Sharp threshold for the appearance of certain spanning trees in random graphs

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Abstract

We prove that a given tree T on n vertices with bounded maximum degree is contained asymptotically almost surely in the binomial random graph $G(n, \frac{(1+\varepsilon)\log n}{n})$ provided that T belongs to one of the following two classes: (1) T has linearly many leaves; (2) T has a path of linear length all of whose vertices have degree two in T.

1 Introduction

In this paper we consider the problem of embedding a copy of a given tree T on n vertices into the binomial random graph $G(n,p)$. We will restrict our attention to the (already challenging enough) case of trees of bounded maximum degree, that is, trees with maximum degree which is bounded from above by a constant which is independent of n.

The problem of embedding large or *nearly* spanning bounded degree trees in random graphs on n vertices (where by a nearly spanning tree we mean a tree T whose number of vertices is at most $(1-c)n$ for some constant $c > 0$) is a rather well studied subject (see, e.g., [8], [1], [10], [9], [12]). In particular, Alon, Sudakov and the second author proved in [2] that for given $\varepsilon > 0$ and integer d there exists $C = C(d, \varepsilon) > 0$ such that a.a.s.[1] the random graph $G(n,p)$ with $p = C/n$ admits a copy of a tree T on $(1-\varepsilon)n$ vertices with maximum degree at most d (in fact, it was proved in [2] that such a random graph contains a.a.s. a copy of *every* such tree). A better bound on the aforementioned constant C and the resilience version of this result have been obtained in [4] and in [5], respectively.

In contrast, apart from some sporadic special cases, not much is known about the case of embedding *spanning* trees. Of course, no spanning tree appears until the random graph becomes connected, which happens at $p(n) = \log n/n$ (in fact, $p = \log n/n$ is known to be the

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[1] An event $\mathcal{E}_n$ occurs asymptotically almost surely, or a.a.s. for brevity, in the probability space $G(n,p)$ if $\lim_{n\to\infty} Pr[G \sim G(n,p) \in \mathcal{E}_n] = 1$.

connectivity threshold in the following very strong sense: if $p(n) = (\log n - \omega(1))/n$, then a.a.s. $G(n,p)$ is not connected, whereas if $p(n) = (\log n + \omega(1))/n$ then a.a.s. $G(n,p)$ is connected; as usual, $\omega(1)$ stands for any function tending to infinity with n arbitrarily slowly). This simple argument provides an immediate lower bound for the edge probability $p(n)$ sufficient for an asymptotically almost sure appearance of any given spanning tree T in $G(n,p)$. Krivelevich [15] proved that for any given bounded degree spanning tree T, if $p(n) > n^{-1+\varepsilon}$ for an arbitrarily small constant $\varepsilon > 0$, then $G(n,p)$ contains a.a.s. a copy of T. (We repeatedly write "a given tree" to stress the order of quantifiers – first a tree is given and only then a random graph $G \sim G(n,p)$ is exposed. Our aim is then to find a copy of that particular tree T in G.[2] Stronger, universality-type statements, which assert that $G(n,p)$ a.a.s. simultaneously contains *every* tree T from a given class, are usually much harder to obtain.) The authors of [2] have observed that if a tree T has at least αn leaves for some constant $\alpha > 0$, then a.a.s. $G(n, C\log n/n)$ contains a copy of T for some sufficiently large $C = C(\alpha) > 0$. The proof is not that hard and utilizes the embedding result for nearly spanning trees from the same paper. Here is a brief sketch. We represent the random graph $G \sim G(n,p)$ as a union of two independent random graphs G_1 and G_2, where $G_i \sim G(n,p_i)$ and $1-p = (1-p_1)(1-p_2)$. We set $p_1 = C_1/n$, where C_1 is a sufficiently large constant. Let L denote the set of leaves of T and let $T' = T \setminus L$. By the embedding results for the nearly spanning case, the first random graph G_1 contains a.a.s. a copy of T'; we fix such a copy. Now we expose the second random graph G_2 and use its edges to embed L and the edges that connect them to their already embedded parents. This can be done using Hall-type arguments provided that $p_2 = C_2 \log n/n$, where $C_2 = C_2(\alpha) > 0$ is a sufficiently large constant. It is instructive to notice that this proof yields the value of $p = C\log n/n$, where the constant $C = C(\alpha)$ has to grow as α becomes smaller (as, in particular, the random graph G_2 has to have an edge connecting every vertex from the set of vertices outside the image of the embedding of T' to one of the vertices slated to serve as a parent of a leaf of T).

