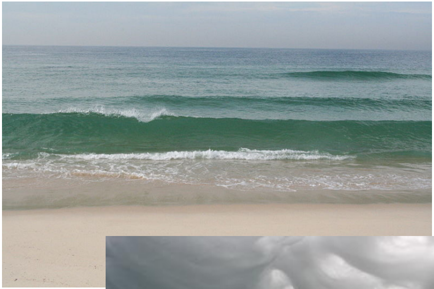
A Semi-Implicit All Froude Number Godunov-type Method for Shallow Water Flows

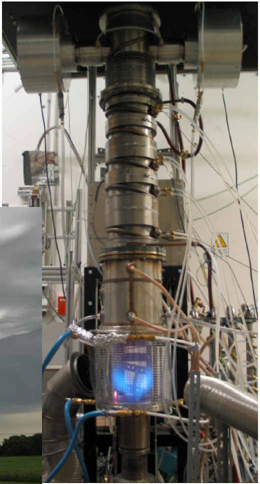
Stefan Vater & Rupert Klein

Department of Mathematics and Computer Science
Freie Universität Berlin

9th Hirschegg Workshop on Conservation Laws
September, 11th 2009

Motivation





Motivation

Tama, Iowa KCCI-TV webcam on 6 May 2007

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Outline

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1. Geophysical Flows
   - Governing Equations
   - Numerical Difficulties
2. Projection Method for Incompressible Flows
3. Weakly Compressible Flows

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Three Dimensional Compressible Flow Equations

Non-dimensional form:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\ (\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\mathsf{M}^2} \nabla p + \frac{1}{\mathsf{Ro}} \boldsymbol{\Omega} \times \rho \boldsymbol{v} &= -\frac{1}{\mathsf{Fr}^2} \rho \boldsymbol{k} \\ (\rho e)_t + \nabla \cdot ([\rho e + p] \boldsymbol{v}) &= 0 \end{aligned}$$

$$\rho e = \frac{p}{\gamma - 1} + \mathsf{M}^2 \frac{\rho \boldsymbol{v}^2}{2}$$

$$\mathsf{M} = \frac{v'_{\text{ref}}}{c'_{\text{ref}}} \approx \frac{5 \text{ m/s}}{300 \text{ m/s}} \ll 1 , \quad \mathsf{Fr} = \frac{v'_{\text{ref}}}{\sqrt{g' \, h'_{\text{sc}}}} \approx \frac{5 \text{ m/s}}{\sqrt{10 \cdot 10000} \text{ m/s}} \ll 1$$

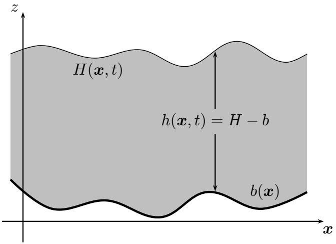
The Shallow Water Equations

Non-dimensional form:

$$h_t + \nabla \cdot (h\boldsymbol{v}) = 0$$

$$(h\boldsymbol{v})_t + \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\mathsf{Fr}^2} \nabla p = -\frac{1}{\mathsf{Fr}^2} h \nabla b$$

$$p = \frac{h^2}{2}$$



- $\mathsf{Fr} = \frac{v'_{\text{ref}}}{\sqrt{g'\, h'_{\text{ref}}}}$
- hyperbolic system of conservation laws
- similar to Euler equations, no energy equation

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Zero Froude Number ("Incompressible") Limit

Limit Equations:

$$\begin{aligned} h_t &+ \nabla \cdot (h\boldsymbol{v}) & &= 0 \\ (h\boldsymbol{v})_t &+ \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) &+ \nabla p^{(2)} &= \boldsymbol{0} \end{aligned}$$

- $h = h_0(t)$ given through boundary conditions
- divergence constraint for velocity field:

$$\int\limits_{\partial V} h\boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = -|V| \frac{dh_0}{dt} \qquad \text{for } V \subset \Omega$$

- $p^{(2)}$: Lagrange multiplier; ensures compliance with divergence constraint

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Computing Low Froude Number Shallow Water Flows

Arising difficulties:

- spatial height (pressure) variations vanish as $\mathrm{Fr} \to 0$, but they do affect the velocity field at leading order
- spatial homogeneity of leading order pressure implies an elliptic divergence constraint for the mass flux
- eigenvalues of the Jacobian flux matrix $\boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v} \cdot \boldsymbol{n} \pm \sqrt{h}/\mathrm{Fr}$ become singular
- explicit methods suffer from a Courant-Friedrichs-Lewy time step restriction with $\delta t \leq \mathcal{O}(\mathrm{Fr})$

Conservative Low Froude Number Numerics

Task: Construct a scheme, which ...

