THE DAUGAVET EQUATION FOR BOUNDED VECTOR VALUED FUNCTIONS

STEFAN BRACH, ENRIQUE A. SÁNCHEZ PÉREZ AND DIRK WERNER

Abstract. Requirements under which the Daugavet equation and the alternative Daugavet equation hold for pairs of nonlinear maps between Banach spaces are analysed. A geometric description is given in terms of nonlinear slices. Some local versions of these properties are also introduced and studied, as well as tests for checking if the required conditions are satisfied in relevant cases.

1. Introduction

I.K. Daugavet [7] proved his eponymous equation in 1963 which establishes the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

for a compact linear operator $T : C[0,1] \to C[0,1]$. This equation was extended to more general classes of linear operators on various spaces over the years. Nowadays investigations on this topic build on the approach of V. Kadets et al. [10] who defined a Banach space X to have the Daugavet property if all rank-1 operators on X satisfy the Daugavet equation. This property can conveniently be characterised in terms of slices of the unit ball, and it can be shown that on a space with the Daugavet property all weakly compact operators and all operators not fixing a copy of ℓ_1 satisfy the Daugavet equation; see [1], [10], [11] or [15].

The Daugavet equation has been extended in a number of other ways as well, replacing the identity operator by a more general reference operator called a Daugavet centre ([3], [4]) or replacing the linear operators T by nonlinear ones ([6], [13], [8]). Here we attempt to combine both these ideas. We study the equation

$$\|\Phi + \Psi\| = \|\Phi\| + \|\Psi\|$$

where Φ and Ψ are bounded maps on the unit ball of some Banach space X having values in some (possibly different) Banach space Y and Ψ is in some sense small with respect to Φ, the norm being the sup norm. Also, the so-called alternative Daugavet equation

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = \|\Phi\| + \|\Psi\|$$

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will be considered. We are going to investigate these equations by means of suitable modifications of the notion of slice continuity introduced in [14]; cf. Definition 3.1 below. We also rely on some techniques from [6] and [13].

The paper is organised as follows. After the preliminary Section 2, we study the Φ-Daugavet equation in the third section, giving complete characterisations using the notion of strong slice continuity introduced below. Likewise, we introduce weak slice continuity in order to deal with the alternative Daugavet equation in Section 4. Finally, Section 5 is devoted to some technical local versions of these Daugavet type properties which are obtained by considering suitable subsets of the ones appearing in the definitions studied before. Some tests that guarantee that the requirements in our main theorems are satisfied are also presented. In particular, examples show their usefulness, especially for the cases of $C(K)$-spaces and $L^1(\mu)$-spaces.

Let us introduce some fundamental definitions and notation. We will write $\mathbb{T}$ for the set of scalars of modulus 1; the field of scalars can be $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We write $\operatorname{Re}\omega$ for the real part, $\operatorname{Im}\omega$ for the imaginary part and $\overline{\omega}$ for the complex conjugate of ω. For a Banach space X, B_X is its closed unit ball, U_X its open unit ball and S_X its unit sphere, and we will denote by X^* its dual space. If L is a Banach lattice, we use the symbol L^+ to denote the positive cone, and B_{L^+} for the set $B_L \cap L^+$. $L(X, Y)$ denotes the space of continuous linear operators from X to Y.

For a bounded mapping $\Phi : B_X \to Y$, we define its norm to be the sup norm, i.e.,

$$\|\Phi\| := \sup_{x \in B_X} \|\Phi(x)\|;$$

the space of all such mappings is denoted by $\ell_\infty(B_X, Y)$. In the scalar case an element of $\ell_\infty(B_X)$ is typically denoted by x'. The symbol $x' \otimes y$ stands for the mapping $x \mapsto x'(x)y$.

Our main characterizations are given in terms of slices. A slice $S(x^*, \varepsilon)$ of B_X defined by a norm one element $x^* \in X^*$ and an $\varepsilon > 0$ is defined by

$$S(x^*, \varepsilon) = \{x \in B_X : \operatorname{Re} x^*(x) \geq 1 - \varepsilon\}.$$

When a nonlinear scalar-valued function is considered, the same definition makes sense; if $p : X \to \mathbb{K}$ is a function with norm ≤ 1, we write

$$S(p, \varepsilon) = \{x \in B_X : \operatorname{Re} p(x) \geq 1 - \varepsilon\}.$$

Note that in this case it may happen that $S(p, \varepsilon) = \emptyset$.

2. Preliminaries

In this section, we prove fundamental characterisations of the Daugavet and the alternative Daugavet equation. The theorems in this section are adapted from results in [6] and [14].

Definition 2.1. Let X, Y be Banach spaces and let $\Phi \in \ell_\infty(B_X, Y)$. We say that $\Psi \in \ell_\infty(B_X, Y)$ satisfies the Φ-*Daugavet equation* if the norm equality

$$\|\Phi + \Psi\| = \|\Phi\| + \|\Psi\| \tag{Φ-DE}$$

holds. If Φ is the restriction of the identity to B_X, we call the above equation the *Daugavet equation* (DE).

To connect the Daugavet equation to a set $V \subset \ell_\infty(B_X, Y)$, we establish the following terminology.

Definition 2.2. Let X, Y be Banach spaces and let $\Phi \in \ell_\infty(B_X, Y)$.

(1) Y has the *Φ-Daugavet property with respect to* $V \subset \ell_\infty(B_X, Y)$ if (Φ-DE) is satisfied by all $\Psi \in V$.
(2) Y has the *Φ-Daugavet property* if $\|\Phi + R\| = \|\Phi\| + \|R\|$ for all $R \in L(X, Y)$ with one-dimensional range.
(3) Y has the *Daugavet property* if (2) holds for $X = Y$ and $\Phi = \mathrm{Id}$.

The following lemma (see e.g. [1, Lemma 11.4] or [15] for a proof) frequently simplifies proofs concerning the Daugavet equation, because we only need to consider maps of norm 1. We will often make use of the lemma without explicitly mentioning it.

Lemma 2.3. *Two vectors u and v in a normed space satisfy $\|u + v\| = \|u\| + \|v\|$ if and only if $\|\alpha u + \beta v\| = \alpha\|u\| + \beta\|v\|$ holds for all $\alpha, \beta \geq 0$. In particular, Ψ satisfies* (Φ-DE) *if and only if $\alpha\Psi$ satisfies* ($\beta\Phi$-DE) *for all $\alpha, \beta \geq 0$.*

To prove the first theorem of this section, we need the following lemma.

Lemma 2.4. *Let X be a Banach space and assume $x' \in \ell_\infty(B_X)$ with $\|x'\| \leq 1$. Let $0 \leq \varepsilon \leq 1$ and $x \in B_X$. Then $\operatorname{Re} x'(x) \geq 1 - \varepsilon$ implies $|1 - x'(x)| \leq \sqrt{2\varepsilon}$.*

Proof. First note that

$$1 \geq |x'(x)|^2 = \big(\operatorname{Im} x'(x)\big)^2 + \big(\operatorname{Re} x'(x)\big)^2 \geq \big(\operatorname{Im} x'(x)\big)^2 + (1-\varepsilon)^2.$$

Hence

$$\big(\operatorname{Im} x'(x)\big)^2 \leq 1 - (1-\varepsilon)^2 = 2\varepsilon - \varepsilon^2.$$

Since $\operatorname{Re} x'(x) \geq 1 - \varepsilon$ and $|x'(x)| \leq 1$, we know that $0 \leq 1 - \operatorname{Re} x'(x) \leq \varepsilon$. Thus

$$\begin{aligned} |1 - x'(x)|^2 &= |1 - \operatorname{Re} x'(x) - i \operatorname{Im} x'(x)|^2 \\ &= (1 - \operatorname{Re} x'(x))^2 + (\operatorname{Im} x'(x))^2 \\ &\leq \varepsilon^2 + 2\varepsilon - \varepsilon^2 \\ &= 2\varepsilon, \end{aligned}$$

i.e., $|1 - x'(x)| \leq \sqrt{2\varepsilon}$. □

Theorem 2.5. *Let X, Y be Banach spaces. Let $\Phi \in \ell_\infty(B_X, Y)$ and consider a norm one map $x' \in \ell_\infty(B_X)$ and $y \in Y \backslash \{0\}$. Then the following are equivalent:*

(1) $\|\Phi + x' \otimes y\| = \|\Phi\| + \|y\|$.
(2) *For every $\varepsilon > 0$ there are $x \in B_X$ and $\omega \in \mathbb{T}$ such that*

$$\operatorname{Re} \omega x'(x) \geq 1 - \varepsilon \quad \text{and} \quad \left\| \omega\Phi(x) + \frac{y}{\|y\|} \right\| \geq \|\Phi\| + 1 - \varepsilon.$$

Proof. (1) $\Rightarrow$ (2): By Lemma 2.3, we can assume $y \in S_Y$. Hence there is an element $x \in B_X$ such that

$$\begin{aligned}\|\Phi\| + 1 - \frac{\varepsilon}{2} &\le \|\Phi(x) + x'(x)y\| \\ &\le \|\Phi(x)\| + |x'(x)|\|y\| \\ &\le \|\Phi\| + |x'(x)|.\end{aligned}$$

Thus $|x'(x)| \ge 1 - \frac{\varepsilon}{2}$. Writing $\omega = |x'(x)|/x'(x) \in \mathbb{T}$ we have

$$\operatorname{Re} \omega x'(x) = |x'(x)| \ge 1 - \varepsilon.$$

Moreover

$$\begin{aligned}\|\Phi\| + 1 - \frac{\varepsilon}{2} &\le \|\Phi(x) + x'(x)y\| \\ &= \|\omega\Phi(x) + \omega x'(x)y\| \\ &\le \|\omega\Phi(x) + y\| + \|\omega x'(x)y - y\| \\ &= \|\omega\Phi(x) + y\| + |\omega x'(x) - 1|\|y\| \\ &= \|\omega\Phi(x) + y\| + |\ |x'(x)| - 1| \\ &\le \|\omega\Phi(x) + y\| + \frac{\varepsilon}{2},\end{aligned}$$

and (2) follows.

