A Semi-Implicit Projection Method for the Zero Froude Number Shallow Water Equations

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Outline

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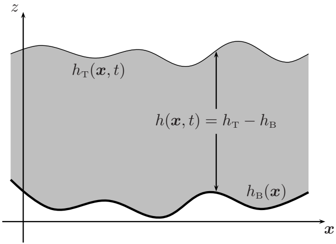
The Shallow Water Equations

Non-dimensional form:

$$
\begin{aligned}
h_t \quad &+ \quad \nabla \cdot (h\boldsymbol{v}) &&= \quad 0 \\
(h\boldsymbol{v})_t \quad &+ \quad \nabla \cdot \left(h\boldsymbol{v} \circ \boldsymbol{v} + \frac{1}{2\,\mathsf{Fr}^2}\, h^2\, \boldsymbol{I} \right) &&= \quad \frac{1}{\mathsf{Fr}^2}\, h \nabla h_{\mathrm{B}}
\end{aligned}
$$



- $\mathsf{Fr} = \dfrac{v'_{\mathrm{ref}}}{\sqrt{g'\, h'_{\mathrm{ref}}}}$
- hyperbolic system of conservation laws
- similar to Euler equations, no energy equation

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The "Incompressible" Limit
(as Fr $\to 0$)

Zero Froude number shallow water equations:

$$\begin{aligned} h_t &+ \nabla \cdot (h\boldsymbol{v}) & &= 0 \\ (h\boldsymbol{v})_t &+ \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) &+ h\nabla h^{(2)} &= \mathbf{0} \end{aligned}$$

- $h = h_0(t)$ is given through boundary conditions.
- mass conservation becomes a divergence constraint for the velocity field:

$$\int\limits_{\partial V} h(\boldsymbol{v} \cdot \boldsymbol{n})\, d\sigma = -|V| \frac{dh_0}{dt} \qquad \text{for } V \subset \Omega$$

Formulation of the Numerical Scheme

Consider a FV method in conservation form:

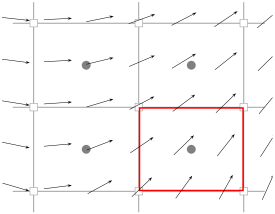
$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

$$\mathbf{F}_I(\mathbf{U}_I, \boldsymbol{n}_I) := \begin{pmatrix} h(\boldsymbol{v} \cdot \boldsymbol{n}) \\ h\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n}) \quad + \quad h_0 \, h^{(2)} \, \boldsymbol{n} \end{pmatrix}_I$$

Construction of numerical fluxes:

- advective fluxes from standard explicit FV scheme (applied to an auxiliary system)
- (MAC)-projection corrects advection velocity divergence
- second (exact) projection adjusts new time level divergence of cell-centered velocities

Correction of the Fluxes



1. Projection:

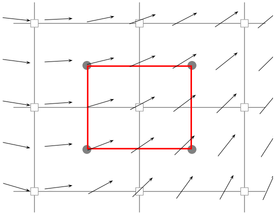
$$(h\boldsymbol{v})_I = (h\boldsymbol{v})_I^* - \frac{\delta t}{2} h_0 (\nabla h^{(2)})_I$$

corrects convective fluxes on boundary of control volume

2. Projection:

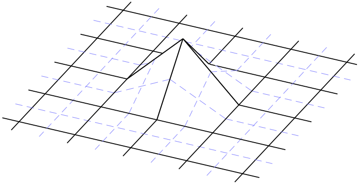
$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t (h_0 \nabla h^{(2)})^{n+1/2}$$

adjusts momentum to obtain correct divergence for new velocity field



The (Second) Projection
Discretization of the Poisson-Type Problem

Consider a Petrov-Galerkin FE discretization [Süli, 1991]:



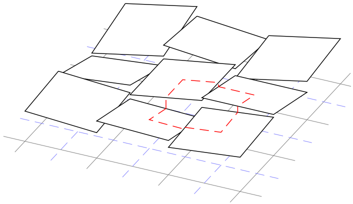
- bilinear trial functions for the unknown $h^{(2)}$
- piecewise constant test functions on the dual discretization