- ... allows time steps independent of the Froude number
- ... conserves mass, momentum, total energy:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

- ... is second order accurate in time and space
- ... uses machinery of Godunov-type methods
- ... requires the solution of at most linear, scalar equations
- ... ensures, that for Fr $= 0$, advection velocities and final momentum satisfy divergence constraint

Outline

Construction of the Projection Method

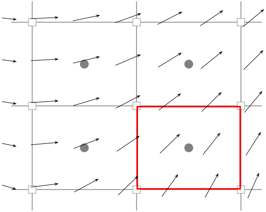
method in conservation form:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

$$\mathbf{F}_I := \mathbf{F}_I^* + \mathbf{F}_I^{\mathsf{MAC}} + \mathbf{F}_I^{\mathsf{P2}}$$

- advective fluxes $\mathbf{F}_I^*$ from second order Godunov-type method (applied to auxiliary system)
- $\mathbf{F}_I^{\mathsf{MAC}}$ from (MAC) projection, which corrects advection velocity divergence
- $\mathbf{F}_I^{\mathsf{P2}}$ from second projection, which adjusts new time level divergence of cell-centered velocities

Correction of the Fluxes

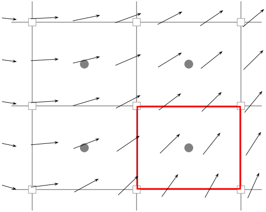


1. (MAC) Projection:

$$(h\boldsymbol{v})_I = (h\boldsymbol{v})_I^* - \frac{\delta t}{2} \nabla p_I^{(2)}$$

corrects advective fluxes on boundary of control volume

Correction of the Fluxes



1. (MAC) Projection:

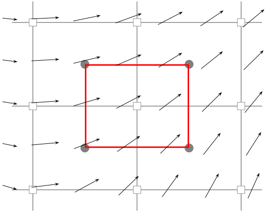
$$(h\boldsymbol{v})_I = (h\boldsymbol{v})^*_I - \frac{\delta t}{2}\nabla p^{(2)}_I$$

corrects advective fluxes on boundary of control volume

2. Projection:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t \nabla p^{(2),n+1/2}$$

adjusts momentum to obtain correct divergence for new velocity field



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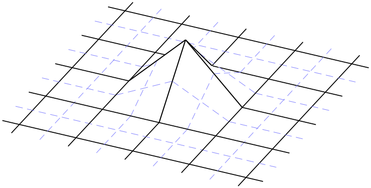
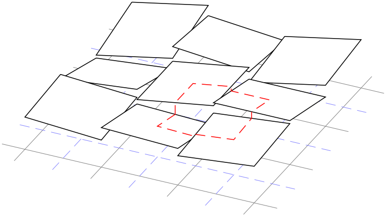
Discretization of the Poisson-Type Problems

Petrov-Galerkin FE discretization [Süli, 1991]:



- bilinear trial functions for the unknown $p^{(2)}$
- piecewise constant test functions on the dual discretization



- discrete divergence can be exactly calculated
- Laplacian has compact stencil
- $\mathsf{L} = \mathsf{D}(\mathsf{G}) \rightsquigarrow$ exact projection method

Exact Projection Method

Local updates within each cell V_{ij}:

$$(h\boldsymbol{v})^{n+1}(x,y) = (h\boldsymbol{v})^{**}(x,y) - \delta t\,\nabla p^{(2),n+1/2}(x,y)$$

where

$$\nabla p^{(2),n+1/2}(x,y)|_{ij} = \begin{pmatrix}\partial_x p^{(2)} \\ \partial_y p^{(2)}\end{pmatrix}_{ij} + \begin{pmatrix} y - y_j \\ x - x_i \end{pmatrix} \partial^2_{xy} p^{(2)}_{ij}$$

$\rightsquigarrow$ second projection modifies piecewise linear reconstruction:

$$\partial_x (h\boldsymbol{v})^{n+1}_{ij} = \partial_x (h\boldsymbol{v})^{**}_{ij} - \delta t \begin{pmatrix} 0 \\ \partial^2_{xy} p^{(2)}_{ij} \end{pmatrix}$$

$$\partial_y (h\boldsymbol{v})^{n+1}_{ij} = \partial_y (h\boldsymbol{v})^{**}_{ij} - \delta t \begin{pmatrix} \partial^2_{xy} p^{(2)}_{ij} \\ 0 \end{pmatrix}$$

Stability of the Projection Step

Derive saddle point problem ...

