Assignment 9

Positional Games, Winter 2009-10
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To be discussed on Jan 5th (Tuesday) at 16:15PM

Let $\mathcal{F} \subseteq 2^X$ be a hypergraph on vertex set X. For a hyperedge $D \in \mathcal{F}$ denote by $\Gamma^+_{\mathcal{F}}(D) = \{F \in \mathcal{F} : F \cap D \neq \emptyset\}$ the neighborhood of D in $\mathcal{F}$.
Consider the following algorithm

Algorithm $\mathcal{F}$-$\texttt{color}(\mathcal{D}, c)$

Input hypergraph $\mathcal{D} \subseteq \mathcal{F}$, two-coloring $c : X \to \{red, blue\}$

while there is a monochromatic hyperedge in $\mathcal{D}$ **do**

select arbitrary monochromatic $D \in \mathcal{D}$

change c at each vertex of D independently, with probability 1/2

$\mathcal{F}$-$\texttt{color}(\Gamma^+_{\mathcal{F}}(D), c)$

return c

Problem 1 Let $\mathcal{F} \subseteq 2^X$ be a k-uniform hypergraph with where every hyperedge intersects less than 2^{k-4} other hyperedges. Prove that if $\mathcal{F}$-$\texttt{color}(\mathcal{F}, c)$ returns then it does so with a proper two-coloring. Prove also that the expected running time is bounded by $O(|\mathcal{F}| \log |\mathcal{F}|)$

Problem 2 Prove that there are at most $|X|/k$ top-level iterations in ANY running of $\texttt{color}(\mathcal{F}, c)$.

Problem 3 Prove that on average there are at most $|X|/2^k$ top-level iterations.

Let G be a d-regular graph. In the min-degree game $\mathcal{D}_k$, played on the edge set $E(G)$ Maker wins if in his graph every vertex has degree at least k.

Problem 4 Give a winning strategy for Maker in the min-degree game $\mathcal{D}_k$ with $k = \lfloor d/4 \rfloor$.

Problem 5 Show that this is best possible by exhibiting a 3-regular graph on which Breaker can isolate a vertex in Maker's graph.

Conjecture Maker can win $\mathcal{D}_k$ for every d with $k = d/2 - o(d)$. (Formally: For every $\epsilon > 0$ there is a $d_0(\epsilon)$ such that for every d-regular graph G with

$d > d_0(\epsilon)$, Maker has a winning a strategy in the min-degree game $\mathcal{D}_{(1/2-\epsilon)d}$ played on $E(G)$.)

Problem 6 Prove that the conjecture is true for the complete graph. That is Maker can build a graph with min-degree $n - O(\sqrt{n \log n})$.

Problem 7 Here is a (wrong) proof that Maker can achieve minimum degree $k = d/3 - \epsilon d$ (for arbitrary $\epsilon > 0$):

The strategy of Maker: after Breaker took his edge, Maker chooses uniformly at random one of the endpoints of this edge and takes an arbitrary edge incident to it. We plan to use the Lovász Local Lemma to show this strategy works with non-zero probability (and that implies the *existence* of a deterministic strategy). We define a bad event for each vertex v: let A_v be the event that v has degree less than k in Maker's graph. These events clearly depend on each other only if v and w are neighbors or have a common neighbor. Now, what is the probability of a bad event A_v? There are at least $d - k = (2/3 + \epsilon)d$ edges of Breaker to v; for at least $d - 2k = (1/3 + 2\epsilon)d$ of them, after Breaker took them, Maker decided to take the an edge at the other endpoint. By standard estimates on the tale of the binomial distribution, the probability of this happening is at most

$$\frac{1}{2^{d-k}} \sum_{i=d-2k}^{d-k} \binom{d-k}{i} < e^{-c_\epsilon d}.$$

To conclude: Since $e \cdot e^{-c_\epsilon d} \cdot (\binom{d-1}{2} + d + 1) < 1$ for large d, by the Local Lemma, with positive probability this does not happen for any vertex v.

What's wrong here?