“Almost flat” simplices in Riemannian manifolds

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Question: What is a triangle in a curved 3-manifold? (clear for space forms: convex hull) Generally: What is an \( n \)-simplex in an \( m \)-manifold?

Requirements:
(1.) Facets must not depend on opposite vertex.
(2.) as “flat” (totally geodesic) as possible.

Definition: Let \( \Delta \) be the standard \( n \)-simplex. If \( p_0, \ldots, p_n \) lie in a common convex ball \( B \), let \( x: \Delta \to M, \lambda \mapsto \argmin_{y \in B} E(\lambda, a) \) be the barycentric mapping with respect to vertices \( p_i \) and \( x(\Delta) \) be the corresponding Karcher simplex.

Karcher ’77: \( x(\lambda) \) is “Riemannian centre of mass” with respect to point masses \( \lambda^i \) concentrated at \( p_i \), nowadays called “Karcher mean”.

Properties (Karcher ’77, Rustamov ’10, Sander ’12):
- well-defined and smooth
- edges are mapped to geodesics
- all \( p_i \) in totally geodesic submanifold \( N \) \( \Rightarrow x(\Delta) \subset N \)
- \( \lambda^i = 0 \Rightarrow x(\lambda) \) is independent of \( p_i \)

Applications:
- Galerkin methods for maps \( M \to \mathbb{R} \): replace \( M \) by piecewise flat manifold
- Galerkin methods for maps \( \Omega \subset \mathbb{R}^m \to M \): use barycentric mapping as interpolation (Sander ’12)
- approximate minimal submanifolds (discrete minimal surfaces in manifolds)
- formulae for discrete curvatures stay correct up to an additional error of order \( C_0 h \).

Idea: Define barycentric coordinates via the energy
\[
E(\lambda, a) := \lambda^i d^2(p_i, a)
\]
because in Euclidean space, \( p = \lambda^i p_i \) minimises \( E(\lambda, a) \) over \( a \in \mathbb{R}^n \).

This energy is defined in every complete Riemannian manifold if \( \lambda \) is replaced by geodesic distance.

Theorem (v. D., Glickenstein, Wardetzky ’13): Assume \((M^m, g)\) is complete with sectional curvatures bounded by \( C_0 \), all \( p_i \) lie in a convex ball \( B \), and \( x \) is its barycentric mapping. If edges \( E_{ij} = d(p_i, p_j) < h \) define a Euclidean simplex with volume \( > \alpha h^n \), then \( x \) is injective, and
\[
\|\nabla dx\| \leq c(m, n) \alpha^{-1} C_0 h.
\]
In particular, the mean curvature vector of the Karcher simplex has norm
\[
\|H\| \leq c(m, n) \alpha^{-1} C_0 h.
\]

References