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \nabla(\delta t\, p^{(2)})$$
$$\frac{1}{2}\nabla\cdot\left[(h\boldsymbol{v})^{n+1} + (h\boldsymbol{v})^{n}\right] = -\frac{dh_0}{dt}$$

... and employ theory of mixed finite elements (Nicolaïdes, 1982):

Theorem (V. & Klein 2007)

The generalized mixed formulation has a unique and stable solution $((h\boldsymbol{v})^{n+1}, \delta t\, p^{(2)})$.

- we obtain approximations, in which the solution of the Poisson problem $p^{(2)}$ and the momentum update $(h\boldsymbol{v})^{n+1}$ cannot decouple!

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Weakly Compressible Flows

mimic Zero Froude Number results by decomposing pressure into two components:

$$p(t, \boldsymbol{x}; \mathsf{Fr}) = p_0(t) + \mathsf{Fr}^2\, p'(t, \boldsymbol{x})$$

$$\begin{aligned} h_t + \nabla \cdot (h\boldsymbol{v}) &= 0 \\ (h\boldsymbol{v})_t + \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) + \nabla p' &= 0 \\ p = p_0(t) + \mathsf{Fr}^2\, p'(t, \boldsymbol{x}) &= \frac{h^2}{2} \end{aligned}$$

$$p_t + h\nabla \cdot (h\boldsymbol{v}) = 0$$

Extension of (zero Froude number) projection method by inclusion of local time derivatives of p' into projection steps

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Explicit Predictor for Advection

Auxiliar system:

$$\begin{aligned} h_t + \nabla \cdot (h\boldsymbol{v}) &= 0 \\ (h\boldsymbol{v})_t + \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) + \nabla p &= -(1 - \mathsf{Fr}^2)\nabla p'^{,\text{old}} \\ p &= \frac{h^2}{2} \end{aligned}$$

yields first order accurate prediction:

$$\begin{aligned} (h\boldsymbol{v})^{n+1/2} &= (h\boldsymbol{v})^{*,n+1/2} + \mathcal{O}\Big(\delta t^2\Big) \\ &= (h\boldsymbol{v})^{*,n+1/2} - \frac{\delta t}{2}(1 - \mathsf{Fr}^2)\nabla \delta p' + \mathcal{O}\Big(\delta t^2\Big) \end{aligned}$$

$$\delta p' := p'^{,n+1} - p'^{,n}$$

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First Correction – Advective Fluxes

Pressure equation:

$$\left(\frac{dp_0}{dt}\right)^{n+1/2} + \mathsf{Fr}^2 \left(\frac{\partial p'}{\partial t}\right)^{n+1/2} = -h\,\nabla\cdot\left((h\boldsymbol{v})^* - \frac{\delta t}{2}(1-\mathsf{Fr}^2)\nabla\delta p'\right)$$

Since:

$$\left(\frac{\partial p'}{\partial t}\right)^{n+1/2} = \frac{\delta p'}{\delta t} + \mathcal{O}\left(\delta t^2\right)$$

this yields Helmholtz equation for $\delta p'$:

$$-\frac{\mathsf{Fr}^2}{\delta t}\delta p' + h^*\frac{\delta t}{2}(1-\mathsf{Fr}^2)\Delta\delta p' = \left(\frac{dp_0}{dt}\right)^{n+1/2} - \left(\frac{dp^*}{dt}\right)^{n+1/2}$$

Second Correction – Pressure Term

intermediate momentum update:

$$(h\boldsymbol{v})^{**} \coloneqq (h\boldsymbol{v})^n - \delta t\left[\nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v})^{n+1/2} + \nabla p^{*,n+1/2} + (1 - \mathsf{Fr}^2)\nabla p'^{,n}\right]$$

momentum at the new time level:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \frac{\delta t}{2}(1 - \mathsf{Fr}^2)\nabla \delta p'^{,n} + \mathcal{O}\left(\delta t^2\right)$$