While this was not stated explicitly in [2], one can observe that a very similar approach works for another class of bounded degree spanning trees. To this end, we need to introduce the notion of a bare path.

Definition 1.1 *A path P in a tree T is called* bare *if all vertices of P have degree exactly two in T.*

Assume now that a given tree T on n vertices admits a bare path P of length at least αn, for some constant $\alpha > 0$. Then one can start by embedding the forest $T' = T \setminus P$ in a random graph $G(n, C_1/n)$, and then use the edges of the random graph $G(n, C_2 \log n/n)$ to find a copy of P between its already embedded endpoints. The latter task amounts to finding a.a.s. a Hamilton path between any given pair of vertices in a random graph. This can be achieved using known tools (say, those from [13]). Here too the constant $C_2 = C_2(\alpha)$ has to grow as $\alpha \to 0$.

In this paper, we get rid of the dependence of C on α for the two aforementioned classes of bounded degree spanning trees. In fact, we prove that any given member of either one of these classes appears a.a.s. in $G(n,p)$ already at $p(n) = (1+\varepsilon)\log n/n$, that is, very shortly after

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[2] Formally, we are of course not given a single tree but a sequence of trees $\{T_n\}_{n=1}^{\infty}$, where T_n is a tree on n vertices for every $n \in \mathbb{N}$. Bearing this in mind we henceforth discard this more accurate but cumbersome notation for the sake of clarity of presentation.

the binomial random graph first becomes connected. Our main results are manifested in the following two theorems.

Theorem 1.2 *Let α and ε be positive real numbers and let d be a positive integer. Let T be a tree on n vertices, with maximum degree at most d and with at least αn leaves. Then a.a.s. the random graph $G(n,(1+\varepsilon)\log n/n)$ contains a copy of T.*

Remark 1.3 *Let T be an arbitrary bounded degree tree on n vertices with $\Theta(n)$ leaves (say, the complete d-ary tree on n vertices for some fixed d). Since $G(n,p)$ is a.a.s. disconnected for every $p \leq (1-\varepsilon)\log n/n$, it follows from Theorem 1.2 that there is a sharp threshold at $\log n/n$ for the appearance of T in $G(n,p)$.*

Let G be a graph and let $\mathcal{F}$ be a family of graphs. The graph G is said to be *universal for the family $\mathcal{F}$*, or $\mathcal{F}$-universal for brevity, if G contains every $F \in \mathcal{F}$ as a subgraph.

Theorem 1.4 *Let α and ε be positive real numbers and let d be a positive integer. Let $\mathcal{L}$ be the family of all trees on n vertices with maximum degree at most d which admit a bare path of length αn. Then the random graph $G(n,(1+\varepsilon)\log n/n)$ is a.a.s. $\mathcal{L}$-universal.*

The rest of this paper is organized as follows. In the next section we gather several tools needed for the subsequent proofs of our main results. In Section 3 we prove Theorems 1.2 and 1.4. The final section of the paper is devoted to concluding remarks.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that the number of vertices n is sufficiently large. Moreover, when proving that a certain graph property holds a.a.s. we will sometimes refrain from explicitly denoting the number of vertices of the graph, as it is in fact a sequence of integers which tends to infinity. Throughout the paper, log stands for the natural logarithm, unless stated otherwise. Our graph-theoretic notation is standard and follows that of [18]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A,B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A,B) = |E_G(A,B)|$. For a set $U \subseteq V(G)$ and a vertex $w \in V(G)$, let $N_G(w,U) = \{u \in U : wu \in E(G)\}$ denote the set of neighbors of w in U and let $d_G(w,U) = |N_G(w,U)|$ denote the degree of w into U. For sets $U, W \subseteq V(G)$ let $N_G(W,U) = \bigcup_{w\in W} N_G(w,U)$. We abbreviate $N_G(w,V(G))$ with $N_G(w)$, and let $d_G(w) = |N_G(w)|$ denote the degree of w in G. We also abbreviate $N_G(W,V(G))$ to $N_G(W)$. Often, when there is no risk of confusion, we omit the subscript G from the notations above. The minimum degree and the maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G, induced on the vertices of S. A graph G is said to be *Hamilton connected* if, for every two vertices $u, w \in V(G)$, there is a Hamilton path in G whose endpoints are u and w.

