Corrigendum to our paper
"The Daugavet property for spaces of Lipschitz functions"

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David Dubray has pointed out to us a serious gap in the proof of Lemma 3.2. In order to fix it, one has to modify the assertion of Lemma 3.2 in that τ_1 and τ_2 now also satisfy the inequality

$$\rho(t_1, \tau_j) + \rho(t_2, \tau_j) \le \frac{a}{1-\varepsilon} \quad (j = 1, 2).$$

To achieve this one introduces a fourth auxiliary function into the proof, viz.

$$y_4(t) = \frac{a}{2}\frac{\rho(t_1, t) - \rho(t_2, t)}{\rho(t_1, t) + \rho(t_2, t)}.$$

Here are the details. Please replace Lemma 3.2 by the following.

Lemma 3.2. *Suppose* Lip(K) *has the Daugavet property. Then for every* $t_1, t_2 \in K$ *with* $\rho(t_1, t_2) = a > 0$*, for every* $f \in S_{\mathrm{Lip}(K)}$ *with* $f(t_2) - f(t_1) = a$ *and for every* $\varepsilon > 0$ *there are* $\tau_1 = \tau_1(\varepsilon), \tau_2 = \tau_2(\varepsilon) \in K$ *with the following properties:*

(1) $f(\tau_2) - f(\tau_1) \ge (1-\varepsilon)\rho(\tau_1, \tau_2)$;
(2) $\rho(t_1, \tau_2) - \rho(t_1, \tau_1) \ge (1-\varepsilon)\rho(\tau_1, \tau_2)$,
$\rho(t_2, \tau_1) - \rho(t_2, \tau_2) \ge (1-\varepsilon)\rho(\tau_1, \tau_2)$;
(3) $\rho(\tau_1, \tau_2) \to 0$ *as* $\varepsilon \to 0$;
(4) $\rho(t_1, \tau_j) + \rho(t_2, \tau_j) \le a/(1-\varepsilon)$, $j = 1, 2$.

Proof. We shall abbreviate Lip(K) by X. Consider the following functions $y_i \in X$:

$$y_1 = f,\ y_2(t) = \rho(t_1, t),\ y_3(t) = -\rho(t_2, t),\ y_4(t) = \frac{a}{2}\frac{\rho(t_1, t) - \rho(t_2, t)}{\rho(t_1, t) + \rho(t_2, t)}.$$

For all these functions we have

$$y_i(t_2) - y_i(t_1) = a,\ \|y_i\| = 1 \tag{3.2}$$

(to check that $\|y_4\| \le 1$ one needs the Sublemma given below). Then the arithmetic mean $y = (y_1 + y_2 + y_3 + y_4)/4$ is of norm 1 as well. Consider $x^* \in X^*$, with the action

$$x^*(x) = \frac{1}{a}(x(t_2) - x(t_1)). \tag{3.3}$$

Clearly $\|x^*\| = 1$. Due to the Daugavet property of X there is, by Lemma 1.1, an $x \in S_X$ such that $x^*(x) > 1 - \varepsilon$, i.e.,

$$x(t_2) - x(t_1) > (1-\varepsilon)a, \tag{3.4}$$

and at the same time $\|x-y\| > 2-\varepsilon/4$. The last condition means that there are two distinct points $\tau_1, \tau_2 \in K$ for which

$$(x-y)(\tau_1) - (x-y)(\tau_2) > (2-\varepsilon/4)\rho(\tau_1,\tau_2),$$

i.e.,

$$\frac{1}{4}\sum_{i=1}^{4}((x-y_i)(\tau_1) - (x-y_i)(\tau_2)) > (2-\varepsilon/4)\rho(\tau_1,\tau_2).$$

Since neither of these four summands exceeds $2\rho(\tau_1,\tau_2)$, we get the following four inequalities:

$$(x-y_i)(\tau_1) - (x-y_i)(\tau_2) > (2-\varepsilon)\rho(\tau_1,\tau_2), \quad i = 1,2,3,4. \tag{3.5}$$

Taking into account $x(\tau_1) - x(\tau_2) \le \rho(\tau_1,\tau_2)$ we deduce that

$$y_i(\tau_2) - y_i(\tau_1) > (1-\varepsilon)\rho(\tau_1,\tau_2), \quad i = 1,2,3,4. \tag{3.6}$$

