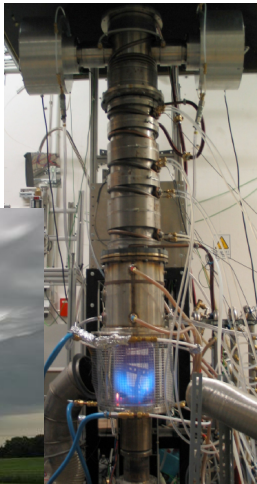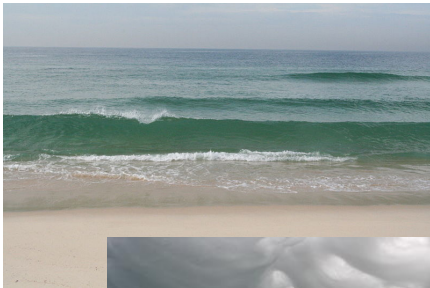# Towards a Semi-Implicit All Froude Number Godunov-type Method for Shallow Water Flows

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Motivation

Outline

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The Shallow Water Equations

Non-dimensional form:

$$h_t + \nabla \cdot (h\boldsymbol{v}) = 0$$

$$(h\boldsymbol{v})_t + \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\mathsf{Fr}^2}\,\nabla p = -\frac{1}{\mathsf{Fr}^2}\,h\nabla h_{\mathrm{B}}$$

$$p = \frac{h^2}{2}$$
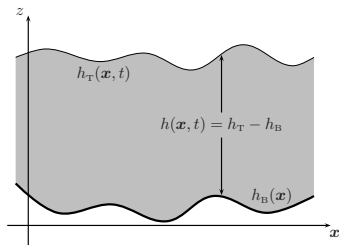
- $\mathsf{Fr} = \dfrac{v'_{\mathrm{ref}}}{\sqrt{g'\,h'_{\mathrm{ref}}}}$
- hyperbolic system of conservation laws
- similar to Euler equations, no energy equation

Zero Froude Number ("Incompressible") Limit

Limit Equations:

$$\begin{aligned}
h_t &+& \nabla \cdot (h\boldsymbol{v}) & & &=& 0 \\
(h\boldsymbol{v})_t &+& \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) &+& h\nabla h^{(2)} &=& \boldsymbol{0}
\end{aligned}$$

- $h = h_0(t)$ given through boundary conditions
- divergence constraint for velocity field:

$$\int\limits_{\partial V} h\boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = -|V| \frac{dh_0}{dt} \qquad \text{for } V \subset \Omega$$

- $h^{(2)}$: Lagrange multiplier; ensures compliance with divergence constraint

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Computing Low Froude Number Shallow Water Flows

Arising difficulties:

- spatial height (pressure) variations vanish as $Fr \to 0$, but they do affect the velocity field at leading order

- spatial homogeneity of leading order pressure implies an elliptic divergence constraint for the mass flux

- eigenvalues of the Jacobian flux matrix $\boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v} \cdot \boldsymbol{n} \pm \sqrt{h}/Fr$ become singular

- explicit methods suffer from a Courant-Friedrichs-Lewy time step restriction with $\delta t \leq \mathcal{O}(Fr)$

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Conservative Low Froude Number Numerics

Task: Construct a scheme, which . . .

- . . . allows time steps independent of the Froude number
- . . . conserves mass, momentum, total energy:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

- . . . is second order accurate in time and space
- . . . uses machinery of Godunov-type methods
- . . . requires the solution of at most linear, scalar equations
- . . . ensures, that for $\mathrm{Fr} = 0$, advection velocities and final momentum satisfy divergence constraint

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# Construction of the Projection Method

method in conservation form:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

$$\mathbf{F}_I \coloneqq \mathbf{F}_I^* + \mathbf{F}_I^{\mathsf{MAC}} + \mathbf{F}_I^{\mathsf{P2}}$$

- advective fluxes $\mathbf{F}_I^*$ from second order Godunov-type method (applied to auxiliary system)
- $\mathbf{F}_I^{\mathsf{MAC}}$ from (MAC) projection, which corrects advection velocity divergence
- $\mathbf{F}_I^{\mathsf{P2}}$ from second projection, which adjusts new time level divergence of cell-centered velocities
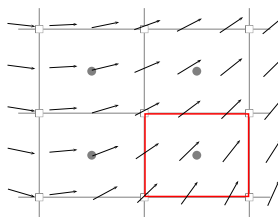
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Correction of the Fluxes
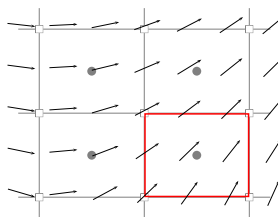
1. (MAC) Projection:

$$(h\boldsymbol{v})_I = (h\boldsymbol{v})_I^* - \frac{\delta t}{2} h_0 (\nabla h^{(2)})_I$$

corrects advective fluxes on boundary of control volume

Correction of the Fluxes
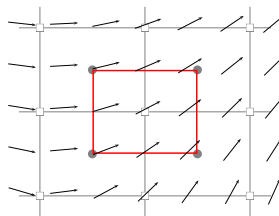

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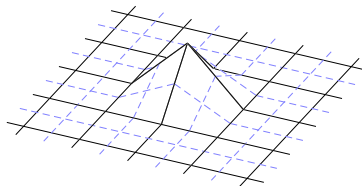corrects advective fluxes on boundary of control volume