2 Preliminaries

In this section we prove several results which will be useful in proving our main theorems. Since there are quite a few such results and they vary in nature, this section is divided into several subsections.

2.1 Properties of $G(n,p)$

In this subsection we prove a few simple and mostly standard properties of the binomial random graph $G(n,p)$. In our proofs we will make use of certain known bounds on the tail of the binomial distribution (see e.g. [14]). In particular we will use the following simple bound:

$$\text{If } X \sim Bin(n,p), \text{ then } Pr(X \geq k) \leq \binom{n}{k} p^k \leq (enp/k)^k \,. \tag{1}$$

The properties of interest are described in the following three lemmas.

Lemma 2.1 *Let $0 < \varepsilon < 1$ and $0 \leq \beta \leq \varepsilon/7$ be real numbers and let $p = p(n) = (1+\varepsilon)\frac{\log n}{n}$. Let U be a given set of size $|U| \leq \beta n$. Then a.a.s. the random graph $G = G(n,p) = ([n], E)$ satisfies all of the following properties:*

- **(P1)** $\Delta(G) \leq 10 \log n$.
- **(P2)** $d_G(u, [n] \setminus U) \geq \eta \log n$ *for every* $u \in [n]$, *where* $0 < \eta = \eta(\varepsilon) < 1/2$ *is a real number satisfying* $\eta \log(6\eta^{-1}) = \varepsilon/3$; *in particular* $\delta(G) \geq \eta \log n$.
- **(P3)** *Every subset* $A \subseteq [n]$ *of cardinality* $|A| \leq \frac{n(\log\log n)^2}{\log n}$ *spans at most* $\frac{|A| \log n}{\log\log n}$ *edges in* G.
- **(P4)** *For every two disjoint subsets* $A, B \subseteq [n]$ *of cardinality* $|A| \leq \frac{n(\log\log n)^2}{\log n}$ *and* $|B| = |A|\sqrt{\log n}$, *we have* $e_G(A,B) \leq \frac{|A| \log n}{\log\log n}$.
- **(P5)** *For every two disjoint subsets* A, B *of* $[n]$ *of cardinality* $|A| = |B| = \frac{n(\log\log n)^{3/2}}{\log n}$, *we have* $e_G(A,B) > 0$.

Proof Properties **(P1)** - **(P5)** follow by standard first moment calculations and standard bounds on the tail of the binomial distribution.

(P1): For a given vertex $v \in [n]$, the degree of v is distributed binomially with parameters $n-1$ and p. Therefore, it follows by (1) that

$$Pr[d_G(v) \geq 10 \log n] \leq \left(\frac{enp}{10 \log n}\right)^{10 \log n} \leq (2e/10)^{10 \log n} = o(1/n)\,.$$

Applying the union bound over all vertices of $[n]$ proves that the required bound on the maximum degree holds a.a.s.

(P2): For a given vertex $u \in [n]$, the degree of u in $[n] \setminus U$ is distributed binomially with parameters $n - |U|$ and p if $u \in U$ and with parameters $n - |U| - 1$ and p if $u \in [n] \setminus U$.

Denote $\delta = \eta \log n$ and fix some vertex $u \in [n]$. Then

$$\begin{aligned} Pr[d_G(u,[n]\setminus U) \leq \eta \log n] &\leq \sum_{i=0}^{\delta} Pr[Bin(n-\beta n-1,p)=i] \\ &\leq (\delta+1)Pr[Bin((1-\beta)n-1,p)=\delta] \\ &\leq \log n \binom{(1-\beta)n}{\delta} p^{\delta}(1-p)^{(1-\beta)n-1-\delta} \\ &\leq \log n \left(\frac{enp}{\delta(1-p)}\right)^{\delta}(1-p)^{(1-\beta)n-1} \\ &\leq \left(\frac{6}{\eta}\right)^{\eta\log n} e^{-p(1-\beta)n} \\ &= \exp\left\{[\eta\log(6/\eta)-(1+\varepsilon)(1-\beta)]\log n\right\} \\ &\leq \frac{1}{n^{1+\varepsilon/3}}, \end{aligned}$$

where the first inequality above follows by the monotonicity of the binomial distribution along its lower tail.

