Assignment 12

Positional Games, Winter 2009-10
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Solutions are due on Jan 26th (Tuesday) at 16:15PM

One of the most basic games, often used also as a tool, is the Box-game. Let $a_1, \dots a_k \geq 0$ integers. In the Box-game $B(a_1, \dots, a_k)$ the winning sets are k pairwise disjoint sets $A_1, \dots, A_k$ of sizes $a_1, \dots, a_k$, respectively. In the Box-game we set the convention that Breaker makes the first move. Maker wins if he completely occupies one of the A_i.

Problem 1 For $k = 3$, characterize the triples (a_1, a_2, a_3), for which Maker wins the $(m : 1)$ Box-game.

Problem 2. We shall say that a hypergraph H is of *type* (k, t) if its edges $A_1, \dots, A_k$ are pairwise disjoint and $\sum |A_i| = t$. If, in addition, the edges are "almost equal" in size (that is, if $|A_i|$ and $|A_j|$ differ by at most one for all choices of i and j) then we shall say that H is *canonical* of type (k, t).

Theorem Maker has a winning strategy in the $(m : 1)$ Box-game $B(a_1, \dots, a_k)$ on a canonical hypergraph of type (k, t) if and only if $t = \sum_{i=1}^{k} a_i \leq f(k, m)$, where $f(k, m)$ is defined recursively by $f(1, m) = 0$, and for $k \geq 2$ with

$$f(k, m) = \left\lfloor \frac{k}{k-1}(f(k-1, m) + m) \right\rfloor .$$

Remark. Note that for all $k \geq 2$ and $m \geq 1$, we have

$$(m-1)k \sum_{i=1}^{k-1} \frac{1}{i} \leq f(k, m) \leq mk \sum_{i=1}^{k-1} \frac{1}{i}.$$

Proof: To prove the "if" part we shall use induction on k. Responding to Breaker's move the Maker can create a canonical hypergraph H^* of type $(k-1, t^*)$ such that $t^* \leq t - \lfloor t/k \rfloor - m$. Since the right hand side of this inequality is at most $f(k-1, m)$, we are done.

To prove the "only if" part, we shall again use induction on k. This time, however, we shall prove a slightly stronger statement: if $t > f(k, m)$, then the Breaker has a winning strategy for the Box-game played on an *arbitrary*, not necessarily canonical, hypergraph of type (k, t). The strategy consists of removing, at each move, the smallest available edge. No matter how the Maker responds, the resulting hypergraph H^* will be of type $(k-1, t^*)$ such that $t^* \geq t - \lfloor t/k \rfloor - m$. Since the right hand side of this inequality is strictly greater than $f(k-1, m)$, we are done again. $\square$

(a) Is this theorem correct?
(b) What's wrong with this proof?

Problem 3. A *tournament* on n vertices is an orientation of the edges of K_n. Let Mr Red and Mr Blue play on K_N such that in each round they orient one of the un-oriented edges of K_N and color it with their own color. Mr Red wins if he occupies a transitive subtournament of order n (A tournament is *transitive* if the oriented edge relation is transitive). Prove that for every n there is an $N_0 = N_0(n)$ such that Mr Red wins the game played on K_N for $N \geq N_0$. (*Hint:* Try to use the biased Weak Win Criterion of Beck (last HW).)