2. Projection:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t (h_0 \nabla h^{(2)})^{n+1/2}$$

adjusts momentum to obtain correct divergence for new velocity field

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Discretization of the Poisson-Type Problems

Petrov-Galerkin FE discretization [SÜLI, 1991]:

- bilinear trial functions for the unknown $h^{(2)}$
- piecewise constant test functions on the dual discretization

- discrete divergence can be exactly calculated
- Laplacian has compact stencil
- $L = D(G) \rightsquigarrow$ exact projection method

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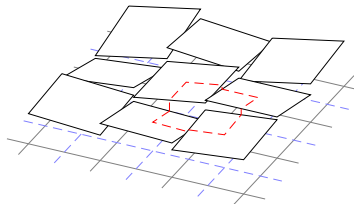
Exact Projection Method

Local updates within each cell V_{ij}:

$$(h\boldsymbol{v})^{n+1}(x,y) = (h\boldsymbol{v})^{**}(x,y) - \delta t\, h_0 \nabla h^{(2)^{n+1/2}}(x,y)$$

where

$$\nabla h^{(2)^{n+1/2}}(x,y)|_{ij} = \begin{pmatrix} \partial_x h^{(2)} \\ \partial_y h^{(2)} \end{pmatrix}_{ij} + \begin{pmatrix} y - y_j \\ x - x_i \end{pmatrix} \partial^2_{xy} h^{(2)}_{ij}$$

⤳ second projection modifies piecewise linear reconstruction:

$$\partial_x(h\boldsymbol{v})^{n+1}_{ij} = \partial_x(h\boldsymbol{v})^{**}_{ij} - \delta t\, h_0 \begin{pmatrix} 0 \\ \partial^2_{xy} h^{(2)}_{ij} \end{pmatrix}$$

$$\partial_y(h\boldsymbol{v})^{n+1}_{ij} = \partial_y(h\boldsymbol{v})^{**}_{ij} - \delta t\, h_0 \begin{pmatrix} \partial^2_{xy} h^{(2)}_{ij} \\ 0 \end{pmatrix}$$

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Stability of the Projection Step

Derive saddle point problem ...

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \nabla_h(\delta t\, h_0 h^{(2)})$$

$$\frac{1}{2}\,\nabla_h \cdot \left[(h\boldsymbol{v})^{n+1} + (h\boldsymbol{v})^n\right] = -\frac{dh_0}{dt}$$

... and employ theory of mixed finite elements (Nicolaïdes, 1982):

Theorem (V. & Klein 2007)

The generalized mixed formulation has a unique and stable solution $((h\boldsymbol{v})^{n+1}, \delta t\, h_0\, h^{(2)})$.

- we obtain approximations, in which the solution of the Poisson problem $h^{(2)}$ and the momentum update $(h\boldsymbol{v})^{n+1}$ cannot decouple!

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Weakly Compressible Flows

mimic Zero Froude Number results by decomposing pressure into two components:

$$p(t, \boldsymbol{x}; \mathsf{Fr}) = p_0(t) + \mathsf{Fr}^2 \, p'(t, \boldsymbol{x})$$

$$
\begin{aligned}
h_t + \nabla \cdot (h\boldsymbol{v}) &= 0 \\
(h\boldsymbol{v})_t + \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) + \nabla p' &= 0 \\
p = p_0(t) + \mathsf{Fr}^2 \, p'(t, \boldsymbol{x}) &= \frac{h^2}{2}
\end{aligned}
$$

$$p_t + h\nabla \cdot (h\boldsymbol{v}) = 0$$

Extension of (zero Froude number) projection method by inclusion of local time derivatives of p' into projection steps

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Explicit Predictor for Advection

Auxiliar system:

$$h_t + \nabla \cdot (h\boldsymbol{v}) = 0$$
$$(h\boldsymbol{v})_t + \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) + \nabla p = -(1 - \mathsf{Fr}^2)\nabla p'^{,\text{old}}$$
$$p = \frac{h^2}{2}$$

yields first order accurate prediction:

$$(h\boldsymbol{v})^{n+1/2} = (h\boldsymbol{v})^{*,n+1/2} + \mathcal{O}\left(\delta t^2\right)$$
$$= (h\boldsymbol{v})^{*,n+1/2} - \frac{\delta t}{2}(1 - \mathsf{Fr}^2)\nabla \delta p' + \mathcal{O}\left(\delta t^2\right)$$

$$\delta p' \coloneqq p'^{,n+1} - p'^{,n}$$

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First Correction – Advective Fluxes

