Assignment 11

Positional Games, Winter 2009-10
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Quiz on Jan 19th (Tuesday) at 16:15PM

Problem 1 Let n be even. Prove that for the critical bias of the perfect matching game $\mathcal{M} = \mathcal{M}_n$ we have

$$b_{\mathcal{M}} \geq (\frac{\ln 2}{2} - o(1))\frac{n}{\ln n}.$$

Here $\mathcal{M} \subseteq \{M \subseteq E(K_n) : \ M \text{ is a perfect matching}\}$.

Problem 2. Prove that Beck's criterion for Breaker's win in a biased positional game is tight. That is, for every pair of positive integers p and q and infinitely many n, construct an n-uniform hypergraph $\mathcal{F} \subseteq 2^X$, such that $\sum_{A\in\mathcal{F}}(1+q)^{-|A|/p} = \frac{1}{1+q}$ and Maker has a winning strategy in the $(p:q)$-game $\mathcal{F}$. (Hint: generalize the binary tree construction for the tightness of the Erdős-Selfridge Criterion.)

Problem 3. (Beck's Criterion for Maker's win in biased game) Prove that Maker has a winning strategy in the $(p:q)$ biased game provided

$$\sum_{A\in\mathcal{F}} \left(\frac{p}{p+q}\right)^{|A|} > \frac{p^2q^2}{(p+q)^3} \cdot \Delta_2(\mathcal{F})|V(\mathcal{F})|.$$

Problem 4. In the non-planarity game $\mathcal{NP} = \mathcal{NP}_n$ Maker's goal is to build a graph that cannot be embedded in the plane without crossing edges. Formally let $\mathcal{NP} = \{E(G) \subseteq E(K_n) : G \text{ is non-planar}\}$. Prove that

$$b_{\mathcal{NP}} \leq \frac{n}{2}.$$

(Planarity is a *decreasing* graph property; this is why it is more natural to consider the *non-planarity* Maker-Breaker game.)