Once again, employ pressure equation to obtain pressure update:

$$\begin{aligned}\left(\frac{\partial p}{\partial t}\right)^{n+1/2} &= -\frac{1}{2}\left[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1}\nabla \cdot (h\boldsymbol{v})^{n+1}\right] \\ &= -\frac{1}{2}\left[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1}\nabla \cdot \left((h\boldsymbol{v})^{**} - \tfrac{\delta t}{2}(1 - \mathsf{Fr}^2)\nabla \delta p'^{,n}\right)\right]\end{aligned}$$

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Second Correction – cont.

1st possibility:

Obtain Helmholtz equation for pressure update:

$$\left(\frac{dp_0}{dt}\right)^{n+1/2} + \frac{\mathsf{Fr}^2}{\delta t}\delta p'^{,n} = -\frac{1}{2}[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1}\nabla \cdot (h\boldsymbol{v})^{**}] + \frac{\delta t}{4}(1 - \mathsf{Fr}^2)h^{n+1}\Delta \delta p'^{,n}$$

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Second Correction – cont.

1st possibility:
Obtain Helmholtz equation for pressure update:

$$\left(\frac{dp_0}{dt}\right)^{n+1/2} + \frac{\mathsf{Fr}^2}{\delta t}\delta p'^{,n} = -\frac{1}{2}[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1}\nabla \cdot (h\boldsymbol{v})^{**}] + \frac{\delta t}{4}(1-\mathsf{Fr}^2)h^{n+1}\Delta \delta p'^{,n}$$

2nd possibility:
Since pressure pressure update already known, derive Poisson-type equation:

$$\left(\frac{\partial p}{\partial t}\right)^{n+1/2} = -\frac{1}{2}[h^n \nabla \cdot (h\boldsymbol{v})^n + h^{n+1}\nabla \cdot (h\boldsymbol{v})^{**}] + \frac{\delta t}{4}(1-\mathsf{Fr}^2)h^{n+1}\Delta \delta p'^{,n}$$

Flux computation

$$F_I = \begin{pmatrix} h(\boldsymbol{v}\cdot\boldsymbol{n}) \\ h\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n}) + p'\boldsymbol{n} \end{pmatrix}_I$$

$$= \underbrace{F_I^*}_{\text{predictor}} \underbrace{- \frac{\delta t}{2}(1-\mathsf{Fr}^2)\begin{pmatrix} \nabla\delta p'^{,n}\cdot\boldsymbol{n} \\ \boldsymbol{v}^*(\nabla\delta p'^{,n}\cdot\boldsymbol{n}) + \nabla\delta p'^{,n}(\boldsymbol{v}^*\cdot\boldsymbol{n}) \end{pmatrix}_I}_{\text{1st advective correction}}$$

$$+ \underbrace{\begin{pmatrix} 0 \\ \frac{1-\mathsf{Fr}^2}{2}\delta p'^{,n}\boldsymbol{n} \end{pmatrix}_I}_{\text{2nd pressure correction}}$$

where

$$F_I^* = \begin{pmatrix} h^*(\boldsymbol{v}^*\cdot\boldsymbol{n}) \\ h^*\boldsymbol{v}^*(\boldsymbol{v}^*\cdot\boldsymbol{n}) + (p^* + (1-\mathsf{Fr}^2)p'^{,n})\boldsymbol{n} \end{pmatrix}_I$$