(2) $\Rightarrow$ (1): Again, by Lemma 2.3, it suffices to consider the case $\|y\| = 1$. Let $\varepsilon > 0$ and take $x \in B_X$ and $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega x'(x) \ge 1 - \varepsilon \quad \text{and} \quad \|\omega\Phi(x) + y\| \ge \|\Phi\| + 1 - \varepsilon.$$

Thus

$$\begin{aligned}\|\Phi\| + 1 - \varepsilon &\le \|\omega\Phi(x) + y\| \\ &= \|\Phi(x) + \overline{\omega}y\| \\ &\le \|\Phi(x) + x'(x)y\| + \|\overline{\omega}y - x'(x)y\| \\ &= \|\Phi(x) + x'(x)y\| + \|y - \omega x'(x)y\| \\ &= \|\Phi(x) + x'(x)y\| + |1 - \omega x'(x)| \\ &\le \|\Phi(x) + x'(x)y\| + \sqrt{2\varepsilon},\end{aligned}$$

where the latter inequality is due to Lemma 2.4. Since ε was arbitrary, (1) holds. $\square$

Next we present analogous results in the setting of the alternative Daugavet equation.

Definition 2.6. Let X, Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$. We say that Ψ satisfies the *alternative Φ-Daugavet equation* if

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = \|\Phi\| + \|\Psi\| \tag{Φ-ADE}$$

is true. In the case where Φ is the identity, we refer to the above equation simply as the alternative Daugavet equation (ADE).

We will also make use of the following definitions regarding a set $V \subset \ell_\infty(B_X, Y)$.

Definition 2.7. Let X, Y be Banach spaces and let $\Phi \in \ell_\infty(B_X, Y)$.

(a) Y has the *alternative Φ-Daugavet property with respect to* $V \subset \ell_\infty(B_X, Y)$ if (Φ-ADE) is satisfied for all $\Psi \in V$.
(b) Y has the *alternative Φ-Daugavet property* if $\max_{|\omega|=1} \|\Phi + \omega R\| = \|\Phi\| + \|R\|$ for all $R \in L(X, Y)$ with one-dimensional range.
(c) Y has the *alternative Daugavet property* if it has the alternative Id-Daugavet property.

Now that the notation is fixed, let us look at how the Daugavet and the alternative Daugavet equation are interrelated.

Remark 2.8. Let X, Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$.

(1) Ψ satisfies (Φ-ADE) if and only if there exists $\omega \in \mathbb{T}$ such that $\omega\Psi$ fulfills (Φ-DE).
(2) (Φ-DE) implies (Φ-ADE), but, in general, the converse is not true. For example, $-\mathrm{Id}$ always satisfies (ADE), but never (DE).
(3) Ψ satisfies (Φ-ADE) if and only if $\alpha\Psi$ satisfies ($\beta\Phi$-ADE) for every $\alpha, \beta \geq 0$. This is a consequence of (1) and Lemma 2.3.

Theorem 2.9. *Let X, Y be Banach spaces. Let $\Phi \in \ell_\infty(B_X, Y)$ and consider a norm one map $x' \in \ell_\infty(B_X)$ and $y \in Y\backslash\{0\}$. Then the following are equivalent:*

(1) $\max_{|\omega|=1} \|\Phi + \omega x' \otimes y\| = \|\Phi\| + \|y\|$.
(2) *For every $\varepsilon > 0$ there exist $\omega_1, \omega_2 \in \mathbb{T}$ and $x \in B_X$ such that*
$$\operatorname{Re} \omega_1 x'(x) \geq 1 - \varepsilon \quad \textit{and} \quad \left\| \omega_2 \Phi(x) + \frac{y}{\|y\|} \right\| \geq \|\Phi\| + 1 - \varepsilon.$$
(3) *For every $\varepsilon > 0$ there exist $\omega \in \mathbb{T}$ and $x \in B_X$ such that*
$$\left| x'(x) \right| \geq 1 - \varepsilon \quad \textit{and} \quad \left\| \omega \Phi(x) + \frac{y}{\|y\|} \right\| \geq \|\Phi\| + 1 - \varepsilon.$$

Proof. (1) $\Rightarrow$ (2): By Remark 2.8(3), we can assume $\|y\| = 1$. According to (1), there exists $\omega \in \mathbb{T}$ such that $\|\Phi + \omega x' \otimes y\| = \|\Phi\| + 1$. Thus, for a given $\varepsilon > 0$, Theorem 2.5 yields $x \in B_X$ and $\omega_2 \in \mathbb{T}$ such that
$$\operatorname{Re} \omega_2 \omega x'(x) \geq 1 - \varepsilon \quad \text{and} \quad \|\omega_2 \Phi(x) + y\| \geq \|\Phi\| + 1 - \varepsilon.$$
Defining $\omega_1 = \omega_2 \omega$, (2) follows.

(2) $\Rightarrow$ (3): If $\operatorname{Re} \omega_1 x'(x) \geq 1 - \varepsilon$, then
$$1 - \varepsilon \leq \operatorname{Re} \omega_1 x'(x) \leq |x'(x)|.$$

(3) $\Rightarrow$ (1): It suffices to consider the case $\|y\| = 1$. For given $\varepsilon > 0$, take $\omega \in \mathbb{T}$ and $x \in B_X$ such that
$$|x'(x)| \geq 1 - \varepsilon \quad \text{and} \quad \|\omega \Phi(x) + y\| \geq \|\Phi\| + 1 - \varepsilon.$$
Denote $\omega_1 = |x'(x)|/x'(x)$ and $\omega_2 = \overline{\omega}\omega_1$. Thus
$$\begin{aligned}
\|\Phi(x) + \omega_2 x'(x) y\| &= \|\Phi(x) + \overline{\omega}\omega_1 x'(x) y\| \\
&= \|\omega \Phi(x) + \omega_1 x'(x) y\| \\
&\geq \|\omega \Phi(x) + y\| - \|y - \omega_1 x'(x) y\| \\
&= \|\omega \Phi(x) + y\| - \big|1 - |x'(x)|\big| \\
&\geq \|\Phi\| + 1 - 2\varepsilon,
\end{aligned}$$
and we are done since $\varepsilon > 0$ was arbitrary. □

3. Strong slice continuity

In [14] the notion of slice continuity was introduced to study when the Daugavet equation holds for a couple of maps Φ and Ψ between Banach spaces, i.e., when

$$\|\Phi + \Psi\| = \|\Phi\| + \|\Psi\|.$$

The functions taken into account were either linear or bilinear bounded maps. In this section, we will extend some of the results from [14] to the case of bounded nonlinear functions.

The following definition is from [14].

Definition 3.1. Let X, Y be Banach spaces and $\Phi \in \ell_\infty(B_X, Y)$.

(a) If $y^* \in Y^*$ with $y^*\Phi \neq 0$, we define $\Phi_{y^*} : B_X \to \mathbb{K}$ by

$$\Phi_{y^*}(x) = \frac{1}{\|y^*\Phi\|} y^*\Phi(x).$$

(b) The *natural set of slices* defined by Φ is given by

$$\mathscr{S}_\Phi = \{S(\Phi_{y^*}, \varepsilon) : 0 < \varepsilon < 1,\ y^* \in Y^*,\ y^*\Phi \neq 0\}.$$

(c) We write $\mathscr{S}_\Psi \leq \mathscr{S}_\Phi$ if for every $S(\Psi_{z^*}, \varepsilon) \in \mathscr{S}_\Psi$ there is $S(\Phi_{y^*}, \mu) \in \mathscr{S}_\Phi$ with

$$S(\Phi_{y^*}, \mu) \subset S(\Psi_{z^*}, \varepsilon).$$

In this instance we say that Ψ is *slice continuous* with respect to Φ.

Now we are ready to introduce the concept of strong slice continuity for bounded nonlinear maps.

Definition 3.2. Let X, Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$. We use the symbol $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ if for every slice $S(\Psi_{z^*}, \varepsilon) \in \mathscr{S}_\Psi$ there is a slice $S(\Phi_{y^*}, \mu) \in \mathscr{S}_\Phi$ such that

$$S(\omega\Phi_{y^*}, \mu) \subset S(\omega\Psi_{z^*}, \varepsilon) \quad \text{for all } \omega \in \mathbb{T}.$$

In this case we say that Ψ is *strongly slice continuous* with respect to Φ.

Note that the above and similar definitions carry over to bounded functions from X to Y by considering the respective restrictions to B_X.

It is clear that strong slice continuity implies slice continuity. The following remark shows that in the case of multilinear maps, the two concepts coincide.

Remark 3.3. Let $X_1, \dots, X_n, Z$ be Banach spaces and $A, B : X_1 \times \cdots \times X_n \to Z$ bounded multilinear maps. Then $\mathscr{S}_A < \mathscr{S}_B$ if and only if $\mathscr{S}_A \leq \mathscr{S}_B$.

Proof. We only need to verify that slice continuity implies strong slice continuity. To this end, let $S(A_{x^*}, \varepsilon) \in \mathscr{S}_A$ be given. Since $\mathscr{S}_A \leq \mathscr{S}_B$, we can find $S(B_{y^*}, \mu) \in \mathscr{S}_B$ with $S(B_{y^*}, \mu) \subset S(A_{x^*}, \varepsilon)$. For a given $\omega \in \mathbb{T}$ and $(x_1, \dots, x_n) \in S(\omega B_{y^*}, \mu)$, we have

$$1 - \mu \leq \operatorname{Re} \omega \frac{y^*B(x_1, \dots, x_n)}{\|y^*B\|} = \operatorname{Re} \frac{y^*B(\omega x_1, \dots, x_n)}{\|y^*B\|},$$

i.e., $(\omega x_1, x_2, \dots, x_n) \in S(B_{y^*}, \mu)$. This ensures $(\omega x_1, x_2, \dots, x_n) \in S(A_{x^*}, \varepsilon)$, and the multilinearity of A leads to $(x_1, x_2, \dots, x_n) \in S(\omega A_{x^*}, \varepsilon)$. $\square$

The canonical example of when the relation $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ holds is given by the case where Ψ is the concatenation of a map Φ and a bounded linear operator.

