Stability of a Cartesian Grid Projection Method for Incompressible Shallow Water Flows

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Outline

The Zero Froude Number ("Incompressible") SWE

Compressible shallow water equations:

$$\begin{aligned} h_t &+ \nabla \cdot (h\boldsymbol{v}) & &= 0 \\ (h\boldsymbol{v})_t &+ \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) &+ \frac{1}{\mathsf{Fr}^2}\, h\nabla h &= \boldsymbol{0} \end{aligned}$$

- $\mathsf{Fr} = \dfrac{v'_{\mathrm{ref}}}{\sqrt{g'\, h'_{\mathrm{ref}}}}$
- hyperbolic system of conservation laws
- similar to Euler equations, no energy equation

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The Zero Froude Number ("Incompressible") SWE

Zero Froude number shallow water equations (as Fr $\to 0$):

$$\begin{aligned} h_t &+ \nabla \cdot (h\boldsymbol{v}) & &= 0 \\ (h\boldsymbol{v})_t &+ \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) &+ \quad h\nabla h^{(2)} &= \mathbf{0} \end{aligned}$$

- $h = h_0(t)$ given through boundary conditions.
- mass conservation becomes divergence constraint for velocity field:

$$\int_{\partial V} h\boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = -|V| \frac{dh_0}{dt} \qquad \text{for } V \subset \Omega$$

- $h^{(2)}$: second order height perturbation; Lagrange multiplier, which ensures compliance with divergence constraint

Construction of the Scheme

- method should be in conservation form:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

- machinery of Godunov-type methods
- second order accuracy
- advection velocities in fluxes and final momentum satisfy divergence constraint

Construction of the Scheme

- Method should be in conservation form:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

$$\mathbf{F}_I \coloneqq \mathbf{F}_I^* + \mathbf{F}_I^{\mathsf{MAC}} + \mathbf{F}_I^{\mathsf{P2}}$$

  - advective fluxes $\mathbf{F}_I^*$ from second order Godunov-type method (applied to auxiliary system)
  - $\mathbf{F}_I^{\mathsf{MAC}}$ from (MAC) projection, which corrects advection velocity divergence
  - $\mathbf{F}_I^{\mathsf{P2}}$ from second projection, which adjusts new time level divergence of cell-centered velocities

Auxiliary System

The auxiliary system

$$\begin{aligned} h_t^* &+ \nabla \cdot (h\boldsymbol{v})^* &= 0 \\ (h\boldsymbol{v})_t^* &+ \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v})^* + h^* \nabla h^* &= \mathbf{0} \end{aligned}$$

enjoys the following properties:

- same convective fluxes as incompressible SWE
- system is hyperbolic.
- having constant height h^* and zero velocity divergence at time t_0, solutions satisfy at $t_0 + \delta t$:

$$\nabla \cdot \boldsymbol{v}^* = \mathcal{O}(\delta t) \ , \quad (h^* \nabla h^*) = \mathcal{O}\big(\delta t^2\big)$$

Correction of Convective Fluxes

Semi-discrete equations (from Taylor series expansion):

$$
\begin{aligned}
h^{n+1} &= h^n - \delta t \; \nabla \cdot (h\boldsymbol{v})^{n+1/2} \\
(h\boldsymbol{v})^{n+1} &= (h\boldsymbol{v})^n - \delta t \left[\nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) \; + (h_0 \nabla h^{(2)})\right]^{n+1/2}
\end{aligned}
$$

Momentum for convective fluxes:

$$
(h\boldsymbol{v})^{n+1/2} = (h\boldsymbol{v})^{*,n+1/2} - \frac{\delta t}{2} \, (h_0 \nabla h^{(2)})^{n+1/4}
$$

Impose divergence constraint (first Poisson type problem):

$$
-\frac{dh_0}{dt}(t^{n+1/2}) = \nabla \cdot (h\boldsymbol{v})^{*,n+1/2} - \frac{\delta t}{2} \, \nabla \cdot (h_0 \nabla h^{(2)})^{n+1/4}
$$

Final Momentum

$$(h\boldsymbol{v})^{n+1} = \underbrace{(h\boldsymbol{v})^n - \delta t\big[\nabla\cdot(h\boldsymbol{v}\circ\boldsymbol{v})}_{=:(h\boldsymbol{v})^{**}} + (h_0\nabla h^{(2)})\big]^{n+1/2}$$

Impose divergence constraint as:

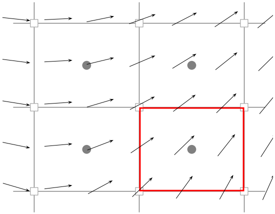
$$\nabla\cdot(h\boldsymbol{v})^{n+1} = -\nabla\cdot(h\boldsymbol{v})^n - 2\,\frac{dh_0}{dt}(t^{n+1/2})$$

Second Poisson type problem:

$$\nabla\cdot(h\boldsymbol{v})^{**} - \delta t\,\nabla\cdot(h_0\nabla h^{(2)})^{n+1/2} = -\nabla\cdot(h\boldsymbol{v})^n - 2\,\frac{dh_0}{dt}(t^{n+1/2})$$

Convective Fluxes
MAC Projection

$$\frac{\delta t}{2} \nabla \cdot (h_0 \nabla h^{(2)}) = \nabla \cdot (h\boldsymbol{v})^* + \frac{dh_0}{dt}$$

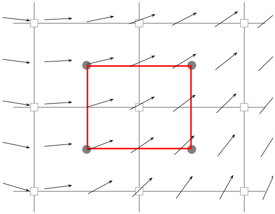


- divergence constraint imposed on each grid cell
- corrects convective fluxes on boundary of control volumes

Final Momentum

Second Projection

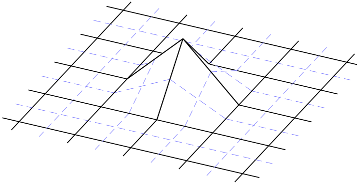
$$\delta t \, \nabla \cdot (h_0 \nabla h^{(2)}) = \nabla \cdot (h\boldsymbol{v})^{**} + \nabla \cdot (h\boldsymbol{v})^{n} + 2 \, \frac{dh_0}{dt}$$



- divergence constraint imposed on dual control volumes
- adjusts momentum to obtain correct divergence for new velocity field

The (Second) Projection
Discretization of the Poisson-Type Problem

Consider a Petrov-Galerkin FE discretization [Süli, 1991]:



- bilinear trial functions for the unknown $h^{(2)}$
- piecewise constant test functions on the dual discretization

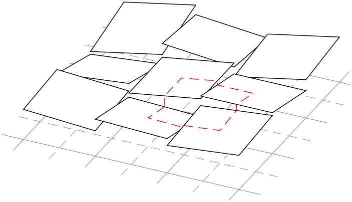
Integration over $\mathbf{\Omega}$ and divergence theorem leads to:

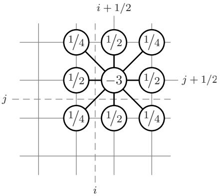
$$\delta t \, h_0 \int_{\partial \bar{V}} \nabla h^{(2)} \cdot \boldsymbol{n} \; d\sigma = \int_{\partial \bar{V}} \left[(h\boldsymbol{v})^{**} + (h\boldsymbol{v})^{n} \right] \cdot \boldsymbol{n} \; d\sigma$$

The (Second) Projection

Discrete Velocity Space

- velocity components at boundary of the dual cells are piecewise linear
- discrete divergence can be exactly calculated





- discrete Laplacian has compact stencil
- discrete divergence, Laplacian and gradient satisfy L = D(G)
- results in exact projection method

Stability of the Projection Step

Generalized Saddle-Point Problems

Find $(u, p) \in (\mathcal{X}_2 \times \mathcal{M}_1)$, such that

$$\begin{cases} a(u,v) + b_1(v,p) & = & \langle f, v \rangle & \forall\, v \in \mathcal{X}_1 \\ b_2(u,q) & = & \langle g, q \rangle & \forall\, q \in \mathcal{M}_2 \end{cases} \tag{1}$$

Theorem (Nicolaïdes, 1982; Bernardi el al., 1988)

If $b_i(\cdot,\cdot)$ $(i = 1,2)$ and similarly $a(\cdot,\cdot)$ satisfy:

$$\inf_{q \in \mathcal{M}_i} \sup_{v \in \mathcal{X}_i} \frac{b_i(v,q)}{\|v\|_{\mathcal{X}_i} \, \|q\|_{\mathcal{M}_i}} \geq \beta_i > 0$$

Then, (1) has a unique solution for all f and g.