Pressure equation:

$$\left(\frac{dp_0}{dt}\right)^{n+1/2} + \mathsf{Fr}^2\left(\frac{\partial p'}{\partial t}\right)^{n+1/2} = -h\,\nabla\cdot\left((h\boldsymbol{v})^* - \frac{\delta t}{2}(1-\mathsf{Fr}^2)\nabla\delta p'\right)$$

Since:

$$\left(\frac{\partial p'}{\partial t}\right)^{n+1/2} = \frac{\delta p'}{\delta t} + \mathcal{O}\left(\delta t^2\right)$$

this yields Helmholtz equation for $\delta p'$:

$$-\frac{\mathsf{Fr}^2}{\delta t}\delta p' + h^*\frac{\delta t}{2}(1-\mathsf{Fr}^2)\Delta\delta p' = \left(\frac{dp_0}{dt}\right)^{n+1/2} - \left(\frac{dp^*}{dt}\right)^{n+1/2}$$

Second Correction – Pressure Term

intermediate momentum update:

$$(h\boldsymbol{v})^{**} := (h\boldsymbol{v})^n - \delta t\, \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v})^{n+1/2}$$

momentum at the new time level:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t\, \nabla p'^{,n+1/2} + \mathcal{O}\!\left(\delta t^2\right)$$

since pressure update already known, obtain Poisson-type equation:

$$
\begin{aligned}
\left(\frac{\partial p}{\partial t}\right)^{n+1/2} &= -\frac{1}{2}\left[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1} \nabla \cdot (h\boldsymbol{v})^{n+1} \right] \\
&= -\frac{1}{2}\left[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1} \nabla \cdot ((h\boldsymbol{v})^{**} - \delta t \nabla p') \right]
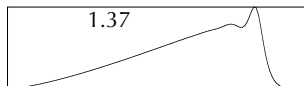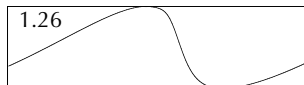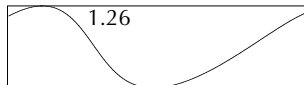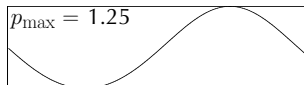\end{aligned}
$$

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Advection of a vortex
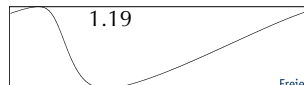
quasi stationary, smooth solution, periodic b.c., Fr ~ 0.1
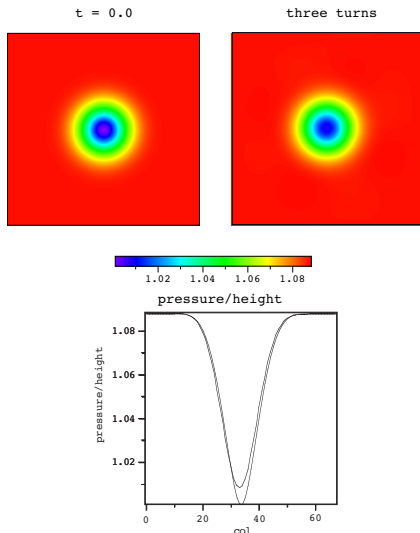
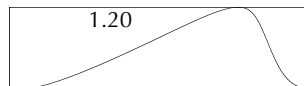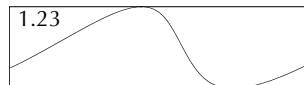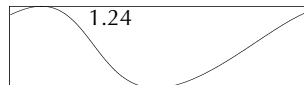# Barotropic Gravity Wave

Fr $\sim$ 0.1, 128 cells, CFL $= 0.9$

1st "projection" only

$p_{\max} = 1.25$

1.26

1.26

1.37

1.59

1st + 2nd "projection"

1.25

1.24

1.23

1.20

1.19
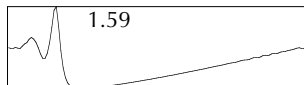
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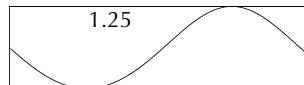
Summary

A Cartesian grid semi-implicit method has been presented.

- conservative method with two flux corrections motivated by zero Froude number projection method
- incorporates Godunov-type method in predictor step
- solution of one Helmholtz and one Poisson-type problem in correction step

- Outlook
  - arbitrary Froude number
  - better representation of gravity waves (incorporation of results from multiscale asymptotic analysis, see Klein (1995))
  - inclusion of source terms (bottom topography)

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For Further Information/Reading

📄 Th. Schneider, N. Botta, K.J. Geratz and R. Klein

Extension of Finite Volume Compressible Flow Solvers to Multi-dimensional, Variable Density Zero Mach Number Flows.

Journal of Computational Physics, 155 : 248–286, 1999.

📄 S. Vater

A New Projection Method for the Zero Froude Number Shallow Water Equations.

PIK Report No. 97, Potsdam Institute for Climate Impact Research, 2005.

📄 S. Vater & R. Klein

Stability of a Cartesian Grid Projection Method for Zero Froude Number Shallow Water Flows.

submitted to: Numerische Mathematik, 2007.
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