Example 3.4. Let X, Y be Banach spaces. Consider $\Phi \in \ell_\infty(B_X, Y)$ and a bounded linear operator $P : Y \to Y$. Denote $\Psi = P \circ \Phi$. Then $S_\Psi < S_\Phi$.

Proof. Let $S(\Psi_{y^*}, \varepsilon)$ be a slice in $\mathscr{S}_\Psi$. First note that since $y^*\Psi \neq 0$, we also have $(y^*P)\Phi = y^*\Psi \neq 0$, and thus $S(\Phi_{y^*P}, \varepsilon) \in \mathscr{S}_\Phi$. Take $\omega \in \mathbb{T}$ and $x \in S(\omega\Phi_{y^*P}, \varepsilon)$, i.e.,

$$\operatorname{Re}\omega \frac{(y^*P)\Phi}{\|(y^*P)\Phi\|}(x) \geq 1 - \varepsilon.$$

By construction,

$$\operatorname{Re}\omega \frac{y^*\Psi}{\|y^*\Psi\|}(x) = \operatorname{Re}\omega \frac{(y^*P)\Phi}{\|(y^*P)\Phi\|}(x) \geq 1 - \varepsilon,$$

and therefore $x \in S(\omega\Psi_{y^*}, \varepsilon)$. $\square$

The next example shows that there are bounded maps Φ, Ψ with $\mathscr{S}_\Psi < \mathscr{S}_\Phi$, but $\Psi \neq P \circ \Phi$ for any bounded linear operator P.

Example 3.5. Let $C[0,1]$ denote the Banach space of continuous functions from $[0,1]$ to $\mathbb{K}$. Let $\Phi : C[0,1] \oplus_1 \mathbb{K} \to C[0,1]$, $\Phi(f, \alpha) = f$, and $\Psi : C[0,1] \oplus_1 \mathbb{K} \to C[0,1]$, $\Psi(f,\alpha) = f + \alpha^2\mathbf{1}$, where $\mathbf{1}$ stands for the constant one function and $\oplus_1$ denotes the direct sum with the 1-norm. Then Ψ and Φ have norm one. The kernel of Φ is not contained in the kernel of Ψ, since $\Phi(0,1) = 0$, but $\Psi(0,1) \neq 0$. Thus we do not have $\Psi = P \circ \Phi$ for any bounded linear operator P. But the slice condition $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ holds. First note that for any $x^* \in C[0,1]^* \setminus \{0\}$, we have $\|x^*\Phi\| = \|x^*\Psi\| = \|x^*\| \neq 0$. Consider some $x^* \in C[0,1]^*$ with $\|x^*\| = 1$, and let $0 < \varepsilon < 1$. We claim $S\left(\omega x^*\Phi, \frac{\varepsilon}{2}\right) \subset S\left(\omega x^*\Psi, \varepsilon\right)$ for all $\omega \in \mathbb{T}$. To prove this, assume $(f, \alpha) \in S\left(\omega x^*\Phi, \frac{\varepsilon}{2}\right)$, i.e., $\operatorname{Re}\omega x^*(f) \geq 1 - \frac{\varepsilon}{2}$. In particular, $\|f\| \geq 1 - \frac{\varepsilon}{2}$, and therefore $|\alpha| \leq \varepsilon/2$. Hence

$$\begin{aligned}
\operatorname{Re}\omega x^*\Psi(f,\alpha) &= \operatorname{Re}\omega x^*(f + \alpha^2\mathbf{1}) \\
&= \operatorname{Re}\omega x^*(f) + \operatorname{Re}\omega x^*(\alpha^2\mathbf{1}) \\
&\geq 1 - \varepsilon.
\end{aligned}$$

Consider now a closed subspace Z of a normed space X. Then $q : X \to X/Z$, $q(x) = x + Z$, sends the open unit ball U_Z of Z onto the open unit ball $U_{X/Z}$ of X/Z. This motivates the following definition.

Definition 3.6. Let X, Y be Banach spaces. We call $\Phi \in \ell_\infty(B_X, Y)$ a *quotient map* if Φ is continuous and $\Phi(U_X) = U_Y$.

Given $\Phi \in \ell_\infty(B_X, Y)$ we set

$$Y^*\Phi \cdot Y = \{y^*\Phi \otimes y : y^* \in Y^*,\ y \in Y\}.$$

Lemma 3.7. *Let X, Y be Banach spaces and assume $\Phi \in \ell_\infty(B_X, Y)$ is a quotient map. Then the following are equivalent:*

1. *Y has the Daugavet property.*
2. *Y has the Φ-Daugavet property with respect to $Y^*\Phi \cdot Y$.*

Proof. This is a consequence of the assumptions that Φ is continuous and $\Phi(U_X) = U_Y$. $\square$

Proposition 3.8. *Let X, Y be Banach spaces and assume Y has the Daugavet property. Consider $\Psi, \Phi \in \ell_\infty(B_X, Y)$ such that Φ is a quotient map and $\|\Psi\| = 1$. Then $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ implies that for every $y \in S_Y$ and $y^* \in Y^*$ with $y^*\Psi \neq 0$*

$$\|\Phi + \Psi_{y^*} \otimes y\| = 2.$$

Proof. By Theorem 2.5, it suffices to show that for every $\varepsilon > 0$ there are $\omega \in \mathbb{T}$ and $x \in S(\omega\Psi_{y^*}, \varepsilon)$ such that

$$\|\omega\Phi(x) + y\| \geq 2 - \varepsilon.$$

Thus, let $\varepsilon > 0$ be given. Since $\mathscr{S}_\Psi < \mathscr{S}_\Phi$, we can find a slice $S(\Phi_{z^*}, \mu) \in \mathscr{S}_\Phi$ with $\mu \leq \varepsilon$ such that $S(\lambda\Phi_{z^*}, \mu) \subset S(\lambda\Psi_{y^*}, \varepsilon)$ for all $\lambda \in \mathbb{T}$. According to Lemma 3.7, $\|\Phi + \Phi_{z^*} \otimes y\| = 2$, therefore Theorem 2.5 gives $\omega \in \mathbb{T}$ and $x \in S(\omega\Phi_{z^*}, \mu)$ satisfying

$$\|\omega\Phi(x) + y\| \geq 2 - \mu \geq 2 - \varepsilon.$$

By construction, $S(\omega\Phi_{z^*}, \mu) \subset S(\omega\Psi_{y^*}, \varepsilon)$, and the proof is complete. $\square$

Remark 3.9. The above proposition is false if the condition $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ is removed. To see this, consider bounded linear operators $\Phi, \Psi : L_1[0,1] \oplus_1 L_1[1,2] \to L_1[0,1]$ given by $\Phi((f,g)) = f$ and $\Psi((f,g)) = (\int_1^2 g\,dx)\cdot\mathbf{1}$, where $(f,g) \in L_1[0,1] \oplus_1 L_1[1,2]$; recall that $L_1[0,1]$ has the Daugavet property. Clearly, Φ is a quotient map and $\|\Psi\| = 1$. But, if $y = \mathbf{1} \in L_1[0,1]$ and $y^* = \mathbf{1} \in L_\infty[0,1]$, then $\|\Phi + y^*\Psi \otimes y\| \leq 1$.

We shall now deal with weakly compact maps. Let us start by recalling the definition of a (nonlinear) weakly compact map.

Definition 3.10. Let X, Y be Banach spaces. A function $\Psi \in \ell_\infty(B_X, Y)$ is called *weakly compact* if the weak closure of $\Psi(B_X)$ is a weakly compact set.

Let us now prove the main result of this section, namely Theorem 3.11.

Theorem 3.11. *Let X, Y be Banach spaces and let $\Phi, \Upsilon, \Psi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = \|\Upsilon\| = \|\Psi\| = 1$. Assume Y has the Φ-Daugavet property with respect to $Y^*\Upsilon \cdot Y$. Then, if $\mathscr{S}_\Psi < \mathscr{S}_\Upsilon$ and Ψ is weakly compact,*

$$\|\Phi + \Psi\| = 2.$$

Proof. Since the set $K = \overline{\text{co}}(\mathbb{T}\Psi(B_X))$ is weakly compact by Krein's theorem, we can conclude that K coincides with the closed convex hull of its strongly exposed points ([5], [2, Cor. 5.18]). Therefore, given $\varepsilon > 0$, we may take a strongly exposed point $y_0 \in K$ with $\|y_0\| > 1 - \varepsilon$. Because y_0 is a strongly exposed point, there are $z^* \in Y^*$ and $\eta > 0$ such that the set

$$\{y \in K : \operatorname{Re} z^*(y) \geq \operatorname{Re} z^*(y_0) - \eta\}$$

has diameter less than ε and $\operatorname{Re} z^*(y_0) > \operatorname{Re} z^*(y)$ for all $y \in K\backslash\{y_0\}$. After defining $y_0^* = z^*/\operatorname{Re} z^*(y_0)$ and $\delta = \min\{\frac{\varepsilon}{2}, \eta/\operatorname{Re} z^*(y_0)\}$, we have found a slice

$$S = \{y \in K : \operatorname{Re} y_0^*(y) \geq 1 - \delta\}$$

containing y_0 and having diameter less than ε. In particular,

$$y \in K,\ \operatorname{Re} y_0^*(y) \geq 1-\delta \quad \Rightarrow \quad \|y-y_0\| < \varepsilon.$$

Also note that since K is balanced,

$$\sup_{y\in K} \operatorname{Re} y_0^*(y) = \sup_{y\in K} |y_0^*(y)| = 1.$$

Denote $\psi := y_0^* \circ \Psi$. We have

$$\|\psi\| = \sup_{x\in B_X} |y_0^*(\Psi(x))| = \sup_{y\in K} |y_0^*(y)| = 1,$$

hence $S(\psi,\delta) \in \mathscr{S}_\Psi$. On account of $\mathscr{S}_\Psi < \mathscr{S}_\Upsilon$, there are $\mu \leq \delta$ and $S(\Upsilon_{z^*},\mu) \in \mathscr{S}_\Upsilon$ such that

$$S(\lambda\Upsilon_{z^*},\mu) \subset S(\lambda\psi,\delta) \quad \text{for all } \lambda \in \mathbb{T}.$$

Since by assumption $\|\Phi + \Upsilon_{z^*} \otimes y_0\| = 1 + \|y_0\|$, Theorem 2.5 yields $\omega \in \mathbb{T}$ and $x \in S(\omega\Upsilon_{z^*},\mu)$ so that

$$\left\|\omega\Phi(x) + \frac{y_0}{\|y_0\|}\right\| \geq 2-\mu \geq 2-\varepsilon.$$

By construction, $x \in S(\omega\Upsilon_{z^*},\mu) \subset S(\omega\psi,\delta)$, and therefore

$$\operatorname{Re} y_0^*(\omega\Psi(x)) = \operatorname{Re} \omega\psi(x) \geq 1-\delta,$$

so the fact that $\omega\Psi(x) \in K$ gives $\|\omega\Psi(x) - y_0\| < \varepsilon$.