Applying the union bound over all vertices of $[n]$ proves that the required bound on the minimum degree holds a.a.s.

(P3): Let $A \subseteq [n]$ be any subset of size $1 \leq a \leq \frac{n(\log\log n)^2}{\log n}$. Let X_A be the random variable that counts the number of edges of G with both endpoints in A. Then $X_A \sim Bin\left(\binom{a}{2},p\right)$ and thus $\mathbb{E}(X_A) = \binom{a}{2}p$. Let E_3 denote the event *"there exists a set $A \subseteq [n]$, of size $1 \leq a \leq \frac{n(\log\log n)^2}{\log n}$, such that $e_G(A) > \frac{a\log n}{\log\log n}$"*. Using the bound (1) we get

$$\begin{aligned} Pr[E_3] &\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \binom{n}{a} Pr\left[X_A \geq \frac{a\log n}{\log\log n}\right] \\ &\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \left[\frac{en}{a}\left(\frac{e\binom{a}{2}p}{a\log n/(\log\log n)}\right)^{\log n/\log\log n}\right]^a \\ &\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \left[\frac{en}{a}\left(\frac{3a\log\log n}{n}\right)^{\log n/\log\log n}\right]^a \\ &\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \left[\exp\left\{1+\log(n/a)-\frac{\log n}{\log\log n}\left(\log(n/a)-2\log\log\log n\right)\right\}\right]^a \\ &= o(1). \end{aligned}$$

(P4): Let $A \subseteq [n]$ be any subset of cardinality $1 \leq a \leq \frac{n(\log\log n)^2}{\log n}$ and let B be any subset of $[n]\setminus A$ of cardinality $b = a\sqrt{\log n}$. Let X_{AB} be the random variable that counts the number of edges of G with one endpoint in A and the other in B. Then $X_{AB} \sim Bin(ab,p)$ and thus $\mathbb{E}(X_{AB}) = abp = a^2p\sqrt{\log n}$. Let E_4 denote the event *"there exist two disjoint subsets $A, B \subseteq [n]$, of sizes $1 \leq a = |A| \leq \frac{n(\log\log n)^2}{\log n}$ and $b = |B| = a\sqrt{\log n}$, such that $e_G(A,B) > \frac{a\log n}{\log\log n}$"*. Using the bound (1) we get

$$
\begin{aligned}
Pr[E_4] &\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \binom{n}{a}\binom{n}{b} Pr\left[X_{AB} \geq \frac{a\log n}{\log\log n}\right] \\
&\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \left[\frac{en}{a}\left(\frac{en}{b}\right)^{\sqrt{\log n}}\left(\frac{ea^2p\sqrt{\log n}}{a\log n/(\log\log n)}\right)^{\log n/\log\log n}\right]^a \\
&\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} \left[\frac{en}{a}\left(\frac{en}{b}\right)^{\sqrt{\log n}}\left(\frac{6a\sqrt{\log n}\log\log n}{n}\right)^{\log n/\log\log n}\right]^a \\
&\leq \sum_{a=1}^{\frac{n(\log\log n)^2}{\log n}} [\exp\{1+\log(n/a)+\sqrt{\log n}\,(1+\log(n/b)) \\
&\quad -\frac{\log n}{\log\log n}\,(\log(n/a) - 0.6\log\log n)\}]^a \\
&= o(1).
\end{aligned}
$$

