# VIZING'S CONJECTURE FOR CHORDAL GRAPHS

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Abstract. Vizing [13] conjectured that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for every pair $G, H$ of graphs, where "$\Box$" is the Cartesian product, and $\gamma(G)$ is the domination number of the graph $G$. Denote by $\gamma^i(G)$ the maximum, over all independent sets $I$ in $G$, of the minimal number of vertices needed to dominate $I$. We prove that $\gamma(G\Box H) \geq \gamma^i(G)\gamma(H)$. Since for chordal graphs $\gamma^i = \gamma$, this proves Vizing's conjecture when $G$ is chordal.

## 1. Introduction

Given two graphs, $G$ and $H$, their *Cartesian product* $G\Box H$ is defined as the graph on $V(G) \times V(H)$, in which every row is a copy of $H$, and every column is a copy of $G$. Namely, the pair $(x, y)$ is connected to the pair $(u, v)$ if either $x = u$ and $(y, v) \in E(H)$ or $y = v$ and $(x, u) \in E(G)$.

The closed neighborhood $N_G[v]$ of a vertex in a graph $G$ is the set consisting of $v$ itself and its neighbors in the graph. A set $A$ of vertices is said to *dominate* a set $B$ if $B \subseteq \bigcup\{N_G[a] \mid a \in A\}$. The minimal size of a set dominating a set $A$ is denoted by $\gamma_G(A)$. A set $D$ of vertices in a graph is called *dominating* if it dominates $V(G)$. We write $\gamma(G)$ for $\gamma_G(V(G))$. The *independence-domination number* $\gamma^i(G)$ is the maximum of $\gamma_G(I)$ over all independent sets $I$ in $G$. This parameter has arisen in the context of matching theory, see e.g. [1, 10]. Obviously, $\gamma^i(G) \leq \gamma(G)$, and in general the gap between the two may be large. For example, in the line graph of the hypergraph consisting of all subsets of size $n$ of a set of size $n^2$ one has $\gamma^i = 1$, while $\gamma = n$ (this example is due to Roy Meshulam, [11]). However, we have:

**Theorem 1.1.** [2] *In chordal graphs $\gamma = \gamma^i$.*

In 1968 Vizing made the following conjecture.

**Conjecture 1.2.** [13] $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ *for every pair $G, H$ of graphs.*

The conjecture was verified for several specific classes of graphs; see the survey paper of Hartnell and Rall [7], or [4, 12] for more recent results. Equality holds, for example, when $G$ and $H$ are graphs with no edges, or when $G = H$ is a graph consisting of a clique $K$ with a vertex $v(x)$ added for each $x \in V(K)$, connected only to $x$. In these constructions both $G$ and $H$ have independent sets containing at least half of their respective vertex sets; for other extremal examples see [6, 7].

For a graph $G$ let $\kappa(G)$ denote the minimal number of cliques in $G$ covering $V(G)$, that is, $\kappa(G) = \chi(\bar{G})$. Obviously, $\gamma(G) \leq \kappa(G)$. We say that $G$ satisfies the Barcalkin-German property (or that it satisfies the B-G property, or that it is B-G), if it is possible to add to $G$ edges so as to obtain a graph $G'$ with $\kappa(G') = \gamma(G') = \gamma(G)$. It is known, for example, that cycles and trees have the B-G property [3] (c.f. [7]). An example of a graph not satisfying the B-G property is the line graph of the complete 2-homogeneous hypergraph on 6 vertices. Its domination number is 3, its clique covering number is 4 (e.g. by Lovász' solution of Kneser's conjecture, [9]), and the addition of any edge reduces the domination number. The following was proved by Barcalkin and German.

**Theorem 1.3.** [3] (c.f.[8]) *If $G$ satisfies the B-G property then Vizing's conjecture is true for $G$, when paired with any graph $H$.*

Clark and Suen [5] used an elaboration of the same idea to prove a result for general graphs.

**Theorem 1.4.** [5] $\gamma(G\square H) \geq \frac{\gamma(G)\gamma(H)}{2}$ *for every pair $G, H$ of graphs.*

We shall adapt their argument to prove the following.