Reformulation of the Poisson-Type Problem

Derive saddle point problem by employing momentum update and divergence constraint:

$$
\begin{aligned}
(h\boldsymbol{v})^{n+1} &= (h\boldsymbol{v})^{**} - \delta t\, h_0 \nabla h^{(2)} \\
\frac{1}{2}\nabla \cdot \left[(h\boldsymbol{v})^{n+1} + (h\boldsymbol{v})^{n}\right] &= -\frac{dh_0}{dt}
\end{aligned}
$$

- variational formulation: multiply with test functions $\boldsymbol{\varphi}$ and ψ and integrate over Ω
- discrete problem with piecewise linear vector and piecewise constant scalar test functions

Existence & Uniqueness
Continuous Problem

- find solution with $(h\boldsymbol{v})^{n+1} \in H_0(\mathrm{div};\Omega)$ and $(\delta t\, h_0\, h^{(2)}) \in H^1(\Omega)/\mathbb{R}$
- test functions in the spaces $[L^2(\Omega)]^2$ and $L^2(\Omega)$
- bilinear forms given by:

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) &:= (\boldsymbol{u},\boldsymbol{v})_0 \\ b_1(\boldsymbol{v},q) &:= (\boldsymbol{v},\nabla q)_0 \\ b_2(\boldsymbol{v},q) &:= (q,\nabla\cdot\boldsymbol{v})_0 \end{aligned}$$

Theorem (V. 2005)

The continuous generalized saddle point problem has a unique solution $((h\boldsymbol{v})^{n+1}, \delta t\, h_0\, h^{(2)})$.

Stability of the Discrete Problem

- find solution with

$$(h\boldsymbol{v})_h^{n+1} \in \mathcal{U}_h \coloneqq \left\{ \boldsymbol{v} \mid \forall\, V : \boldsymbol{v}|_V \in [\mathcal{P}_1(V)]^2 \right\} \not\subseteq H(\mathrm{div};\Omega)$$
$$(\delta t\, h_0\, h_h^{(2)}) \in \{ p \mid \forall\, V : p|_V \in Q_1(V) \} \subset H^1(\Omega)/\mathbb{R}$$

- test functions in the spaces $\mathcal{U}_h \subset [L^2(\Omega)]^2$ and $\mathcal{P}_0 \subset L^2(\Omega)$
- problem: piecewise linear vector functions not in $H(\mathrm{div};\Omega)$ in general (nonconforming finite elements)
- conforming (e.g. Raviart-Thomas) elements do not match with the piecewise linear, discontinuous ansatz functions from the Godunov-Type method

Stability of the Discrete Problem

The nonconforming space $\mathcal{U}_h$

- discrete norm defined by

$$\|\boldsymbol{w}_h\|_{\mathcal{U}^h} \coloneqq \|\boldsymbol{w}_h\|_0 + \sup_{z_h \in \mathcal{Q}^h} \frac{b_{2h}(\boldsymbol{w}_h, z_h)}{\|z_h\|_{\mathcal{Q}}} \quad \text{for } \boldsymbol{w}_h \in \mathcal{U}^h$$

- bilinear form b_2 has to be changed $\rightsquigarrow b_{2h} : \mathcal{U}^h \times \mathcal{Q}^h \to \mathbb{R}$ with

$$b_{2h}(\boldsymbol{v}_h, q_h) \coloneqq \sum_{\bar{V} \in \bar{\mathcal{V}}} q_{h,\bar{V}} \int_{\partial \bar{V}} \boldsymbol{v}_h \cdot \boldsymbol{n} \, d\sigma$$

definition consistent with its continuous counterpart b_2

Inf-Sup Condition for $a(\cdot,\cdot)$
Discrete Problem

To show ("coercivity"):

$$\inf_{\boldsymbol{u}\in\mathcal{K}_2^h} \sup_{\boldsymbol{v}\in\mathcal{K}_1^h} \frac{a(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|} \geq \alpha \quad \text{and} \quad \sup_{\boldsymbol{u}\in\mathcal{K}_2^h} a(\boldsymbol{u},\boldsymbol{v}) > 0 \quad \forall\, \boldsymbol{v} \in \mathcal{K}_1^h \setminus \{0\}$$

$$\boldsymbol{v} \in \mathcal{K}_1^h \Leftrightarrow 0 = \frac{1}{\delta x} f(u_{ij}, v_{ij}) + \frac{1}{6} g(u_{y,ij}, v_{x,ij})$$

$$\boldsymbol{v} \in \mathcal{K}_2^h \Leftrightarrow 0 = \frac{1}{\delta x} f(u_{ij}, v_{ij}) + \frac{1}{4} g(u_{y,ij}, v_{x,ij})$$

$\rightsquigarrow$ one-to-one mapping from $\mathcal{K}_1^h$ to $\mathcal{K}_2^h$ by multiplying partial derivatives of each element with $4/6$

Inf-Sup Condition for $a(\cdot,\cdot)$ (cont.)

