Assignment 10

Positional Games, Winter 2009-10
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Quiz on Jan 12th (Tuesday) at 16:15PM

Problem 1 Let n be even. Prove that the edge set of K_n can be partitioned into $n/2$ spanning trees.

Problem 2. In class we showed that playing the $(1:2)$ biased game on a graph that is the union of three $(= 1+2)$ pairwise edge disjoint spanning trees, Enforcer has a strategy that forces Avoider to build a spanning tree.

Give an example of a graph G containing three pairwise edge-disjoint spanning trees such that Maker **loses** the $(1:2)$ Maker/Breaker game played on $E(G)$. Show that even 100 edge-disjoint spanning trees does not necessarily help Maker to win the $(1:2)$ game.

Recall the general definition of Maker/Breaker (Avoider/Enforcer) positional games: given are a set X, the "board", and a family $\mathcal{F} \subseteq 2^X$, the family of "winning sets". Player Maker (Avoider) wins if he occupies a winning set (avoids occupying winning sets), otherwise player Breaker (Enforcer) wins.

Problem 3. (a) One intuitively thinks that taking **more** edges in a Maker/Breaker game is advantageous for the players. Prove this **formally** and **precisely** in the following form: if Breaker has a winning strategy in the $(1:b)$ positional game $(X, \mathcal{F})$ then he has a winning strategy in the $(1:b+1)$ game as well.
Conclude that one can define the *critical bias* $b_{\mathcal{F}} \in \mathbb{N} \cup \{\infty\}$ of the game $(X, \mathcal{F})$ such that for every $b \in \mathbb{N}$, Maker has a winning strategy in the $(1:b)$ game, if and only if $b < b_{\mathcal{F}}$. ($\mathbb{N}$ denotes the set of non-negative integers.)
What is $b_{\mathcal{F}}$ when $\mathcal{F} = \emptyset$? When is $b_{\mathcal{F}} = \infty$?

(b) Similarly, it is maybe plausible to believe that being able to take **less** edges in an Avoider/Enforcer game is advantageous for the players. Show that this intuition is false in general: Give an infinite sequence of Avoider/Enforcer games such that in each of them the winner of the $(1:a)$ game changes (more or less) according to the *parity* of a.