The case $i = 1$ gives us the requested property (1). The case $i = 4$ yields the inequality

$$\frac{\rho(t_1,\tau_2)-\rho(t_2,\tau_2)}{\rho(t_1,\tau_2)+\rho(t_2,\tau_2)} - \frac{\rho(t_1,\tau_1)-\rho(t_2,\tau_1)}{\rho(t_1,\tau_1)+\rho(t_2,\tau_1)} > \frac{2}{a}(1-\varepsilon)\rho(\tau_1,\tau_2),$$

but the estimate $(*)$ of the following Sublemma says that

$$\begin{aligned}\frac{\rho(t_1,\tau_2)-\rho(t_2,\tau_2)}{\rho(t_1,\tau_2)+\rho(t_2,\tau_2)} &- \frac{\rho(t_1,\tau_1)-\rho(t_2,\tau_1)}{\rho(t_1,\tau_1)+\rho(t_2,\tau_1)} \\ &\le \frac{2\rho(\tau_1,\tau_2)}{\max\{\rho(t_1,\tau_1)+\rho(t_2,\tau_1), \rho(t_1,\tau_2)+\rho(t_2,\tau_2)\}}.\end{aligned}$$

These inequalities imply property (4).

The cases $i = 2, 3$ of (3.6) immediately provide property (2). Finally, substituting the Lipschitz conditions $x(\tau_1) \le x(t_1) + \rho(t_1,\tau_1)$ and $x(\tau_2) \ge x(t_2) - \rho(t_2,\tau_2)$ into (3.5) and applying (3.4) we obtain

$$\begin{aligned}(2-\varepsilon)\rho(\tau_1,\tau_2) &< x(t_1) - x(t_2) + \rho(t_1,\tau_1) + \rho(t_2,\tau_2) + y_i(\tau_2) - y_i(\tau_1) \\ &\le -(1-\varepsilon)\rho(t_1,t_2) + \rho(t_1,\tau_1) + \rho(t_2,\tau_2) + \rho(\tau_1,\tau_2),\end{aligned}$$

so

$$\begin{aligned}(1-\varepsilon)\rho(t_1,t_2) &< \rho(t_1,\tau_1) + \rho(t_2,\tau_2) - (1-\varepsilon)\rho(\tau_1,\tau_2) \\ &\le (2-\varepsilon)\left(\rho(t_1,\tau_1) + \rho(t_2,\tau_2)\right) - (1-\varepsilon)\rho(t_1,t_2)\end{aligned}$$

by the triangle inequality; hence

$$2\rho(t_1,\tau_1) + 2\rho(t_2,\tau_2) > 4(1-\varepsilon)/(2-\varepsilon)\rho(t_1,t_2).$$

Adding to this inequality both inequalities from property (2) we obtain

$$\begin{aligned}&\rho(t_1,\tau_1) + \rho(t_2,\tau_2) + \rho(t_1,\tau_2) + \rho(t_2,\tau_1) \\ &\qquad \ge 4(1-\varepsilon)/(2-\varepsilon)\rho(t_1,t_2) + 2(1-\varepsilon)\rho(\tau_1,\tau_2).\end{aligned}$$

Since by (4) the left hand side is not greater than $2\rho(t_1,t_2)/(1-\varepsilon)$ we deduce

$$2(1-\varepsilon)\rho(\tau_1,\tau_2) \le \left(\frac{2}{(1-\varepsilon)} - 4\frac{1-\varepsilon}{2-\varepsilon}\right)\rho(t_1,t_2)$$

which gives property (3). □

Sublemma. *For arbitrary positive numbers u_1, v_1, u_2, v_2 the function q defined by $q(u,v) = (u-v)/(u+v)$ satisfies the condition*

$$|q(u_1,v_1) - q(u_2,v_2)| \le 2\frac{\max\{|u_1-u_2|,|v_1-v_2|\}}{\max\{|u_1+v_1|,|u_2+v_2|\}}. \qquad (*)$$

Proof. We have

$$\begin{aligned} |q(u_1,v_1) - q(u_2,v_2)| &= \left|\frac{2u_1v_2 - 2u_2v_1}{(u_1+v_1)(u_2+v_2)}\right| \\ &= 2\left|\frac{(u_1-u_2)v_2 + u_2(v_2-v_1)}{(u_1+v_1)(u_2+v_2)}\right| \\ &\le 2\frac{\max\{|u_1-u_2|,|v_1-v_2|\}}{u_1+v_1}. \end{aligned}$$

Changing the roles of the pairs (u_1,v_1) and (u_2,v_2) one obtains

$$|q(u_1,v_1) - q(u_2,v_2)| \le 2\frac{\max\{|u_1-u_2|,|v_1-v_2|\}}{u_2+v_2},$$

which together with the previous inequality gives $(*)$. □

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