(P5): Let E_5 denote the event: *"there exist two disjoint subsets $A, B \subseteq [n]$ of size $|A| = |B| = \frac{n(\log\log n)^{3/2}}{\log n}$ such that $e_G(A,B) = 0$"*. Then

$$
\begin{aligned}
Pr[E_5] &\leq \binom{n}{|A|}\binom{n}{|B|}(1-p)^{|A||B|} \\
&\leq \binom{n}{\frac{n(\log\log n)^{3/2}}{\log n}}^2 e^{-p|A||B|} \\
&\leq \left(\frac{e\log n}{(\log\log n)^{3/2}}\right)^{\frac{2n(\log\log n)^{3/2}}{\log n}} \exp\left\{-\frac{(1+\varepsilon)\log n}{n}\cdot\frac{n^2(\log\log n)^3}{(\log n)^2}\right\} \\
&\leq \exp\left\{\frac{2n(\log\log n)^{5/2}}{\log n} - \frac{n(\log\log n)^3}{\log n}\right\} \\
&= o(1).
\end{aligned}
$$

□

Lemma 2.2 *Let $0 < \beta_1 < \beta_2 \leq 1$ be real numbers and let $p = p(n) = (1-\beta_1)\frac{\log n}{n}$. Let $W \subseteq [n]$ be a given subset of size $|W| \leq n^{1-\beta_2}$. Then a.a.s. the random graph $G = G(n,p) = ([n], E)$ satisfies all of the following properties:*

(Q1) $\Delta(G) \leq 10 \log n$.

(Q2) *Let* $0 < \gamma < 1/2$ *be a real number satisfying* $\gamma \log(3/\gamma) = (\beta_2 - \beta_1)/3$, *then* $d_G(w, [n] \setminus W) \geq \gamma \log n$ *for every* $w \in W$.

(Q3) $|\{u \in N_G(w) \setminus W : N_G(u) \cap (W \setminus \{w\}) \neq \emptyset\}| \leq 2/\beta_2$ *for every* $w \in W$.

Proof Properties **(Q1)** - **(Q3)** follow by standard first moment calculations and standard bounds on the tail of the binomial distribution.

(Q1): This follows immediately from Property **(P1)** of Lemma 2.1.

(Q2): It suffices to prove this under the assumption $|W| = n^{1-\beta_2}$. For a given vertex $w \in W$, the degree of w in $[n] \setminus W$ is distributed binomially with parameters $n - n^{1-\beta_2}$ and p.

Denote $\delta = \gamma \log n$ and fix some vertex $w \in W$. Then

$$\begin{aligned} Pr[d_G(w, [n] \setminus W) \leq \gamma \log n] &= \sum_{i=0}^{\delta} Pr[Bin(n - n^{1-\beta_2}, p) = i] \\ &\leq (\delta + 1) Pr[Bin(n - n^{1-\beta_2}, p) = \delta] \\ &\leq \log n \binom{n - n^{1-\beta_2}}{\delta} p^{\delta} (1-p)^{n - n^{1-\beta_2} - \delta} \\ &\leq \log n \left(\frac{enp}{\delta(1-p)} \right)^{\delta} (1-p)^{n - n^{1-\beta_2}} \\ &\leq \left(\frac{3}{\gamma} \right)^{\gamma \log n} e^{-(1-o(1))pn} \\ &= \exp\{[\gamma \log(3/\gamma) - (1 - \beta_1 - o(1))] \log n\} \\ &\leq \frac{1}{n^{1-(\beta_1+\beta_2)/2}} , \end{aligned}$$

where the first inequality above follows by the monotonicity of the binomial distribution along its lower tail.

Since $\beta_1 < \beta_2$, applying the union bound over all vertices of W proves that the required lower bound on the degree a.a.s. holds for every $w \in W$.

(Q3): Fix some $w \in W$ and let $u \in [n] \setminus W$ be an arbitrary vertex. It follows that

$$\begin{aligned} Pr[uw \in E(G),\ N_G(u) \cap (W \setminus \{w\}) \neq \emptyset] &< p \cdot |W| p \\ &\leq n^{-1-\beta_2} \log^2 n \\ &< n^{-1-\beta_2/2} . \end{aligned}$$

The above events are mutually independent for distinct $u \in [n] \setminus W$ as they involve disjoint sets of edges. Hence, the probability that there are at least $2/\beta_2$ such vertices u is at most

$$\binom{n}{2/\beta_2} \left(n^{-1-\beta_2/2} \right)^{2/\beta_2} \leq 1/n .$$

Applying the union bound over all vertices $w \in W$ yields the desired result. □

Figure 1: Tree transformation.