Discrete Problem

- the following estimates can be given for corresponding elements $\boldsymbol{v} \in \mathcal{K}_1^h$ and $\boldsymbol{u} \in \mathcal{K}_2^h$ (with $\bar{\boldsymbol{u}} = \bar{\boldsymbol{v}}$ and $\nabla \tilde{\boldsymbol{u}} = 2/3 \nabla \tilde{\boldsymbol{v}}$):

$$\frac{4}{9}\, a(\boldsymbol{v}, \boldsymbol{v}) \le a(\boldsymbol{u}, \boldsymbol{u}) \le a(\boldsymbol{u}, \boldsymbol{v})$$

- This gives for each $\boldsymbol{u} \in \mathcal{K}_2^h$, $\|\boldsymbol{u}\|_{\mathcal{U}^h} = \|\boldsymbol{u}\|_0 \neq 0$

$$\sup_{\boldsymbol{v} \in \mathcal{K}_1^h} \frac{a(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_0} \ge \frac{a(\boldsymbol{u}, \boldsymbol{u})}{\frac{3}{2}\|\boldsymbol{u}\|_0} = \frac{2}{3}\, \|\boldsymbol{u}\|_{\mathcal{U}^h}$$

and for $\boldsymbol{v} \in \mathcal{K}_1^h \setminus \{0\}$ we obtain

$$\sup_{\boldsymbol{u} \in \mathcal{K}_2^h} a(\boldsymbol{u}, \boldsymbol{v}) \ge \frac{4}{9} a(\boldsymbol{v}, \boldsymbol{v}) > 0$$

Inf-Sup Condition for $b_1(\cdot,\cdot)$

Discrete Problem

- for piecewise bilinear $p \in \mathcal{H}^h \subset H^1(\Omega)/\mathbb{R}$ it follows that $\nabla p \in \mathcal{U}^h$; i.e. piecewise linear
- thus, for arbitrary $p \in \mathcal{H}^h$, we have

$$\sup_{\boldsymbol{v}\in\mathcal{U}^h} \frac{b_1(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_0} \geq \frac{b_1(\nabla p,p)}{\|\nabla p\|_0} = \frac{(\nabla p,\nabla p)_0}{\|\nabla p\|_0} = |p|_1$$

- analogous to continuous case

Inf-Sup Condition for $b_{2h}(\cdot,\cdot)$

Discrete Problem

- introduce lumping operator $\mathtt{L} : \mathcal{H}^h \to \mathcal{Q}^h$ with

$$\mathtt{L} r_h \coloneqq \sum_{\bar{V} \in \bar{\mathcal{V}}} \chi_{\bar{V}} \, r_h(x_{\bar{V}}, y_{\bar{V}}) \quad \forall r_h \in \mathcal{H}^h$$

- have to show

$$\sup_{\boldsymbol{w}_h \in \mathcal{U}^h} \frac{b_{2h}(\boldsymbol{w}_h, q_h)}{\|\boldsymbol{w}_h\|_{\mathcal{U}^h}} \geq \beta_2^* \, \|q_h\|_{\mathcal{Q}^h} \quad \forall q_h$$

- proof is done by definition of an auxiliary mapping $G_h : \mathcal{Q}^h \to \mathcal{U}^h$, where $G_h q_h \coloneqq \nabla r_h$ and $r_h \in \mathcal{H}^h$ is the solution of

$$b_{2h}(\nabla r_h, z_h) = (q_h, z_h)_{0,\Omega} \quad \forall z_h \in \mathcal{Q}^h$$

Stability of the Discrete Problem

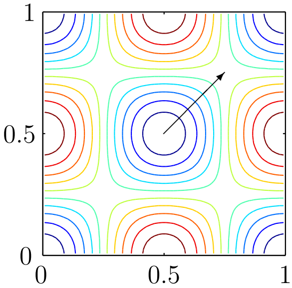
Theorem (V. & Klein 2007)

The generalized mixed formulation has a unique and stable solution $((h\boldsymbol{v})_h^{n+1}, \delta t\, h_0\, h_h^{(2)})$.

- we obtain approximations, in which the solution of the Poisson problem $h^{(2)}$ and the momentum update $(h\boldsymbol{v})^{n+1}$ cannot decouple!
- former version [Schneider et al. 1999] can also be formulated as mixed method; but unstable!