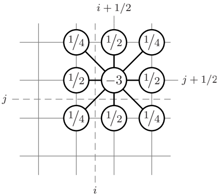
Integration over $\boldsymbol{\Omega}$ and divergence theorem leads to:

$$\delta t \, h_0 \int_{\partial \bar{V}} \nabla h^{(2)} \cdot \boldsymbol{n} \; d\sigma = \int_{\partial \bar{V}} \left[(h\boldsymbol{v})^{**} + (h\boldsymbol{v})^{n} \right] \cdot \boldsymbol{n} \; d\sigma$$

The (Second) Projection
Discrete Velocity Space

- velocity components at boundary of the dual cells are piecewise linear!
- discrete divergence can be exactly calculated





- discrete divergence, Laplacian and gradient satisfy $L = D(G)$
- discrete Laplacian has compact stencil

Approximate vs. Exact Projection

- discrete divergence also affected by partial derivatives u_y and v_x
- using just the mean values to correct momentum:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t\, h_0\, \overline{\mathsf{G}(h^{(2)})}$$

  we obtain $\mathsf{D}(\boldsymbol{v}^{n+1}) = \mathcal{O}(\delta t\, \delta x^2)$; approximate projection method
- additional correction of derivatives and their employment in the reconstruction of the predictor step: exact projection method

Generalized Saddle-Point Problems

NICOLAÏDES [1982] and BERNARDI EL AL. [1988]

Find $(u, p) \in (\mathcal{X}_2 \times \mathcal{M}_1)$, such that

$$\left\{ \begin{array}{lcll} a(u,v) + b_1(v,p) & = & \langle f, v \rangle & \forall\, v \in \mathcal{X}_1 \\ b_2(u,q) & = & \langle g, q \rangle & \forall\, q \in \mathcal{M}_2 \end{array} \right. \qquad (1)$$

Theorem

If $b_i(\cdot,\cdot)$ $(i = 1, 2)$ and similarly $a(\cdot,\cdot)$ satisfy:

$$\inf_{q \in \mathcal{M}_i} \sup_{v \in \mathcal{X}_i} \frac{b_i(v,q)}{\|v\|_{\mathcal{X}_i} \, \|q\|_{\mathcal{M}_i}} \geq \beta_i > 0$$

Then, (1) has a unique solution for all f and g.

Reformulation of the Poisson-Type Problem

Derive saddle point problem by employing momentum update and divergence constraint:

$$
\begin{aligned}
(h\boldsymbol{v})^{n+1} &= (h\boldsymbol{v})^{**} - \delta t \left(h_0 \nabla h^{(2)}\right) \\
\frac{1}{2}\nabla \cdot \left[(h\boldsymbol{v})^{n+1} + (h\boldsymbol{v})^{n}\right] &= -\frac{dh_0}{dt}
\end{aligned}
$$

- variational formulation: multiply with test functions $\boldsymbol{\varphi}$ and ψ and integrate over Ω
- discrete problem with piecewise linear vector and piecewise constant scalar test functions

Existence & Uniqueness
Continuous Problem

- find solution with $(h\boldsymbol{v})^{n+1} \in H_0(\mathrm{div};\Omega)$ and $h^{(2)} \in H^1(\Omega)/\mathbb{R}$
- test functions in the spaces $[L^2(\Omega)]^2$ and $L^2(\Omega)$
- bilinear forms given by:

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &\coloneqq (\boldsymbol{u},\boldsymbol{v})_0 \\
b_1(\boldsymbol{v},q) &\coloneqq \delta t\, h_0\, (\boldsymbol{v},\nabla q)_0 \\
b_2(\boldsymbol{v},q) &\coloneqq (q,\nabla\cdot\boldsymbol{v})_0
\end{aligned}
$$

Theorem (V. 2005)

The continuous generalized saddle point problem has a unique solution $((h\boldsymbol{v})^{n+1}, h^{(2)})$.

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Stability of the Discrete Problem?

- $a(\cdot,\cdot)$ and $b_1(\cdot,\cdot)$ satisfy discrete inf-sup conditions, open question for $b_2(\cdot,\cdot)$
- problem: piecewise linear vector functions not in $H(\mathrm{div};\Omega)$ in general (nonconforming finite elements)
- conforming (e.g. Raviart-Thomas) elements do not match with the piecewise linear, discontinuous ansatz functions from the Godunov-Type method
- former version [SCHNEIDER ET AL. 1999] can also be formulated as saddle point problem; but unstable!