We calculate

$$\begin{aligned}\|y_0 + \omega\Phi(x)\| &\geq \left\|\omega\Phi(x) + \frac{y_0}{\|y_0\|}\right\| - \left\|y_0 - \frac{y_0}{\|y_0\|}\right\| \\ &= \left\|\omega\Phi(x) + \frac{y_0}{\|y_0\|}\right\| - |\,\|y_0\| - 1| \\ &\geq 2-2\varepsilon.\end{aligned}$$

Finally,

$$\begin{aligned}\|\Phi+\Psi\| &\geq \|\Phi(x)+\Psi(x)\| \\ &= \|\omega\Phi(x)+\omega\Psi(x)\| \\ &\geq \|\omega\Phi(x)+y_0\| - \|\omega\Psi(x)-y_0\| \\ &\geq 2-3\varepsilon.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we conclude that Ψ satisfies (Φ-DE). □

Remark 3.12. The requirement on the weak compactness of the function Ψ can be substituted in the result above by the more general notion of Radon-Nikodým function, which fits exactly with what is needed; see the definition and how to use it in this setting for example in [4]. One way of defining the Radon-Nikodým property for a closed convex set A is that every closed convex subset $B \subset A$ is the closed convex hull of its strongly exposed points. (See [2, Th. 5.8 and Th. 5.17].) So, a function is said to be a Radon-Nikodým function if the closure of $T(B_X)$ has the Radon-Nikodým property.

Corollary 3.13. *Let X, Y be Banach spaces and assume Y has the Daugavet property. Consider $\Phi, \Psi \in \ell_\infty(B_X, Y)$ such that Φ is a quotient map and $\|\Psi\| = 1$. If $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ and Ψ is weakly compact, then*

$$\|\Phi + \Psi\| = 2.$$

Proof. In Lemma 3.7 we observed that if Y has the Daugavet property and Φ is a quotient map, then Y has the Φ-Daugavet property with respect to $Y^*\Phi \cdot Y$. Thus, Theorem 3.11 yields $\|\Phi + \Psi\| = 2$. □

Remark 3.14. The above corollary is not valid if the condition $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ is removed. For instance, consider $\Phi, \Psi : L_1[0,1] \oplus_1 L_2[1,2] \to L_1[0,1]$ given by $\Phi((f,g)) = f$ and $\Psi((f,g))(x) = g(x+1)$, where $(f,g) \in L_1[0,1] \oplus_1 L_2[1,2]$. Then Ψ is weakly compact and $\|\Phi\| = \|\Psi\| = 1$, but $\|\Phi + \Psi\| \leq 1$.

Theorem 3.15. *Let X, Y be Banach spaces and let $\mathscr{Z}$ be a linear subspace of $\ell_\infty(B_X)$. Assume $\Phi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = 1$. Then the following are equivalent:*

(1) *For every $x' \in \mathscr{Z}$ and every $y \in Y$, $x' \otimes y$ satisfies* (Φ-DE).
(2) *For every $x' \in S_{\mathscr{Z}}$, every $y \in S_Y$, and every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $x \in B_X$ such that*

$$\operatorname{Re} \omega x'(x) \geq 1 - \varepsilon \quad \text{and} \quad \|\omega \Phi(x) + y\| \geq 2 - \varepsilon.$$

(3) *Every weakly compact $\Psi \in \ell_\infty(B_X, Y)$ such that $y^* \circ \Psi \in \mathscr{Z}$ for all $y^* \in Y^*$ satisfies* (Φ-DE).

Proof. (1) ⇔ (2): This equivalence follows from Theorem 2.5.

(1) ⇒ (3): Let Ψ be as in (3). Because of (1), Y has the Φ-Daugavet property with respect to $Y^*\Psi \cdot Y$. Since trivially $\mathscr{S}_\Psi < \mathscr{S}_\Psi$, Theorem 3.11 gives (3).

(3) ⇒ (1): Given $x' \in \mathscr{Z}$ and $y \in Y$, $x' \otimes y$ has finite-dimensional range and consequently is a weakly compact map. □

For completeness we note the n-linear version of [14, Cor. 3.10].

Corollary 3.16. *Let $X_1, \dots, X_n, Y$ be Banach spaces and consider a continuous multilinear map $B_0 : X_1 \times \cdots \times X_n \to Y$ satisfying $B_0(U_{X_1 \times \cdots \times X_n}) = U_Y$. Consider the subsets R, C and WC of $L(Y,Y)$ of rank one, compact and weakly compact linear operators. Denote $R \circ B_0 = \{T \circ B_0 : T \in R\}$, $C \circ B_0 = \{T \circ B_0 : T \in C\}$ and $WC \circ B_0 = \{T \circ B_0 : T \in WC\}$. Then the following are equivalent:*

(1) *Y has the Daugavet property.*
(2) *Y has the B_0-Daugavet property with respect to $R \circ B_0$.*
(3) *Y has the B_0-Daugavet property with respect to $C \circ B_0$.*
(4) *Y has the B_0-Daugavet property with respect to $WC \circ B_0$.*

Proof. The equivalence of (1) and (2) follows from Lemma 3.7. (2) and (4) are equivalent by letting $\mathscr{Z} = \{y^* \circ B_0 : y^* \in Y^*\}$ in Theorem 3.15. The implications (4) ⇒ (3) ⇒ (2) are due to the inclusions $R \subset C \subset WC$. □

4. Weak slice continuity

In the previous section we defined the notion of strong slice continuity and related it to the Daugavet equation. This section is the analogue of Section 3 for the alternative Daugavet equation. We introduce the concept of weak slice continuity to further investigate when two maps Ψ, Φ satisfy the alternative Daugavet equation, i.e., when

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = \|\Phi\| + \|\Psi\|.$$

Definition 4.1. Let X be a Banach space, $x' \in \ell_\infty(B_X)$ with $\|x'\| = 1$ and $\varepsilon > 0$. We write

$$S'(x', \varepsilon) = \{x \in B_X : |x'(x)| \geq 1 - \varepsilon\}$$

for the *weak slice* of B_X determined by x' and ε.

In a second step we extend the above definition to Banach space valued functions.

Definition 4.2. Let X, Y be Banach spaces and $\Phi \in \ell_\infty(B_X, Y)$. The *natural set of weak slices* defined by Φ is given by

$$\mathscr{S}'_\Phi = \{S'(\Phi_{y^*}, \varepsilon) : 0 < \varepsilon < 1,\ y^* \in Y^*,\ y^*\Phi \neq 0\}.$$

Now we are in a position to define weak slice continuity in analogy to strong slice continuity; cf. Definition 3.2.

Definition 4.3. Let X, Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$. We write $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$ if for every weak slice $S'(\Psi_{z^*}, \varepsilon) \in \mathscr{S}'_\Psi$ there is a weak slice $S'(\Phi_{y^*}, \mu) \in \mathscr{S}'_\Phi$ such that

$$S'(\Phi_{y^*}, \mu) \subset S'(\Psi_{z^*}, \varepsilon).$$

In this case we say that Ψ is *weakly slice continuous* with respect to Φ.

If Φ, Ψ are two maps such that Ψ is strongly slice continuous with respect to Φ, then Ψ is also slice continuous with respect to Φ. Let us check that a similar implication holds for strong and weak slice continuity.

Remark 4.4. Let X, Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$. Then $\mathscr{S}_\Psi < \mathscr{S}_\Phi$ implies $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$.

Proof. Assume $S'(\Psi_{z^*}, \varepsilon) \in \mathscr{S}'_\Psi$. Since $\mathscr{S}_\Psi < \mathscr{S}_\Phi$, there is $S(\Phi_{y^*}, \mu) \in \mathscr{S}_\Phi$ satisfying $S(\lambda\Phi_{y^*}, \mu) \subset S(\lambda\Psi_{z^*}, \varepsilon)$ for all $\lambda \in \mathbb{T}$. We claim $S'(\Phi_{y^*}, \mu) \subset S'(\Psi_{z^*}, \varepsilon)$. To prove this, let $x \in B_X$ with $|\Phi_{y^*}(x)| \geq 1 - \mu$ and denote $\omega = |\Phi_{y^*}(x)|/\Phi_{y^*}(x)$. Then $\operatorname{Re} \omega\Phi_{y^*}(x) = |\Phi_{y^*}(x)| \geq 1 - \mu$ and therefore $\operatorname{Re} \omega\Psi_{z^*}(x) \geq 1 - \varepsilon$. In particular, $|\Psi_{z^*}(x)| \geq 1 - \varepsilon$, i.e., $x \in S'(\Psi_{z^*}, \varepsilon)$. $\square$

The next example shows that the reverse implication in the above remark does not hold.

Example 4.5. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Psi(x) = \begin{cases} 1 & \text{if } x = 0, \\ -|x| & \text{if } x \neq 0. \end{cases}$$

Then Ψ is weakly slice continuous with respect to the identity, but Ψ is not strongly slice continuous with respect to the identity.

