$\rho(t,z_0) + \rho(z_0,\tau) = \rho(t,\tau). \tag{2.17}$$

As remarked earlier this implies the metric convexity of the compact space K.

Let us now come to the details. Fix t, τ and ε; we may suppose that $\rho(t,\tau) = 1$. Assume for a contradiction that we cannot achieve (2.14) and (2.16) simultaneously. Let

$$K_0 = \{z \in K\colon \text{(2.14) and (2.15) hold}\}.$$

Since $K_0 \ne \{\tau\}$ by property (Z'), there is some $u \in K_0$ such that $\rho(u,t) < 1$, and therefore $\alpha := \min\{\rho(z,t)\colon z \in K_0\}$ is attained at some $u_0 \in K_0 \setminus \{\tau\}$. Then $(1-\varepsilon)\rho(\tau,u_0) + \rho(u_0,t) \le 1$ by (2.14). Now define $0 \le \tilde\varepsilon \le \varepsilon$ by

$$(1-\tilde\varepsilon)\rho(\tau,u_0) + \rho(u_0,t) = 1. \tag{2.18}$$

If $\tilde\varepsilon = 0$, we have already found a point as in (2.17), and we are done. So we assume that $\tilde\varepsilon > 0$ in the sequel. Then we can apply (2.14) and (2.15), i.e., property (Z), with t, u_0 and $\tilde\varepsilon$ in place of t, τ and ε. This yields some $\tilde z \in K \setminus \{u_0\}$ with

$$(1-\tilde\varepsilon)\rho(u_0,\tilde z) + \rho(t,\tilde z) \le \rho(t,u_0), \tag{2.19}$$

$$\rho(u_0,\tilde z) \le \rho(t,\tilde z). \tag{2.20}$$

Next, add (2.18) and (2.19) to obtain

$$(1-\tilde\varepsilon)(\rho(\tau,u_0)+\rho(u_0,\tilde z)) + \rho(t,\tilde z) \le 1. \tag{2.21}$$

But $\rho(t,\tilde z) < \rho(t,u_0) = \alpha$, since $\tilde z \neq u_0$ in (2.19); hence $\tilde z \notin K_0$. Now the previous inequality, (2.21) and $\tilde\varepsilon \le \varepsilon$ show that $\tilde z$ satisfies (2.14); therefore it must fail (2.15), i.e.,

$$\rho(\tau,\tilde z) > \rho(t,\tilde z). \tag{2.22}$$

Also, recall that u_0 satisfies (2.14) and that we have assumed that (2.14) and (2.16) do not hold simultaneously. This implies that

$$\rho(\tau,u_0) < 1/10 \quad\text{or}\quad \rho(\tau,u_0) > 9/10$$

and

$$\rho(\tau,\tilde z) < 1/10 \quad\text{or}\quad \rho(\tau,\tilde z) > 9/10.$$

If $\rho(\tau,u_0) > 9/10$, then $\rho(t,u_0) > 9/10$ by (2.15); recall that $u_0 \in K_0$. Then (2.18) furnishes the contradiction

$$1 = (1-\tilde\varepsilon)\rho(\tau,u_0) + \rho(u_0,t) > (2-\tilde\varepsilon)\frac{9}{10} > 1$$

if, say, $\varepsilon \le 1/4$. The conclusion at this point is

$$\rho(\tau,u_0) < 1/10. \tag{2.23}$$

On the other hand, if $\rho(\tau,\tilde z) < 1/10$, then $\rho(t,\tilde z) > 9/10$ by the triangle inequality, which contradicts (2.22). Consequently

$$\rho(\tau,\tilde z) > 9/10. \tag{2.24}$$

If we now use that $\tilde z$ satisfies (2.19) and (2.20), we derive, for $\varepsilon \le 1/4$, that

$$\rho(u_0,\tilde z) \le \rho(t,\tilde z) \le 1 - (1-\varepsilon)\rho(\tau,\tilde z) \le \frac{13}{40}$$

and hence the contradiction

$$\rho(\tau,t) \le \rho(\tau,u_0) + \rho(u_0,\tilde z) + \rho(\tilde z,t) < 1.$$

This completes the proof of the remark.

We do not know any example of a compact space with (Z) that is not metrically convex.

3. Locality and the Daugavet property

We can now prove a sufficient criterion for $\mathrm{Lip}(K)$ to have the Daugavet property. In particular it turns out that for closed convex subsets of Banach spaces $\mathrm{Lip}(K)$ has the Daugavet property.

Theorem 3.1. *If K is a spreadingly local metric space (in particular if K is a metrically convex or a compact local metric space), then* $\mathrm{Lip}(K)$ *has the Daugavet property.*

Proof. For short write $X = \mathrm{Lip}(K)$. Due to Lemma 1.1 it is sufficient to prove that for every $\varepsilon \in (0, 1/4]$, and for every $f, g \in S_X$ the closed convex hull of the set $W = \{u \in (1+\varepsilon)B_X \colon \|f + u\| \geq 2 - \varepsilon\}$ contains g.