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Discrete Inf-Sup Condition for $a(\cdot,\cdot)$

To show ("coercivity"):

$$\inf_{\boldsymbol{u}\in\mathcal{K}_2^h} \sup_{\boldsymbol{v}\in\mathcal{K}_1^h} \frac{a(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|} \geq \alpha \quad \text{and} \quad \sup_{\boldsymbol{u}\in\mathcal{K}_2^h} a(\boldsymbol{u},\boldsymbol{v}) > 0 \quad \forall\, \boldsymbol{v} \in \mathcal{K}_1^h \setminus \{0\}$$

$$\boldsymbol{v} \in \mathcal{K}_1^h \Leftrightarrow 0 = \frac{1}{\delta x} f(u_{ij}, v_{ij}) + \frac{1}{6} g(u_{y,ij}, v_{x,ij})$$

$$\boldsymbol{v} \in \mathcal{K}_2^h \Leftrightarrow 0 = \frac{1}{\delta x} f(u_{ij}, v_{ij}) + \frac{1}{4} g(u_{y,ij}, v_{x,ij})$$

⇝ one-to-one mapping from $\mathcal{K}_1^h$ to $\mathcal{K}_2^h$ by multiplying partial derivatives of each element with $4/6$

Discrete Inf-Sup Condition for $a(\cdot,\cdot)$ (cont.)

- the following estimates can be given for corresponding elements $\boldsymbol{v} \in \mathcal{K}_1^h$ and $\boldsymbol{u} \in \mathcal{K}_2^h$ (with $\bar{\boldsymbol{u}} = \bar{\boldsymbol{v}}$ and $\nabla\tilde{\boldsymbol{u}} = 2/3\nabla\tilde{\boldsymbol{v}}$):

$$\frac{4}{9}\,a(\boldsymbol{v},\boldsymbol{v}) \leq a(\boldsymbol{u},\boldsymbol{u}) \leq a(\boldsymbol{u},\boldsymbol{v})$$

- This gives for each $\boldsymbol{u} \in \mathcal{K}_2^h$, $\|\boldsymbol{u}\|_{\mathrm{div},\mathcal{V}} = \|\boldsymbol{u}\|_0 \neq 0$

$$\sup_{\boldsymbol{v}\in\mathcal{K}_1^h} \frac{a(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_0} \geq \frac{a(\boldsymbol{u},\boldsymbol{u})}{\frac{3}{2}\,\|\boldsymbol{u}\|_0} = \frac{2}{3}\,\|\boldsymbol{u}\|_{\mathrm{div},\mathcal{V}}$$

and for $\boldsymbol{v} \in \mathcal{K}_1^h \setminus \{0\}$ we obtain

$$\sup_{\boldsymbol{u}\in\mathcal{K}_2^h} a(\boldsymbol{u},\boldsymbol{v}) \geq \frac{4}{9}a(\boldsymbol{v},\boldsymbol{v}) > 0$$

Discrete Inf-Sup Condition for $b_1(\cdot,\cdot)$

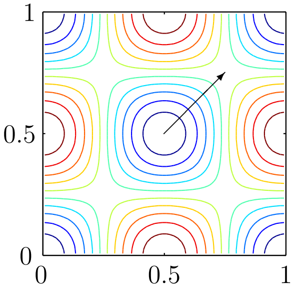
- for piecewise bilinear $p \in \mathcal{H}^h \subset H^1(\Omega)/\mathbb{R}$ it follows that $\nabla p \in \mathcal{U}^h$; i.e. piecewise linear
- thus, for arbitrary $p \in \mathcal{H}^h$, we have

$$
\begin{aligned}
\sup_{\boldsymbol{v} \in \mathcal{U}^h} \frac{b_1(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_0} &\geq \frac{b_1(\nabla p, p)}{\|\nabla p\|_0} \\
&= \frac{\delta t\, h_0\, (\nabla p, \nabla p)_0}{\|\nabla p\|_0} \\
&= \delta t\, h_0\, |p|_1
\end{aligned}
$$

Convergence Studies
Taylor Vortex

Originally proposed by Minion [1996] and Almgren et al. [1998] for the incompressible flow equations

- smooth velocity field
- nontrivial solution for $h^{(2)}$
- solved on unit square with periodic BC
- 32×32, 64×64 and 128×128 grid cells
- error to exact solution at $t = 3$