Proof. Consider the slice $S(\Psi, 1/2) \in \mathscr{S}_\Psi$. Then $S(c\,\mathrm{Id}, \varepsilon) \not\subset S(\Psi, 1/2)$ for any $c \in \{-1, 1\}$ and $0 < \varepsilon < 1$. Thus Ψ is not strongly slice continuous with respect to Id. But if $c \in \{-1, 1\}$ and $0 < \varepsilon < 1$ are given, then $S'(\mathrm{Id}, \varepsilon) \subset S'(c\Psi, \varepsilon)$. Therefore Ψ is weakly slice continuous with respect to Id. □

Example 4.6. Let X, Y be Banach spaces. Consider $\Phi \in \ell_\infty(B_X, Y)$ and a bounded linear operator $P : Y \to Y$. Denote $\Psi = P \circ \Phi$. Then $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$.

Proof. According to Example 3.4, the assumptions imply $\mathscr{S}_\Psi < \mathscr{S}_\Phi$. Hence $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$ by Remark 4.4. □

Note that we have shown in Example 4.5 that there are bounded maps Φ, Ψ with $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$, but $\Psi \neq P \circ \Phi$ for any bounded linear operator P.

Recall from Definition 3.6 that a quotient map is a continuous function mapping the open unit ball of its domain onto the open unit ball of its range space. These properties allow for the following lemma.

Lemma 4.7. *Let X, Y be Banach spaces and assume $\Phi \in \ell_\infty(B_X, Y)$ is a quotient map. Then the following are equivalent:*

(1) *Y has the alternative Daugavet property.*
(2) *Y has the alternative Φ-Daugavet property with respect to $Y^*\Phi \cdot Y$.*

Proposition 4.8. *Let X, Y be Banach spaces and assume Y has the alternative Daugavet property. Consider $\Psi, \Phi \in \ell_\infty(B_X, Y)$ such that Φ is a quotient map and $\|\Psi\| = 1$. Then $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$ implies that for every $y \in S_Y$ and $y^* \in Y^*$ with $y^*\Psi \neq 0$*

$$\max_{|\omega|=1} \|\Phi + \omega\Psi_{y^*} \otimes y\| = 2.$$

Proof. We will use Theorem 2.9, i.e., we need to show that for every $\varepsilon > 0$ there exist $\omega \in \mathbb{T}$ and $x \in S'(\Psi_{y^*}, \varepsilon)$ such that

$$\|\omega\Phi(x) + y\| \geq 2 - \varepsilon.$$

Since $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$, there is a slice $S'(\Phi_{z^*}, \mu) \in \mathscr{S}'_\Phi$ such that $S'(\Phi_{z^*}, \mu) \subset S'(\Psi_{y^*}, \varepsilon)$ and $\mu \leq \varepsilon$. The alternative Daugavet property of Y in conjunction with Lemma 4.7 yields the norm equality $\max_{|\omega|=1} \|\Phi + \omega\Phi_{y^*} \otimes y\| = 2$. Hence another application of Theorem 2.9 gives $\omega \in \mathbb{T}$ and $x \in S'(\Phi_{y^*}, \mu)$ such that

$$\|\omega\Phi(x) + y\| \geq 2 - \mu \geq 2 - \varepsilon.$$

Because of $S'(\Phi_{z^*}, \mu) \subset S'(\Psi_{y^*}, \varepsilon)$, we also have $x \in S'(\Psi_{y^*}, \varepsilon)$, which completes the proof. □

Remark 4.9. In the above proposition, the assumption $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$ cannot be removed. This can be shown by using the functions from Remark 3.9.

Theorem 4.10. *Let X, Y be Banach spaces and let $\Phi, \Upsilon, \Psi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = \|\Upsilon\| = \|\Psi\| = 1$. Assume Y has the alternative Φ-Daugavet property with respect to $Y^*\Upsilon \cdot Y$. Then, if $\mathscr{S}'_\Psi < \mathscr{S}'_\Upsilon$ and Ψ is weakly compact,*

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = 2.$$

Proof. Denote $K = \overline{\mathrm{co}}(\mathbb{T}\Psi(B_X))$ and let $\varepsilon > 0$ be given. In the same way as in the proof of Theorem 3.11, we may find $y_0 \in K$ with $\|y_0\| > 1 - \varepsilon$, $\delta \in (0, \varepsilon/2)$ and $y_0^* \in Y^*$ such that

$$y \in K,\ \mathrm{Re}\, y_0^*(y) \geq 1 - \delta \quad \Rightarrow \quad \|y - y_0\| < \varepsilon$$

and

$$\sup_{y \in K} |y_0^*(y)| = 1.$$

Setting $\psi := y_0^* \circ \Psi$, we get

$$\|\psi\| = \sup_{x \in B_X} |y_0^*(\Psi(x))| = \sup_{y \in K} |y_0^*(y)| = 1,$$

i.e., $S'(\psi, \delta) \in \mathscr{S}'_\Psi$. From $\mathscr{S}'_\Psi < \mathscr{S}'_\Upsilon$ we deduce the existence of $\mu \leq \delta$ as well as $S'(\Upsilon_{z^*}, \mu) \in \mathscr{S}'_\Upsilon$ satisfying $S'(\Upsilon_{z^*}, \mu) \subset S'(\psi, \delta)$. Since Y has the alternative Φ-Daugavet property with respect to $Y^*\Upsilon \cdot Y$, we can use Theorem 2.9 to get $\omega_1 \in \mathbb{T}$ and $x \in S'(\Upsilon_{z^*}, \mu)$ such that

$$\left\| \omega_1 \Phi(x) + \frac{y_0}{\|y_0\|} \right\| \geq 2 - \mu \geq 2 - \varepsilon.$$

In particular, $x \in S'(\Upsilon_{z^*}, \mu) \subset S'(\psi, \delta)$. Writing $\omega_2 = |\psi(x)|/\psi(x)$ we observe

$$\mathrm{Re}\, y_0^*(\omega_2 \Psi(x)) = \mathrm{Re}\, \omega_2 \psi(x) = |\psi(x)| \geq 1 - \delta,$$

so the fact that $\omega_2 \Psi(x) \in K$ gives

$$\|\omega_2 \Psi(x) - y_0\| < \varepsilon.$$

On the other hand,

$$\begin{aligned}
\|y_0 + \omega_1 \Phi(x)\| &\geq \left\| \omega_1 \Phi(x) + \frac{y_0}{\|y_0\|} \right\| - \left\| y_0 - \frac{y_0}{\|y_0\|} \right\| \\
&= \left\| \omega_1 \Phi(x) + \frac{y_0}{\|y_0\|} \right\| - \big|\, \|y_0\| - 1 \big| \\
&\geq 2 - 2\varepsilon.
\end{aligned}$$

Altogether

$$\begin{aligned}
\max_{|\omega|=1} \|\Phi + \omega \Psi\| &\geq \|\Phi + \overline{\omega_1} \omega_2 \Psi\| \\
&\geq \|\Phi(x) + \overline{\omega_1} \omega_2 \Psi(x)\| \\
&= \|\omega_1 \Phi(x) + \omega_2 \Psi(x)\| \\
&\geq \|\omega_1 \Phi(x) + y_0\| - \|\omega_2 \Psi(x) - y_0\| \\
&\geq 2 - 3\varepsilon,
\end{aligned}$$

which proves the assertion because $\varepsilon > 0$ was chosen arbitrarily. □

Corollary 4.11. *Let X, Y be Banach spaces and assume Y has the alternative Daugavet property. Consider $\Phi, \Psi \in \ell_\infty(B_X, Y)$ such that Φ is a quotient map and $\|\Psi\| = 1$. If $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$ and Ψ is weakly compact, then*

$$\max_{|\omega|=1} \|\Phi + \omega \Psi\| = 2.$$

Proof. Y has the alternative Φ-Daugavet property with respect to $Y^*\Phi \cdot Y$ by Lemma 4.7. Therefore Ψ satisfies the alternative Φ-Daugavet equation according to Theorem 4.10. □

Remark 4.12. In the above corollary, the assumption $\mathscr{S}'_\Psi < \mathscr{S}'_\Phi$ cannot be dropped. For instance, this follows with the help of the functions constructed in Remark 3.14.

Theorem 4.13. *Let X, Y be Banach spaces and let $\mathscr{X}$ be a linear subspace of $\ell_\infty(B_X)$. Assume $\Phi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = 1$. Then the following are equivalent:*

(1) *For every $x' \in \mathscr{X}$ and every $y \in Y$, $x' \otimes y$ satisfies* (Φ-ADE).

(2) *For every $x' \in S_{\mathscr{X}}$, every $y \in S_Y$, and every $\varepsilon > 0$, there exist $\omega_1, \omega_2 \in \mathbb{T}$ and $y \in B_X$ such that*
$$\operatorname{Re} \omega_1 x'(x) \geq 1 - \varepsilon \quad \text{and} \quad \|\omega_2 \Phi(x) + y\| \geq 2 - \varepsilon.$$

(3) *For every $x' \in S_{\mathscr{X}}$, every $y \in S_Y$, and every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $x \in B_X$ such that*
$$\left|x'(x)\right| \geq 1 - \varepsilon \quad \text{and} \quad \|\omega \Phi(x) + y\| \geq 2 - \varepsilon.$$

(4) *Every weakly compact $\Psi \in \ell_\infty(B_X, Y)$ such that $y^* \circ \Psi \in \mathscr{X}$ for all $y^* \in Y^*$ satisfies* (Φ-ADE).

Proof. The equivalence of (1), (2) and (3) is a consequence of Theorem 2.9. The implication (1) $\Rightarrow$ (4) follows from Theorem 4.10 since trivially $\mathscr{S}'_\Psi < \mathscr{S}'_\Psi$ for any Ψ as in (4). The direction (4) $\Rightarrow$ (1) is true because finite-rank maps are weakly compact. $\square$

The following corollary is analogous to Corollary 3.16.