Originally proposed by Minion [1996] and Almgren et al. [1998] for the incompressible flow equations

- smooth velocity field
- nontrivial solution for $h^{(2)}$
- solved on unit square with periodic BC
- 32×32, 64×64 and 128×128 grid cells
- error to exact solution at $t = 3$



Convergence Studies

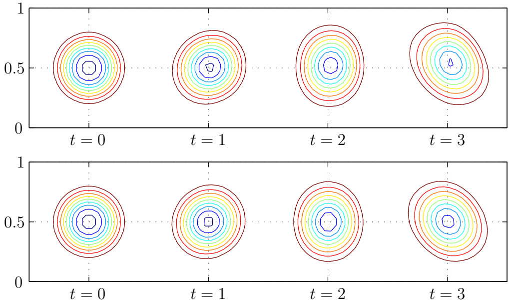
Errors and Convergence Rates

Method	Norm	32x32	Rate	64x64	Rate	128x128
SCHNEIDER ET AL.	L^2	0.2929	2.16	0.0656	2.16	0.0146
	L^∞	0.4207	2.15	0.0945	2.18	0.0209
new exact projection	L^2	0.0816	2.64	0.0131	2.17	0.0029
	L^∞	0.1277	2.45	0.0234	2.32	0.0047

- second order accuracy is obtained in the L^2 and the L^∞ norms
- absolute error obtained with the new exact projection method about four times smaller on fixed grids

Advection of a Vortex
Results for the New Projection Method

Exact projection, central differences (no limiter):



SCHNEIDER ET AL.

new exact projection

Less deviation from the center line of the channel, loss in vorticity is slightly reduced.

Summary

A Cartesian grid projection method has been presented.

- conservative and exact projection method with two projections based on a FE formulation
- second projection stable in the sense of generalized inf-sup / Babuška-Brezzi theory; no local decoupling of the pressure gradient
- numerical results of the new method show considerable accuracy improvements on fixed grids compared to the old formulation

- Outlook
  - convergence of the mixed formulation
  - extention to weakly compressible case (incorporation of results from asymptotic analysis)
  - inclusion of bottom topography

For Further Information/Reading

Th. Schneider, N. Botta, K.J. Geratz and R. Klein.

Extension of Finite Volume Compressible Flow Solvers to Multi-dimensional, Variable Density Zero Mach Number Flows.

Journal of Computational Physics, 155 : 248–286, 1999.

S. Vater.

A New Projection Method for the Zero Froude Number Shallow Water Equations.

PIK Report No. 97, Potsdam Institute for Climate Impact Research, 2005.

S. Vater & R. Klein.

Stability of a Cartesian Grid Projection Method for Zero Froude Number Shallow Water Flows.

submitted to: Numerische Mathematik, 2007.
also available as ZIB-Report No. 07-13 (`http://www.zib.de`)

Approximate vs. Exact Projection

- discrete divergence also affected by partial derivatives u_y and v_x
- using just the mean values to correct momentum:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t\, h_0\, \overline{\mathsf{G}(h^{(2)})}$$

  we obtain $\mathsf{D}(\boldsymbol{v}^{n+1}) = \mathcal{O}(\delta t\, \delta x^2)$; approximate projection method
- additional correction of derivatives and their employment in the reconstruction of the predictor step: exact projection method

Inf-Sup Condition for $a(\cdot,\cdot)$
Continuous Problem

- an orthogonal decomposition of $(L^2(\Omega))^2$ is given by

$$\{\boldsymbol{v} \in H_0(\mathrm{div};\Omega) \mid \nabla\cdot\boldsymbol{v} = 0\} \oplus \{\nabla q \mid q \in H^1(\Omega)\}$$

- $\Rightarrow\ \mathcal{K}_1 = \{\boldsymbol{v} \in H_0(\mathrm{div};\Omega) \mid \nabla\cdot\boldsymbol{v} = 0\} = \mathcal{K}_2$
- for each $\boldsymbol{u} \in \mathcal{K}_2$, $\|\boldsymbol{u}\|_{0,\Omega} \neq 0$, $a(\cdot,\cdot)$ satisfies

$$\sup_{\boldsymbol{v}\in\mathcal{K}_1} \frac{a(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{0,\Omega}} \geq \frac{a(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|_{0,\Omega}} = \frac{\|\boldsymbol{u}\|_{0,\Omega}^2}{\|\boldsymbol{u}\|_{0,\Omega}} = \|\boldsymbol{u}\|_{\mathrm{div},\Omega}$$