2.2 Splitting trees and random graphs

In order to embed a spanning tree T in $G(n, p)$, we will want to split both T and $G(n, p)$ into several parts. Starting with the former, we prove the following.

Lemma 2.3 *For every positive real numbers α and ε, there exist an integer $n_0 = n_0(\alpha, \varepsilon)$ and a real number $\beta = \beta(\alpha, \varepsilon) > 0$ such that the following holds. For every tree $T = (V, E)$ with $n \geq n_0$ vertices and at least αn leaves, there exist a vertex $u \in V$ and subtrees T_1 and T_2 of T such that $V(T_1) \cup V(T_2) = V$, $V(T_1) \cap V(T_2) = \{u\}$, $|V(T_1)| \leq \varepsilon n$ and the number of leaves of T_1 is at least βn.*

Note that we do not assume any restrictions on $\Delta(T)$ in Lemma 2.3. While this makes the proof a little more complicated, we do this for two reasons; first, the result is more general, and second, it will be needed in the concluding remarks section where we discuss embedding random trees in $G(n, p)$.

In the proof of Lemma 2.3 we will make use of the notion of separators. Let $G = (V, E)$ be a graph on n vertices and let $0 < \gamma < 1$ be a real number. An $(f(n), \gamma)$-separator of G is a set $S \subseteq V$ of size $|S| \leq f(n)$ such that every connected component of $G[V \setminus S]$ is of size at most γn. It is well known (and easy) that any tree T has a $(1, 1/2)$-separator. Indeed, pick an arbitrary non-leaf vertex $u \in V(T)$; clearly it is a cut vertex. If u is a $(1, 1/2)$-separator of T, then we are done. Otherwise, there is exactly one component of $T \setminus u$ of size greater than $n/2$. Let F be this component and let v be the unique vertex of F such that $uv \in E(T)$. We repeat this process with v instead of u, thus decreasing the order of the largest component. Continuing this way, we will end up with a $(1, 1/2)$-separator.

Proof of Lemma 2.3 Transform T into a binary tree $T_B = (V \cup F, E_B)$ as follows. Root T at some arbitrary vertex $r \in V$. Let $u_1, \ldots, u_t$ denote the children of r in T. Replace the star $(\{r, u_1, \ldots, u_t\}, \{ru_i : 1 \leq i \leq t\})$ with a rooted (not necessarily complete) binary tree whose root is r, whose leaves are $u_1, \ldots, u_t$ and whose other vertices all have degree 3 (see Figure 1). The new vertices which were created in this process are put into F. Repeat the same process with every $w \in V$ to obtain T_B. It follows by the construction of T_B that every leaf of T_B is in V and that $|F| \leq |V|$, entailing $v(T_B) \leq 2n$.

Let $k = k(\varepsilon)$ be the smallest positive integer for which $2^{-k+1} < \varepsilon$ holds. Let u be a $(1, 1/2)$-separator of T_B and let $v_1, \ldots, v_d$ be the neighbors of u in T_B; note that $d \leq 3$. For every $1 \leq i \leq d$ let T^i denote the tree of $T_B \setminus u$ which contains v_i (we consider v_i to be the root of T^i). Since T_B has at least αn leaves (recall that T and T_B have the same leaves) and each of its leaves is in exactly one T^i there must exist a $1 \leq j \leq d$ such that the number of leaves of T^j is at least $\alpha n/3$. Moreover, $|V(T^j)| \leq |V(T_B)|/2$ holds by the choice of u. Repeating this process k times (each time with an appropriate tree T^j) we obtain a rooted subtree $\hat{T}_1$ of T_B with at most $2^{-k} v(T_B) \leq 2^{-k} \cdot 2n < \varepsilon n$ vertices and with at least $\alpha n/3^k$ leaves.