Corollary 4.14. *Let $X_1, \dots, X_n, Y$ be Banach spaces and consider a continuous multilinear map $B_0 : X_1 \times \cdots \times X_n \to Y$ satisfying $B_0(U_{X_1 \times \cdots \times X_n}) = U_Y$. Consider the subsets R, C and WC of $L(Y, Y)$ of rank one, compact and weakly compact operators. Denote $R \circ B_0 = \{T \circ B_0 : T \in R\}$, $C \circ B_0 = \{T \circ B_0 : T \in C\}$ and $WC \circ B_0 = \{T \circ B_0 : T \in WC\}$. Then the following are equivalent:*

(1) *Y has the alternative Daugavet property.*
(2) *Y has the alternative B_0-Daugavet property with respect to $R \circ B_0$.*
(3) *Y has the alternative B_0-Daugavet property with respect to $C \circ B_0$.*
(4) *Y has the alternative B_0-Daugavet property with respect to $WC \circ B_0$.*

Proof. The equivalence of (1) and (2) is due to Lemma 4.7. (2) and (4) are equivalent by letting $\mathscr{X} = \{y^* \circ B_0 : y^* \in Y^*\}$ in Theorem 4.13. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) follow from the inclusions $R \subset C \subset WC$. $\square$

5. Local Φ-Daugavet type properties and applications

After the explanation of our main results given in the previous part of the paper, we are ready to present more technical versions of the tools obtained there. All of them can be proved using the same arguments and are useful for applications. Essentially, we introduce the notion of norm determining set $\Gamma \subset B_X$ for a class of functions and some new elements that allow to define the notion of Φ-Daugavet property with respect to particular sets of scalar functions and vectors in Y, with a norm that can be defined as the supremum of the evaluation of the functions involved just for some subset of vectors in S_X.

In this section all Banach spaces are supposed to be $\mathbb{R}$-vector spaces for simplicity of notation.

Let X and Y be Banach spaces, and let $V \subset \ell_\infty(B_X, Y)$. We say that a subset $\Gamma \subset B_X$ is *norm determining* for V if

$$\|\Psi\| = \|\Psi\|_\Gamma := \sup_{x\in\Gamma} \|\Psi(x)\|$$

for all $\Psi \in V$.

Let us start by formulating a version of Theorem 2.5 considering norm determining subsets for the functions involved. Its proof follows the same lines as the proof of that theorem, so we omit it.

Proposition 5.1. *Let X and Y be Banach spaces and let $\Gamma \subset B_X$. Let $\Phi \in \ell_\infty(B_X, Y)$ be a norm one map, and consider a norm one function $x' \in \ell_\infty(B_X)$. Let $y \in S_Y$. The following assertions are equivalent.*

(1) $\|\Phi + x' \otimes y\|_\Gamma = 2$.

(2) *For every $\varepsilon > 0$ there is some $\omega \in \mathbb{T}$ and an element $x \in S(\omega x', \varepsilon) \cap \Gamma$ such that*

$$\|\omega\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Remark 5.2. Notice that the condition in the result above implies that for a norm one scalar function $x' \in \ell_\infty(B_X, Y)$,

$$2 \leq \|x' \otimes y + \Phi\|_\Gamma \leq \|x'\|_\Gamma \|y\| + \|\Phi\| \leq \|x'\|_\Gamma + 1,$$

and so $\|x'\|_\Gamma = \|x'\| = 1$. Thus Γ is norm determining for x'; the same argument gives that it is so for Φ.

Let us define now some sort of "local version" of the notion of Φ-Daugavet property.

Definition 5.3. Let X and Y be Banach spaces and let $\Phi : B_X \to Y$ be a norm one function. Let $\Gamma \subset B_X$ be a norm determining set for Φ and consider subsets $\mathscr{W} \subset \ell_\infty(B_X)$ and $\Delta \subset S_Y$. We say that Y has the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$ if for every $x' \in \mathscr{W}$ and $y \in \Delta$

$$\sup_{x\in\Gamma} \|\Phi(x) + x'(x)y\| = 2.$$

The reader can notice that this definition is related to the one of Daugavet centre given in Definition 1.2 of [4] and that of the almost Daugavet property from [9].

Let us provide a concrete example of a function Φ and sets Γ, $\mathscr{W}$ and Δ for which every Banach space has the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$.

Example 5.4. Let X be a real Banach space and take $Y = X$. Consider the sets $\Gamma = B_X$,

$$\mathscr{W} = \{x' \in \ell_\infty(B_X) : |x'(x)| = 1 \text{ and } x'(x) = x'(-x) \text{ for all } x \in S_X\},$$

and $\Delta = S_X$. Let $\Phi : B_X \to X$ be a norm one function such that $\Phi(S_X) = S_X$ and $\Phi(-x) = -\Phi(x)$ for all x. Take $\varepsilon > 0$. Fix a norm one function $x' \in \mathscr{W}$. If $y \in S_X$, take $x_0 \in S_X$ such that $\Phi(x_0) = y$. If $x'(x_0) = 1$, then

$$\sup_{x\in\Gamma} \|\Phi(x) + x'(x)y\| \geq \|\Phi(x_0) + x'(x_0)y\| \geq 2\|y\| = 2.$$

If $x'(x_0) = -1 = x'(-x_0)$, then $\Phi(-x_0) = -\Phi(x_0) = -y$, and thus

$$\sup_{x\in\Gamma} \|\Phi(x) + x'(x)y\| \geq \|\Phi(-x_0) + x'(-x_0)y\| \geq \|-y-y\| = 2.$$

Therefore, X has the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$.

The space ℓ^∞ and the function $\Phi(x) = x^3$ show an example of this situation, although ℓ^∞ does not have the Daugavet property.

The proof of the following result is a direct application of Proposition 5.1.

Corollary 5.5. *Let X and Y be Banach spaces and consider Φ, Γ, $\mathscr{W}$ and Δ as in Definition 5.3. The following statements are equivalent.*

(1) *Y has the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$.*
(2) *For every $y \in \Delta$, for every $x' \in \mathscr{W}$ of norm one and for every $\varepsilon > 0$ there are $\omega \in \mathbb{T}$ and an element $x \in S(\omega x', \varepsilon) \cap \Gamma$ such that*

$$\|\omega\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Remark 5.6. Let us show that, under the assumption that the function Φ maps B_X onto B_Y, the Daugavet property for Y implies the Φ-Daugavet property with respect to $\Gamma = B_X$, $\mathscr{W} = \{x' : X \to \mathbb{R} : x' = y^* \circ \Phi,\ y^* \in S_{Y^*}\}$ and $\Delta = S_Y$. This case is canonical, and in a sense also trivial, since the result is a consequence of some simple computations. However, there are more examples that show that not all the cases can be obtained in this way, i.e., there are families of functions $\mathscr{W}$ whose elements are not compositions of a given Φ and the functionals of S_{Y^*} for which Φ satisfies the Daugavet equation.

(1) Let us first show the statement announced above. Let Y be a Banach space with the Daugavet property and let $\Phi : B_X \to Y$ satisfy $\Phi(B_X) = B_Y$. Let us show that then Y has the Φ-Daugavet property with respect to $(B_X, \mathscr{W}, S_Y)$, where $\mathscr{W} = \{x' : X \to \mathbb{R} : x' = y^* \circ \Phi,\ y^* \in S_{Y^*}\}$.

To see this, suppose that $\Phi : B_X \to Y$ satisfies $\Phi(B_X) = B_Y$. Then we claim that for each $\varepsilon > 0$, $y^* \in S_{Y^*}$ and $y \in S_Y$ there is $x \in S(y^* \circ \Phi, \varepsilon)$ such that

$$\|\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Indeed, let $\varepsilon > 0$, $y \in S_Y$ and $y^* \in S_{Y^*}$. Then by the Daugavet property for Y there is an element $z \in S(y^*, \varepsilon)$ such that $\|z + y\| \geq 2 - 2\varepsilon$. Since Φ maps B_X onto B_Y, we find $x \in B_X$ such that $\Phi(x) = z \in S(y^*, \varepsilon)$, and so $\langle \Phi(x), y^* \rangle = y^* \circ \Phi(x) > 1 - \varepsilon$ and $\|\Phi(x) + y\| \geq 2 - 2\varepsilon$. An application of Corollary 5.5 gives the result.

(2) There are also other families of functions $\mathscr{W}$ for which the Daugavet equation is satisfied with a function Φ, but they cannot be defined by composition as in (1). For example, take $X = Y = C(K)$, where K is a perfect compact Hausdorff space, and define $\mathscr{W}$ as the set of continuous linear functionals on $C(K)$. Consider the function $x \mapsto \Phi(x) = x^3$. Clearly, a linear functional cannot be written as a composition of Φ and some other linear functional. However, for each norm one element $y \in S_{C(K)}$ we find an element $x \in S_{C(K)}$ such that $x^3 = y$. This, together with the Daugavet property of $C(K)$, implies (2) in Corollary 5.5. To see this, just take into account that by the Daugavet property of $C(K)$, for each $\varepsilon > 0$, each $y \in S_{C(K)}$ and

each $y^* \in S_{C(K)^*}$ there is $x \in S(y^*, \varepsilon/2)$ such that

$$\|x + y\| > 2 - 2(\varepsilon/2) = 2 - \varepsilon > 2 - 2\varepsilon.$$

Take $z \in S_{C(K)}$ such that $z^3 = x$, and so $\|z^3 + y\| > 2 - 2\varepsilon$. Let us show that $z \in S(y^*, \varepsilon)$ too, that is, (2) in Corollary 5.5 holds. Consider the measurable sets defined by setting $A^+ := \{w \in K : z(w) \geq 0\}$ and $A^- := \{w \in K : z(w) < 0\}$. Take the decomposition of the measure μ that defines the functional y^* as a difference of positive disjointly supported measures $\mu = \mu^+ - \mu^-$. Then, using that $|z^3| \leq |z|$, we get

$$\begin{aligned} 1 - \varepsilon/2 &\leq \int_K z^3 \, d\mu \\ &= \int_{A^+} |z^3| \, d\mu^+ + \int_{A^-} |z^3| \, d\mu^- - \int_{A^+} |z^3| \, d\mu^- - \int_{A^-} |z^3| \, d\mu^+ \\ &\leq \int_{A^+} |z| \, d\mu^+ + \int_{A^-} |z| \, d\mu^- \\ &\leq \mu^+(A^+) + \mu^-(A^-) \leq 1. \end{aligned}$$

Hence

$$\mu^+(A^-) + \mu^-(A^+) \leq \varepsilon/2.$$

Consequently,

$$\begin{aligned} 1 &\geq \int_K z \, d\mu \\ &= \int_{A^+} |z| \, d\mu^+ + \int_{A^-} |z| \, d\mu^- - \int_{A^+} |z| \, d\mu^- - \int_{A^-} |z| \, d\mu^+ \\ &\geq (1 - \varepsilon/2) - (\mu^-(A^+) + \mu^+(A^-)) \\ &\geq 1 - 2(\varepsilon/2) = 1 - \varepsilon. \end{aligned}$$

Then $z \in S(y^*, \varepsilon)$, and we get the result.