**Theorem 1.5.** *For any graphs $G$ and $H$ there holds: $\gamma(G\square H) \geq \gamma^i(G)\gamma(H)$.*

This proves Vizing's conjecture for graphs for which $\gamma^i = \gamma$, in particular for chordal graphs, when paired with any other graph. Another class of graphs for which $\gamma^i = \gamma$ is that of cycles of length divisible by 3. But as noted above, all cycles satisfy the B-G property and thus the fact that they satisfy the Vizing conjecture is known. We do not know, and do not even have an intelligent guess, whether chordal graphs are necessarily B-G.

**Problem 1.6.** *Does every chordal graph satisfy the B-G property?*

The proof technique of Theorem 1.5 can also be used to prove the validity of Vizing's conjecture for the graph parameter $\gamma^i$.

**Theorem 1.7.** *For any graphs $G$ and $H$ it holds that $\gamma^i(G\square H) \geq \gamma^i(G)\gamma^i(H)$.*

## 2. Proof of the theorem

*Proof of Theorem 1.5.* If $v$ is an isolated vertex in $G$, then the validity of the theorem for $G - v$ easily implies its validity for $G$. Thus we may assume that $G$ contains no isolated vertices.

Write $p$ for $\gamma^i(G)$, and $q$ for $\gamma(H)$. Let $I$ be an independent subset of $V(G)$ requiring at least $p$ vertices to dominate it in $G$. We shall prove something a bit stronger than claimed in the theorem, namely that $\gamma_{G\square H}(I \times V(H)) \geq pq$.

Let $D \subseteq V(G) \times V(H)$ be a set dominating $I \times V(H)$ in $G\square H$. Our aim is to show that $|D| \geq pq$.

Let $X = \{x_1, x_2, \ldots, x_q\}$ be a dominating set in $H$ of size $q$. Partition $V(H)$ into sets $W_i,\ i = 1, 2, \ldots, q$ such that $x_i$ dominates $W_i$. By the minimality of $q$, we have:

**Lemma 2.1.** *For every subset $J$ of $\{1, \ldots, q\}$ there holds: $\gamma_H(\bigcup_{j\in J} W_j) \geq |J|$.*

Let $\mathcal{S}$ be the family of all sets of the form $\{v\} \times W_j$ ($v \in I,\ j \leq q$), all of whose elements are dominated by $D$ within *the copy of $H$ they are in.* That is, $\{v\} \times W_j \in \mathcal{S}$ if for every $u \in W_j$ there is a vertex $w = w(u) \in V(H)$ such that $(v, w) \in D$ and $uw \in E(H)$.

For every vertex $v \in I$ let $\mathcal{S}_v$ be the family of those members of $\mathcal{S}$ which are of the form $\{v\} \times W_j$, and for every $j \leq q$ denote by $\mathcal{S}^j$ the family of those members of $\mathcal{S}$ which are of the form $\{v\} \times W_j$.

By Lemma 2.1, for every $v \in I$ we have:

$$|D \cap (\{v\} \times V(H))| \geq |\mathcal{S}_v| \tag{1}$$

Summing over all $v \in I$ we obtain:

$$|D \cap (I \times V(H))| \geq |\mathcal{S}| \tag{2}$$

Fix an index $j \leq q$. Each set $\{v\} \times W_j$ not belonging to $\mathcal{S}$ contains a vertex $(v, w)$ dominated in $G\Box H$ by a vertex $(u, w) \in D$, where $u = u(v) \in V(G)$ and $uv \in E(G)$. Note that $u(v) \notin I$ since $I$ is independent. Thus the set $\{u(v) | \{v\} \times W_j \notin S\}$ dominates $|I| - |\mathcal{S}^j|$ vertices in $I$. This means that it can be completed to a set dominating $I$ by adding to it $|\mathcal{S}^j|$ vertices, and thus its size is at least $p - |\mathcal{S}^j|$. Summing over all $j$, and keeping in mind that the vertices $u(v)$ do not belong to $I$, we obtain:

$$|D \cap ((V(G) \setminus I) \times V(H))| \geq pq - |\mathcal{S}| \tag{3}$$

Adding up Equations 2 and 3 yields the desired inequality on $|D|$.

$\square$

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