We would like to transform $\hat{T}_1$ into a tree $T_1 \subseteq T$ with the desired properties. Let v denote the root of $\hat{T}_1$. If $v \in V$, then we can take T_1 to be the subtree of T which is rooted at v. Indeed, T_1 and $\hat{T}_1$ have the same leaves and $V(T_1) = V \cap V(\hat{T}_1)$. Otherwise, let $u \in V$ be the nearest ancestor of v in $\hat{T}_1$ (that is, v was created when we transformed the star centered at u). Let $T_1 = T[\{u\} \cup (V \cap V(\hat{T}_1)]$. Note that T_1 is a tree, that it has the same leaves as $\hat{T}_1$ and that $v(T_1) \leq |V(\hat{T}_1) \cup \{u\}| \leq \varepsilon n$.

In either case, adding the root of T_1 to $T \setminus T_1$ yields the desired second tree T_2. $\square$

The following lemma handles splitting $G(n, p)$.

Lemma 2.4 (Clustered Local Lemma) *Let $G = (V, E)$ be a graph on n vertices with maximum degree Δ. Let $Y \subseteq V$ be a set of $m = a + b$ vertices where a and b are positive integers. Assume that $d_G(v, Y) \geq \delta$ holds for every $v \in V$. If $\Delta^2 \cdot \left\lceil \frac{m}{\min\{a,b\}} \right\rceil \cdot 2 \cdot e^{1 - \frac{\min\{a,b\}^2}{5m^2} \cdot \delta} < 1$, then there exists a partition $Y = A \cup B$ of Y such that*

- *(i)* $|A| = a$ *and* $|B| = b$.
- *(ii)* $d_G(v, A) \geq \frac{a}{3m} d_G(v, Y)$ *for every* $v \in V$.
- *(iii)* $d_G(v, B) \geq \frac{b}{3m} d_G(v, Y)$ *for every* $v \in V$.

In the proof of Lemma 2.4 we will make use of the following well known results.

Lemma 2.5 (Lovász Local Lemma (see e.g. [3])) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $Pr(\bigwedge_{i=1}^n \bar{A}_i) > 0$.*

Lemma 2.6 *[16, Theorem 2.5] Let $X_1, \ldots, X_n$ be independent random variables and let $S = \sum_{i=1}^n X_i$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers such that $a_i \leq X_i \leq b_i$ holds for every $1 \leq i \leq n$. Then for every $t > 0$*

$$Pr(|S - \mathbb{E}(S)| \geq t) \leq 2 \exp\left\{ -\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2} \right\}.$$

Proof of Lemma 2.4 Assume without loss of generality that $a \leq b$. Let $Y = Q_1 \cup \ldots \cup Q_a$ be an arbitrary partition of Y into a parts of nearly equal size (that is, $|Q_i| = \left\lfloor \frac{m}{a} \right\rfloor$ or $|Q_i| = \left\lceil \frac{m}{a} \right\rceil$

for every $1 \leq i \leq a$, in particular no Q_i is empty). We construct the set A by selecting one vertex from every Q_i, independently and uniformly at random. The size of A is then clearly precisely a and the size of $B := Y \setminus A$ is precisely b. Hence Property (i) is satisfied. For every $v \in V$, let E_v denote the *bad event* "v violates Property (ii) or Property (iii)", that is, $d_G(v, A) < \frac{a}{3m} d_G(v, Y)$ or $d_G(v, B) < \frac{b}{3m} d_G(v, Y)$.

In order to bound from above the probability of a bad event E_v we consider the random variables $d_G(v, A)$ and $d_G(v, B)$. Let $\Gamma_v := \{1 \leq i \leq a : N_G(v) \cap Q_i \neq \emptyset\}$; note that $|\Gamma_v| \leq d_G(v, Y)$. For every $i \in \Gamma_v$ let X_i be the indicator random variable for the event "$N_G(v, A) \cap Q_i \neq \emptyset$". Then $Pr[X_i = 1] = \frac{|N_G(v) \cap Q_i|}{|Q_i|}$ for every $i \in \Gamma_v$ and $d_G(v, A) = \sum_{i \in \Gamma_v} X_i$. It follows that $\mathbb{E}(d_G(v, A)) = \sum_{i \in \Gamma_v} \frac{|N_G(v) \cap Q_i|}{|Q_i|}$. Hence