(3) Surjectivity of Φ is sometimes not needed if the sets Γ, $\mathscr{W}$ and Δ are adequately chosen. Take now $X = Y = C(K)$, $\Phi(x) = |x|^{1/4}$ and $\mathscr{W}$ the set of probability measures $\mathscr{P}(K) \subset C(K)^*$. Take also $\Gamma = B_{C(K)^+}$ and $\Delta = S_{C(K)^+}$. Then the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$ is satisfied, as a consequence of Corollary 5.5. To see this, note that if $y \in S_{C(K)^+}$ and $\mu \in \mathscr{P}(K)$, then for $\omega = 1$ we obtain by the Daugavet property of $C(K)$, given $\varepsilon > 0$, a (positive) function x of norm one in $S_{C(K)}$ such that $\int_K x \, d\mu \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - 2\varepsilon$. Then since $1 \geq x^{1/4} \geq x$ we obtain

$$\|x^{1/4} + y\| \geq \|x + y\| \geq 2 - 2\varepsilon,$$

i.e., the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$ is satisfied. Again, the elements of $\mathscr{P}(K)$ cannot be factored through Φ.

The following result gives the main tool for extending the Daugavet equation to other functions not belonging to the set of products of scalar functions of $\mathscr{W}$ and elements of the unit sphere of Y. In particular, well-known arguments provide the condition of the following theorem, concerning the inclusion of the image of a slice in a small ball, for the big class of the strong Radon-Nikodým operators. Notably, this class contains the weakly compact operators (see for example the first part of [10], or Theorem 1.1 in [6] for a version directly related with the present paper).

Theorem 5.7. *Let $\Psi : B_X \to Y$ be a norm one function. If the Banach space Y has the Φ-Daugavet property with respect to $(\Gamma, \mathscr{W}, \Delta)$ for $\mathscr{W} \subset \ell_\infty(B_X)$, and for all $\varepsilon > 0$ there are $x' \in \mathscr{W}$, $\delta > 0$ and $y \in \Delta$ such that for all $\omega \in \mathbb{T}$, $\Psi(S(\omega x', \delta) \cap \Gamma) \subset B_\varepsilon(\overline{\omega} y)$, then*

$$\|\Phi + \Psi\|_\Gamma = 2.$$

Proof. Fix $\varepsilon > 0$. By the hypothesis there are $x'_0 \in \mathscr{W}$ and $y \in \Delta$ such that for every $\omega \in \mathbb{T}$, $\|\Psi(x) - \overline{\omega} y\| < \varepsilon$ for all $x \in S(\omega x'_0, \delta) \cap \Gamma$.

By Corollary 5.5, for x'_0 and y there are $\omega_0 \in \mathbb{T}$ and $x_0 \in S(\omega_0 x'_0, \delta) \cap \Gamma$ such that $\|\omega_0 \Phi(x_0) + y\| \geq 2 - 2\varepsilon$. Then,

$$\begin{aligned} \|\Phi + \Psi\|_\Gamma &\geq \|\Phi(x_0) + \Psi(x_0)\| \\ &\geq \|\Phi(x_0) + \overline{\omega_0} y\| - \|\Psi(x_0) - \overline{\omega_0} y\| \\ &= \|\omega_0 \Phi(x_0) + y\| - \|\Psi(x_0) - \overline{\omega_0} y\| \\ &\geq 2 - 2\varepsilon - \varepsilon = 2 - 3\varepsilon. \end{aligned}$$

Since this happens for each $\varepsilon > 0$, we obtain that $\|\Phi + \Psi\|_\Gamma = 2$. □

The proof of the following corollary is just an application of Theorem 5.7 for $\mathscr{W} := \{x' = y^* \circ \Psi : y^* \in Y^*, \|y^* \circ \Psi\| = 1\}$, $\Gamma = B_X$ and $\Delta = S_X$, together with the argument in the proof of Theorem 3.11 regarding weakly compact sets that gives the condition for applying Theorem 5.7. The same comments regarding Radon-Nikodým functions given in Remark 3.12 apply in the present case.

Corollary 5.8. *Let $\Phi : B_X \to Y$ be a norm one function such that $\Phi(B_X) = B_Y$ and let $\Psi : B_X \to Y$ be a norm one weakly compact function. Suppose Y has the Φ-Daugavet property. Then $\|\Phi + \Psi\| = 2$.*

We finish the paper by showing some special new tools for obtaining applications in the case of $C(K)$-spaces and $L^1(\mu)$-spaces.

5.1. **A general test for the Φ-Daugavet property: the case of functions on $C(K)$-spaces.** The requirement $\Psi(S(\omega x', \delta) \cap \Gamma) \subset B_\varepsilon(\overline{\omega} y)$ in Theorem 5.7 seems to be a difficult property to check directly . The next result provides a simpler test that can be used in some cases. We will use this new tool to analyse the Daugavet equation for functions on $C(K)$-spaces.

Proposition 5.9. *Let X be a Banach space. Let $z \in S_X$, $K > 0$ and let $\Phi, \Psi : B_X \to X$ be norm one functions. Take a subset $B \subset B_X$. The following statements are equivalent.*

(1) *There is a w^*-compact convex set $V \subset X^*$ such that for all finite sequences $x_1, \ldots, x_n \in B$ and positive scalars $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^n \alpha_i = 1$ we have*

$$\sum_{i=1}^n \alpha_i \|\Psi(x_i) - z\| \leq K \sup_{x^* \in V} \Big(1 - \Big\langle \sum_{i=1}^n \alpha_i \Phi(x_i), x^* \Big\rangle\Big).$$

(2) *For each $\varepsilon > 0$ there exists $x_0^* \in V$ such that*

$$\|\Psi(x) - z\| \leq K(1 - \langle \Phi(x), x_0^* \rangle)$$

for all $x \in B_X$.

These equivalent properties imply that for each $\varepsilon > 0$ there exists $x_0^* \in V$ such that $\Psi(S(x_0^* \circ \Phi, \varepsilon) \cap B) \subset B_{K\varepsilon}(z)$.

Proof. We shall obtain this result as a consequence of Ky Fan's lemma (see [12, p. 40]), so it is in essence a consequence of the Hahn-Banach theorem.

We only sketch the proof. Consider the concave set of convex functions $\Upsilon : V \to \mathbb{R}$ defined by

$$\Upsilon(x^*) := \sum_{i=1}^{n} \alpha_i \|\Psi(x_i) - z\| - K\Big(1 - \Big\langle \sum_{i=1}^{n} \alpha_i \Phi(x_i), x^* \Big\rangle\Big),$$

where $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$ and $x_1, \ldots, x_n \in B$. The inequality in (1) provides for such a Υ an element $x_\Upsilon^* \in V$ such that $\Upsilon(x_\Upsilon^*) \leq 0$. Ky Fan's Lemma gives an element $x_0^* \in V$ such that $\Upsilon(x_0^*) \leq 0$ for all the functions Υ in the family. This proves (1) $\Rightarrow$ (2), and the converse is obvious.

On the other hand, if $x \in S(x_0^* \circ \Phi, \varepsilon) \cap B$, then

$$\|R(x) - z\| \leq K(1 - \langle \Phi(x), x_0^* \rangle) \leq K\varepsilon.$$

This proves the final statement. □

Example 5.10. Let us show an application of the criterion given in Proposition 5.9. Let $X = C(K)$ and $V = B_{C(K)^*}$. Take a positive norm one function f in $C(K)$. Define the class of functions C by

$$C = \{g \in B_{C(K)} : g^2 \leq f \leq |g|\}.$$

Let us see that the requirements of Proposition 5.9 are satisfied for $B = C$ and Φ and Ψ defined by $\Phi(g) = g^2$ and $\Psi(g) = |g|$. Note that for all positive functions $h \in B_{C(K)}$, $\mathbf{1} - h \leq \mathbf{1} - h^2$. Then for all $g_1, \ldots, g_n \in C$ and positive $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i \||g_i| - \mathbf{1}\| &\leq \sum_{i=1}^{n} \alpha_i \|\mathbf{1} - f\| = \|\mathbf{1} - f\| \leq \sum_{i=1}^{n} \alpha_i \|\mathbf{1} - g_i^2\| \\ &\leq \sup_{x^* \in B_{C(K)^*}} \Big(1 - \Big\langle \sum_{i=1}^{n} \alpha_i g_i^2, x^* \Big\rangle\Big). \end{aligned}$$

Consequently, an application of the proposition shows that for each $\varepsilon > 0$ there exists $x_0^* \in C(K)^*$ such that $\Psi(S(x_0^* \circ \Phi, \varepsilon) \cap C) \subset B_{K\varepsilon}(\mathbf{1})$.

Note that for applying Proposition 5.9 in a nontrivial way, it must be assumed that $S(x^* \circ \Phi, \varepsilon) \cap B \neq \emptyset$. For example, in the next corollary the requirement is satisfied, since $B = B_X$. Note also that the requirement on Φ of being surjective from B_X to B_X ensures that the slices $S(x^* \circ \Phi, \varepsilon)$ are not empty themselves.

Corollary 5.11. *Let $\Phi, \Psi : B_X \to X$ be norm one functions. If there exist $z \in S_X$ and $K > 0$ such that for all $x_1, \ldots, x_n \in B_X$ and $\alpha_1, \ldots, \alpha_n \geq 0$*

such that $\sum_{i=1}^n \alpha_i = 1$ *there is an element* $x \in B_X$ *such that the inequality*

$$\sum_{i=1}^n \alpha_i \|\Psi(x_i) - z\| \le K \Big\| x - \sum_{i=1}^n \alpha_i \Phi(x_i) \Big\|$$

holds, then for each $\varepsilon > 0$ *there exist* $\delta > 0$ *and* $x_0^* \in S_{X^*}$ *such that* $\Psi(S(x_0^* \circ \Phi, \delta)) \subset B_{K\varepsilon}(z)$.