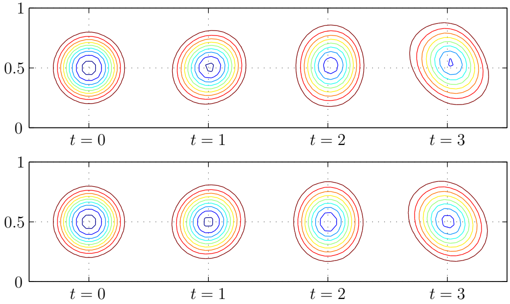
Convergence Studies
Errors and Convergence Rates

Method	Norm	32x32	Rate	64x64	Rate	128x128
Schneider et al.	L^2	0.2929	2.16	0.0656	2.16	0.0146
	L^∞	0.4207	2.15	0.0945	2.18	0.0209
new exact projection	L^2	0.0816	2.64	0.0131	2.17	0.0029
	L^∞	0.1277	2.45	0.0234	2.32	0.0047

- second order accuracy is obtained in the L^2 and the L^∞ norms
- absolute error obtained with the new exact projection method about four times smaller on fixed grids

Advection of a Vortex
Results for the New Projection Method

Exact projection, central differences (no limiter):



SCHNEIDER ET AL.

new exact projection

Less deviation from the center line of the channel, loss in vorticity is slightly reduced.

Summary

A new projection method has been presented.

- it is an exact projection method with a projection based on a FE formulation
- numerical results of the new method show considerable accuracy improvements on fixed grids compared to the old formulation
- results supported by theoretical analysis; no local decoupling of the gradient in the 2nd projection

- Outlook
  - stability of the discrete method; inf-sup for $b_2(\cdot, \cdot)$
  - additional degrees of freedom through partial derivatives u_y, v_x and u_x, v_y
  - include additional terms (Coriolis etc.)

- related talk: M. Oevermann, Wed. 15:10 h (Sect 18, Session 5)

For Further Information/Reading

Th. Schneider, N. Botta, K.J. Geratz and R. Klein.
Extension of Finite Volume Compressible Flow Solvers to Multi-dimensional, Variable Density Zero Mach Number Flows.
Journal of Computational Physics, 155 : 248–286, 1999.

S. Vater.
A New Projection Method for the Zero Froude Number Shallow Water Equations.
PIK Report No. 97, Potsdam Institute for Climate Impact Research, 2005.

Auxiliary System

The auxiliary system

$$\begin{array}{rcll} h_t^* & + & \nabla \cdot (h\boldsymbol{v})^* & = & 0 \\ (h\boldsymbol{v})_t^* & + & \nabla \cdot \Big((h\boldsymbol{v} \circ \boldsymbol{v})^* + \frac{1}{2}(h^*)^2 \boldsymbol{I}\Big) & = & \boldsymbol{0} \end{array}$$

enjoys the following properties:

- It has the same convective fluxes as the zero Froude number shallow water equations.
- The system is hyperbolic.
- Having constant height h^* and a zero velocity divergence at time t_0, solutions satisfy at $t_0 + \delta t$:

$$\nabla \cdot \boldsymbol{v}^* = \mathcal{O}(\delta t) \ , \quad (h^* \nabla h^*) = \mathcal{O}\big(\delta t^2\big)$$

Inf-Sup Condition for $a(\cdot,\cdot)$

- an orthogonal decomposition of $(L^2(\Omega))^2$ is given by

$$\{\boldsymbol{v} \in H_0(\mathrm{div};\Omega) \mid \nabla\cdot\boldsymbol{v} = 0\} \oplus \{\nabla q \mid q \in H^1(\Omega)\}$$

- $\Rightarrow \ \mathcal{K}_1 = \{\boldsymbol{v} \in H_0(\mathrm{div};\Omega) \mid \nabla\cdot\boldsymbol{v} = 0\} = \mathcal{K}_2$
- for each $\boldsymbol{u} \in \mathcal{K}_2$, $\|\boldsymbol{u}\|_{0,\Omega} \neq 0$, $a(\cdot,\cdot)$ satisfies

$$\sup_{\boldsymbol{v}\in\mathcal{K}_1} \frac{a(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{0,\Omega}} \geq \frac{a(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|_{0,\Omega}} = \frac{\|\boldsymbol{u}\|_{0,\Omega}^2}{\|\boldsymbol{u}\|_{0,\Omega}} = \|\boldsymbol{u}\|_{\mathrm{div},\Omega}$$