In order to do this fix an $n \in \mathbb{N}$ and select $\varepsilon/2$-points $s_1, \dots, s_n$ of f. Let $r \in (0, \varepsilon/4]$ be a number so small that the balls $U_i = B_K(s_i, r)$, $i = 1, \dots, n$, are disjoint. Fix a $\delta < r^2/16$, and select $t_i, \tau_i \in B_K(s_i, \delta)$ such that

$$f(\tau_i) - f(t_i) > (1 - \varepsilon/2)\rho(t_i, \tau_i). \tag{3.1}$$

Consider $K_i = (K \setminus U_i) \sqcup \{t_i, \tau_i\}$ as a subspace of the metric space K. Define $u_i \colon K_i \to \mathbb{R}$ as follows: $u_i(t_i) = g(t_i)$, $u_i(\tau_i) = g(t_i) + f(\tau_i) - f(t_i)$ and $u_i(s) = g(s)$ on the rest of K_i. It follows from Lemma 2.5 that u_i satisfies a Lipschitz condition on K_i with the constant $1 + r/2 < 1 + \varepsilon/2$. Extend u_i to a function on K preserving the Lipschitz constant, still denoted by u_i.

Note that each u_i belongs to W. In fact $\|u_i\| \leq 1 + \varepsilon$ by construction and

$$\|f + u_i\| \geq \frac{(f + u_i)(\tau_i) - (f + u_i)(t_i)}{\rho(\tau_i, t_i)} = 2\frac{f(\tau_i) - f(t_i)}{\rho(\tau_i, t_i)} > 2 - \varepsilon.$$

On the other hand the arithmetic mean of the u_i (the simplest convex combination) approximates g, for

$$\left\| g - \frac{1}{n}\sum_{i=1}^{n} u_i \right\| = \frac{1}{n}\left\| \sum_{i=1}^{n}(u_i - g) \right\| \leq \frac{4 + 2\varepsilon}{n}.$$

The last inequality follows from the fact that each $u_i - g$ has norm $\leq \|u_i\| + \|g\| \leq 2 + \varepsilon$ and their supports U_i are disjoint. □

Finally we address the question in how far our locality conditions are necessary for the Daugavet property; for compact spaces, this will turn out to be the case (Theorem 3.3 below). The bulk of the technical work will be done in the following lemma.

Lemma 3.2. *Suppose* $\mathrm{Lip}(K)$ *has the Daugavet property. Then for every $t_1, t_2 \in K$ with $\rho(t_1, t_2) = a > 0$, for every $f \in S_{\mathrm{Lip}(K)}$ with $f(t_2) - f(t_1) = a$ (i.e., f attains its norm at the pair t_1, t_2) and for every $\varepsilon > 0$ there are $\tau_1 = \tau_1(\varepsilon), \tau_2 = \tau_2(\varepsilon) \in K$ with the following properties:*

1. *$f(\tau_2) - f(\tau_1) \geq (1 - \varepsilon)\rho(\tau_1, \tau_2)$;*
2. *$\rho(t_1, \tau_2) - \rho(t_1, \tau_1) \geq (1 - \varepsilon)\rho(\tau_1, \tau_2)$,*
   $\rho(t_2, \tau_1) - \rho(t_2, \tau_2) \geq (1 - \varepsilon)\rho(\tau_1, \tau_2)$;

(3) $\rho(\tau_1, \tau_2) \to 0$ *as* $\varepsilon \to 0$.

Proof. We shall abbreviate $\mathrm{Lip}(K)$ by X. Consider the following functions $y_i \in X$:

$$y_1 = f,\ y_2(t) = \rho(t_1, t),\ y_3(t) = -\rho(t_2, t).$$

For all these functions we have

$$y_i(t_2) - y_i(t_1) = a,\ \|y_i\| = 1. \tag{3.2}$$

Then the arithmetic mean $y = (y_1+y_2+y_3)/3$ is of norm 1 as well. Consider $x^* \in X^*$, with the action

$$x^*(x) = \frac{1}{a}(x(t_2) - x(t_1)). \tag{3.3}$$

Clearly $\|x^*\| = 1$. Due to the Daugavet property of X there is, by Lemma 1.1, an $x \in S_X$ such that $x^*(x) > 1 - \varepsilon$, i.e.,

$$x(t_2) - x(t_1) > (1 - \varepsilon)a, \tag{3.4}$$

and at the same time $\|x - y\| > 2 - \varepsilon/3$. The last condition means that there are two distinct points $\tau_1, \tau_2 \in K$ for which

$$(x - y)(\tau_1) - (x - y)(\tau_2) > (2 - \varepsilon/3)\rho(\tau_1, \tau_2),$$

i.e.,

$$\frac{1}{3}\sum_{i=1}^{3}((x - y_i)(\tau_1) - (x - y_i)(\tau_2)) > (2 - \varepsilon/3)\rho(\tau_1, \tau_2).$$