Proof. Fix some $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n$ and consider the element $x \in B_X$ given in the statement. Using the inequality we obtain

$$\begin{aligned}\sum_{i=1}^n \alpha_i \|\Psi(x_i) - z\| &\le K \sup_{x^* \in B_{X^*}} \Big\langle x - \sum_{i=1}^n \alpha_i \Phi(x_i), x^* \Big\rangle \\ &\le K \sup_{x^* \in B_{X^*}} \Big(1 - \Big\langle \sum_{i=1}^n \alpha_i \Phi(x_i), x^* \Big\rangle \Big).\end{aligned}$$

An application of Proposition 5.9 gives the result. □

Example 5.12. Take $X = C(K)$ for a perfect K, $\Phi(x) = x^2$ and $\Psi(x) = (\int_K x^2 \, d\mu) y$ for a probability measure on K and a fixed function $y \in S_{C(K)}$. Then taking $z = y$ we get

$$\begin{aligned}\sum_{i=1}^n \alpha_i \Big\| \Big(\int_K x_i^2 \, d\mu \Big) y - z \Big\| &\le \sum_{i=1}^n \alpha_i \Big(1 - \int_K x_i^2 \, d\mu \Big) \|z\| \\ &= \int_K d\mu - \sum_{i=1}^n \alpha_i \int_K x_i^2 \, d\mu \\ &\le \Big\| \mathbf{1} - \sum_{i=1}^n \alpha_i x_i^2 \Big\|\end{aligned}$$

for each finite set of functions $x_1, \dots, x_n \in B_{C(K)}$ and $0 \le \alpha_1, \dots, \alpha_n$ such that $\sum_{i=1}^n \alpha_i = 1$.

Consequently, the result holds and for each $\varepsilon > 0$ there is a slice $S(x_0^* \circ \Phi, \delta)$ such that $\Psi(S(x_0^* \circ \Phi, \delta)) \subset B_\varepsilon(z)$. However, observe that the slices $S(x_0^* \circ \Phi, \delta)$ can be empty in this case, and so the Daugavet equation cannot be assured in general by applying Remark 5.6(1). In fact, the equation does not hold if one takes for example $y = -\mathbf{1}$; in this case,

$$\sup_{x \in B_{C(K)}} \Big\| x^2 + \Big(\int_K x^2 \, d\mu \Big) (-\mathbf{1}) \Big\| \le 1.$$

However, if we take $y = \mathbf{1}$, we obtain $\sup_{x \in B_{C(K)}} \|x^2 + (\int_K x^2 \, d\mu) \mathbf{1}\| = 2$, and the Daugavet equation holds.

Note that Remark 5.6(1) provides the Daugavet equation for the "order 3 version" of this result, since $\Phi(x) = x^3$ satisfies $\Phi(B_{C(K)}) = B_{C(K)}$. Therefore, due to the Daugavet property of $C(K)$, for every $\mu \in S_{C(K)^*}$ and $y \in S_{C(K)}$ we have

$$\sup_{x \in B_{C(K)}} \Big\| x^3 + \Big(\int_K x^3 \, d\mu \Big) y \Big\| = 2.$$

5.2. **The case of $L^1(\mu)$-spaces for non-atomic measures μ.** In this subsection we analyse several functions Φ that are natural candidates for being functions Φ on (the unit ball of) L^1 in the results exposed in the previous sections.

Some cases that are in a sense canonical for applying our results are the following. The first one given by the function $\Phi_0(f) := |f|$, $f \in L^1(\mu)$. The second case is the function $\Phi_* := B_{L^1[0,1]} \to B_{L^1[0,1]}$ given by the expression $\Phi_*(f) = |f| * |f|$, where $*$ denotes the convolution in $L^1[0,1]$; the third one is given by the formula $\Phi_2(f) := (\int_\Omega |f|\, d\mu) \cdot f$. Adapting the proof of Theorem 2.6 and Proposition 2.7 in [13] that is based in some classical arguments for the Daugavet property in $L^1(\mu)$, we obtain the following results, which can be applied to these examples.

Lemma 5.13. *Let (Ω, Σ, μ) be a non-atomic measure space. Let $\mathscr{W}$ be a set of norm one scalar functions in $\ell_\infty(B_{L^1(\mu)})$. Let $\Phi : B_{L^1(\mu)} \to L^1(\mu)$ be a norm one function such that $\|\Phi(z)\| = 1$ for each $z \in S_{L^1(\mu)}$ and satisfying also that for each $\delta, \varepsilon > 0$ and $x' \in \mathscr{W}$ we can find a norm one simple function z such that $\mu(\operatorname{supp} \Phi(z)) < \delta$ and $|x'(\Phi(z))| > 1 - \varepsilon$. Then*

$$\|\Phi + x' \otimes y\| = 2$$

for all $x' \in \mathscr{W}$, $y \in S_{L^1(\mu)}$.

Proof. We use Proposition 5.1. Let $\varepsilon > 0$, $x' \in \mathscr{W}$ and $y \in S_{L^1(\mu)}$. Let us show that we can find ω and an element $x \in S(\omega x', \varepsilon)$ such that

$$\|\omega \Phi(x) + y\| > 2 - 2\varepsilon.$$

First note that there exists $\delta > 0$ such that $\int_A |y|\, d\mu < \varepsilon$ for each $A \in \Sigma$ such that $\mu(A) < \delta$. By the requirement on Φ for these $\delta > 0$ and $\varepsilon > 0$ and choosing an $\omega \in \mathbb{T}$ such that $\omega x'(z) = |x'(z)|$, we have that $z \in S(\omega x', \varepsilon)$. Thus we obtain

$$\begin{aligned} \|y + \omega\Phi(z)\| &= \int_{\Omega \setminus \operatorname{supp} \Phi(z)} |y|\, d\mu + \int_{\operatorname{supp} \Phi(z)} |y + \omega \Phi(z)|\, d\mu \\ &\geq \|y\| - \int_{\operatorname{supp} \Phi(z)} |y|\, d\mu + \|\Phi(z)\| - \int_{\operatorname{supp} \Phi(z)} |y|\, d\mu \\ &> 2 - 2\varepsilon. \end{aligned}$$

Proposition 5.1 gives the result. □

Lemma 5.14. *Let (Ω, Σ, μ) be a non-atomic measure space. Let $\mathscr{W}$ be a set of norm one scalar functions from $L^1(\mu)$ that are weakly sequentially continuous. Let $\Phi : B_{L^1(\mu)} \to L^1(\mu)$ be a norm one map that maps $S_{L^1(\mu)}$ onto $S_{L^1(\mu)}$. Then*

$$\|\Phi + x' \otimes y\| = 2$$

for all $x' \in \mathscr{W}$, $y \in S_{L^1(\mu)}$.

Proof. Let $x' \in \mathscr{W}$ and let $\delta, \varepsilon > 0$. Since it is weakly sequentially continuous, by Lemma 2.5 in [13], we can find a norm one simple function x such that $\mu(\operatorname{supp} x) < \delta$ and $|x'(\Phi(x))| > 1 - \varepsilon$. The surjectivity of Φ provides an element $z \in S_{L^1(\mu)}$ such that $\Phi(z) = x$. This z satisfies the requirement for Φ in Lemma 5.13; hence the result holds. □

In order to adapt the results on weak sequential continuity that are shown to be useful in the case of the polynomial Daugavet property for $L^1(\mu)$ (see [13]), there are two requirements on Φ that are useful and are included in the following definition.

In the next proposition, we call a function $\Phi : B_{L^1(\mu)} \to L^1(\mu)$ *admissible* if the following requirements are satisfied.

(i) Φ must send functions of small support to functions of small support, i.e., for each $\delta > 0$ there is a $\delta' > 0$ such that for a function $f \in L^1(\mu)$ with support satisfying $\mu(\operatorname{supp} f) < \delta'$, we have that $\mu(\operatorname{supp}_{\Phi(f)}) < \delta$.

(ii) For all $f \in S_{L^1(\mu)}$, $\|\Phi(f)\| = 1$.

Note that the mappings Φ_0, Φ_* and Φ_2 mentioned at the beginning of this subsection are admissible.

Proposition 5.15. *Let (Ω, Σ, μ) be a non-atomic measure space. Let $\Phi : B_{L^1(\mu)} \to L^1(\mu)$ be a norm one admissible function. Let $\mathscr{W} \subset \ell_\infty(B_{L^1(\mu)})$ be a set of norm one scalar functions from $B_{L^1(\mu)}$ to $\mathbb{K}$ such that $x' \circ \Phi$ is norm one and weakly sequentially continuous for each $x' \in \mathscr{W}$. Then*

$$\|\Phi + x' \otimes y\| = 2$$

for all $x' \in \mathscr{W}$, $y \in S_{L^1(\mu)}$.

Proof. We use Lemma 5.13. Let $\varepsilon, \delta > 0$ and $p \in \mathscr{W}$. Note that since Φ is admissible, there is a $\delta' > 0$ such that if $f \in L^1(\mu)$ and $\mu(\operatorname{supp} f) < \delta'$, we have that $\mu(\operatorname{supp} \Phi(f)) < \delta$.

Since $x' \circ \Phi$ is weakly sequentially continuous, by Lemma 2.5 in [13], we can find a norm one simple function z such that $\mu(\operatorname{supp} z) < \delta'$ and $|x'(\Phi(z))| > 1 - \varepsilon$. Finally, notice that we also have that $\mu(\operatorname{supp} \Phi(z)) < \delta$, by the admissibility of Φ. Lemma 5.13 gives the result. $\square$

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Department of Mathematics, Freie Universität Berlin, Arnimallee 6, D-14 195 Berlin, Germany
E-mail address: brach.stefan@gmail.com

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain.
E-mail address: easancpe@mat.upv.es

Department of Mathematics, Freie Universität Berlin, Arnimallee 6, D-14 195 Berlin, Germany
E-mail address: werner@math.fu-berlin.de