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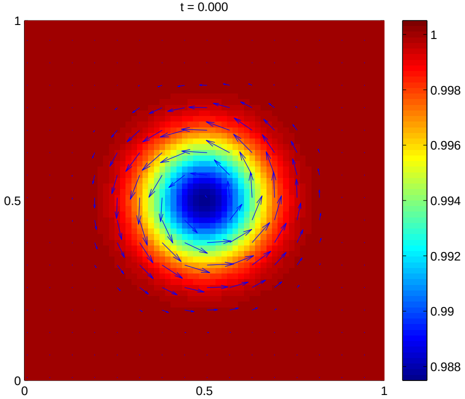
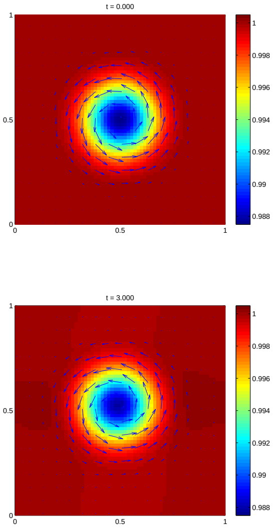
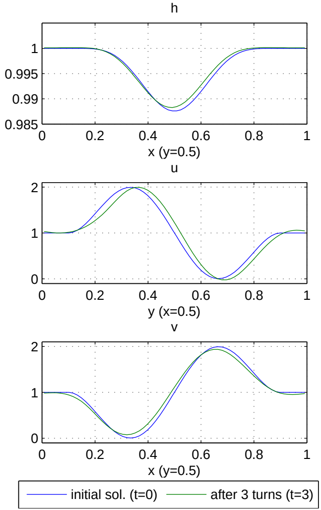
Advection of a vortex

quasi stationary, smooth solution, 64x64 cells, periodic b.c., Fr $= 0.1$



Advection of a vortex





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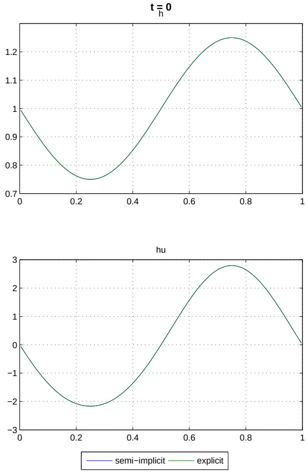
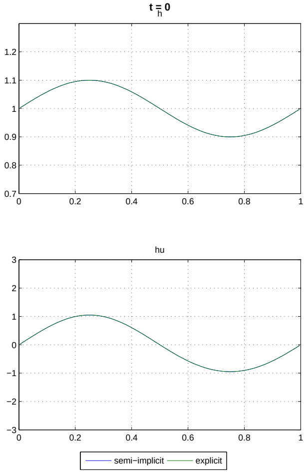
Simple Wave / Barotropic Gravity Wave

Fr $= 0.1$, 128 cells, $\text{CFL}_{\text{adv.}} = 0.9$



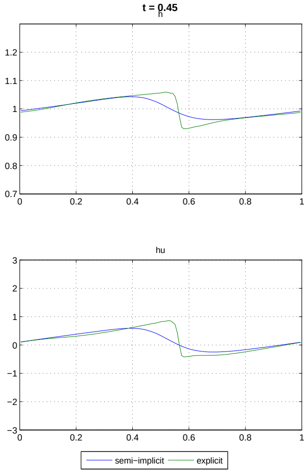


$\text{CFL}_{\text{sound}} \approx 4.3$

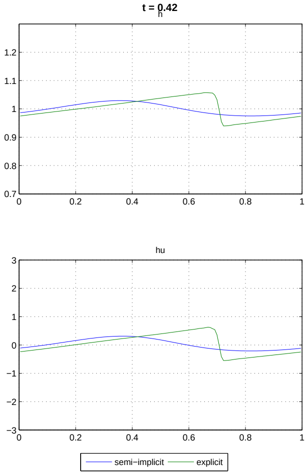
$\text{CFL}_{\text{sound}} \approx 10$

Simple Wave / Barotropic Gravity Wave

Fr $= 0.1$, 128 cells, CFL$_{\mathrm{adv.}} = 0.9$



CFL$_{\mathrm{sound}} \approx 4.3$



CFL$_{\mathrm{sound}} \approx 10$

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Summary

A Cartesian grid semi-implicit method has been presented.

- conservative method with two flux corrections motivated by zero Froude number projection method
- incorporates Godunov-type method in predictor step
- solution of two Helmholtz (one Helmholtz and one Poisson-type) problem in correction step

- Outlook
  - arbitrary Froude number
  - better representation of gravity waves (incorporation of results from multiscale asymptotic analysis, see Klein (1995))
  - inclusion of source terms (bottom topography)

For Further Information/Reading

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  Numerische Mathematik, 113(1) : 123–161, 2009.
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  Theoretical and Computational Fluid Dynamics, 23 : 161-195, 2009.