Since neither of these three summands exceeds $2\rho(\tau_1, \tau_2)$, we get the following three inequalities:

$$(x - y_i)(\tau_1) - (x - y_i)(\tau_2) > (2 - \varepsilon)\rho(\tau_1, \tau_2), \quad i = 1, 2, 3. \tag{3.5}$$

Taking into account $x(\tau_1) - x(\tau_2) \le \rho(\tau_1, \tau_2)$ we deduce that

$$y_i(\tau_2) - y_i(\tau_1) > (1 - \varepsilon)\rho(\tau_1, \tau_2), \quad i = 1, 2, 3. \tag{3.6}$$

The case $i = 1$ gives us the requested property (1), and the cases $i = 2, 3$ of (3.6) immediately provide property (2). Finally, substituting the Lipschitz conditions $x(\tau_1) \le x(t_1) + \rho(t_1, \tau_1)$ and $x(\tau_2) \ge x(t_2) - \rho(t_2, \tau_2)$ into (3.5) and applying (3.4) we obtain

$$\begin{aligned}(2 - \varepsilon)\rho(\tau_1, \tau_2) &< x(t_1) - x(t_2) + \rho(t_1, \tau_1) + \rho(t_2, \tau_2) + y_i(\tau_2) - y_i(\tau_1)\\ &\le -(1 - \varepsilon)\rho(t_1, t_2) + \rho(t_1, \tau_1) + \rho(t_2, \tau_2) + \rho(\tau_1, \tau_2),\end{aligned}$$

so

$$\begin{aligned}(1 - \varepsilon)\rho(t_1, t_2) &< \rho(t_1, \tau_1) + \rho(t_2, \tau_2) - (1 - \varepsilon)\rho(\tau_1, \tau_2)\\ &\le (2 - \varepsilon)\left(\rho(t_1, \tau_1) + \rho(t_2, \tau_2)\right) - (1 - \varepsilon)\rho(t_1, t_2)\end{aligned}$$

by the triangle inequality; hence

$$2\rho(t_1, \tau_1) + 2\rho(t_2, \tau_2) > 4(1 - \varepsilon)/(2 - \varepsilon)\rho(t_1, t_2).$$

Adding to this inequality both inequalities from property (2) we obtain

$$\rho(t_1,\tau_1)+\rho(t_2,\tau_2)+\rho(t_1,\tau_2)+\rho(t_2,\tau_1)$$
$$\geq 4(1-\varepsilon)/(2-\varepsilon)\rho(t_1,t_2)+2(1-\varepsilon)\rho(\tau_1,\tau_2).$$

Since the left hand side is not greater than $2\rho(t_1,t_2)$ we deduce

$$2(1-\varepsilon)\rho(\tau_1,\tau_2)\leq\left(2-4\frac{1-\varepsilon}{2-\varepsilon}\right)\rho(t_1,t_2)$$

which gives property (3). □

We can now deduce the main theorem of this paper.

Theorem 3.3. *If K is a compact metric space, then* $\mathrm{Lip}(K)$ *has the Daugavet property if and only if K is local.*

Proof. The "if" part has already been proved in Theorem 3.1. Let us prove the "only if" part. Assume K is not local. Then there is a function $f\in\mathrm{Lip}(K)$, $\|f\|=1$, and there is an $r>0$ such that

$$f(\tau_2)-f(\tau_1)<(1-r)\rho(\tau_1,\tau_2) \tag{3.7}$$

for every $\tau_1,\tau_2\in K$ with $\rho(\tau_1,\tau_2)<r$. Hence by a compactness argument there is a pair of points $t_1,t_2\in K$ with $\rho(t_1,t_2)>0$ on which f attains its norm, i.e., with $f(t_2)-f(t_1)=\rho(t_1,t_2)$. If nevertheless $\mathrm{Lip}(K)$ has the Daugavet property, then applying Lemma 3.2 to f and these t_1,t_2 with $\varepsilon\to 0$ entails a contradiction between (3.7) and properties (1) and (3) from the lemma. □

The space $\mathrm{Lip}(K)$ has a canonical predual, called the Arens-Eells space in [13] and the Lipschitz free space in [4] and [7]. Since we have used in (3.3), in the proof of Lemma 3.2, a functional from that predual, i.e., a weak* open slice, the lemma works under the assumption that the Lipschitz free space on K has the Daugavet property. Consequently, for a compact metric space $\mathrm{Lip}(K)$ has the Daugavet property if and only if its Lipschitz free space has.

In the setting of subsets of certain Banach spaces like L_p, $1<p<\infty$, we can rephrase Theorem 3.3 as follows, using Corollary 2.10.

Corollary 3.4. *If K is a compact subset of a smooth locally uniformly rotund Banach space, then* $\mathrm{Lip}(K)$ *has the Daugavet property if and only if K is convex.*

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