$$\mathbb{E}(d_G(v, A)) \geq \sum_{i \in \Gamma_v} \frac{|N_G(v) \cap Q_i|}{\lceil m/a \rceil} = \frac{d_G(v, Y)}{\lceil m/a \rceil} \geq \frac{a}{m + a} d_G(v, Y) \, ,$$

and similarly

$$\mathbb{E}(d_G(v, A)) \leq \sum_{i \in \Gamma_v} \frac{|N_G(v) \cap Q_i|}{\lfloor m/a \rfloor} = \frac{d_G(v, Y)}{\lfloor m/a \rfloor} \leq d_G(v, Y)/2 \, ,$$

where the last inequality follows since $a \leq m/2$.

Clearly $d_G(v, B) = d_G(v, Y) - d_G(v, A)$ and thus $\mathbb{E}(d_G(v, B)) = d_G(v, Y) - \mathbb{E}(d_G(v, A))$. Hence, applying Lemma 2.6 with $n = |\Gamma_v|$ and with $a_i = 0$ and $b_i = 1$ for every $i \in \Gamma_v$ we obtain

$$\begin{aligned}
Pr(E_v) \quad &\leq \quad Pr\left(d_G(v, A) < \frac{a}{3m} d_G(v, Y)\right) + Pr\left(d_G(v, B) < \frac{b}{3m} d_G(v, Y)\right) \\
&\leq \quad Pr\left(d_G(v, A) < \frac{a}{3m} d_G(v, Y)\right) + Pr\left(d_G(v, A) > \frac{2m + a}{3m} d_G(v, Y)\right) \\
&\leq \quad Pr\left(d_G(v, A) - \mathbb{E}(d_G(v, A)) < \left(\frac{a}{3m} - \frac{a}{m + a}\right) d_G(v, Y)\right) \\
&+ \quad Pr\left(d_G(v, A) - \mathbb{E}(d_G(v, A)) > \left(\frac{2}{3} + \frac{a}{3m} - \frac{1}{2}\right) d_G(v, Y)\right) \\
&\leq \quad Pr\left(|d_G(v, A) - \mathbb{E}(d_G(v, A))| > \frac{a}{3m} d_G(v, Y)\right) \\
&\leq \quad 2 \exp\left\{-\frac{2a^2 d_G(v, Y)^2}{9m^2 |\Gamma_v|}\right\} \\
&\leq \quad 2 \exp\left\{-\frac{a^2 \delta}{5m^2}\right\} ,
\end{aligned}$$

where the fourth inequality follows since $a \leq m/2$.

Next, we bound from above the maximum degree of the dependency graph of bad events. Let $v, w \in V$ be two distinct vertices. It is clear that if $\Gamma_v \cap \Gamma_w = \emptyset$, then E_v and E_w are independent events. As previously noted $|\Gamma_v| \leq d_G(v, Y) \leq \Delta$. Since $|Q_i| \leq \lceil \frac{m}{a} \rceil$ for every $i \in \Gamma_v$, it follows that E_v is independent of all but at most $\sum_{i \in \Gamma_v} \sum_{u \in Q_i} (d_G(u) - 1) \leq \Delta^2 \lceil \frac{m}{a} \rceil - 1$ of the events $E_u : u \in V \setminus \{v\}$.

The existence of the required partition $Y = A \cup B$ thus follows by Lemma 2.5. $\square$

2.3 Embedding almost spanning trees in random and pseudo-random graphs

As noted in the previous section, in order to embed a spanning tree T in $G(n,p)$, we will first want to embed a large subtree of T in a certain subgraph of $G(n,p)$. Moreover, we will want this embedding to cover certain "problematic" vertices of $G(n,p)$. We prove the following embedding statement which might be of independent interest.

Proposition 2.7 *Let β_1 and $0 < \beta_2 \leq 1$ be real numbers such that $\beta_2 > 2\beta_1$. Let $0 < a \leq b < 1$ be real numbers and let d be a positive integer. Let $T = (V,E)$ be a rooted tree with maximum degree d, where $an \leq |V| \leq bn$. Let $r' \in V$ be the root of T, let $W \subseteq [n]$ be a given subset of size $|W| \leq n^{1-\beta_2}$ and let $r \in [n] \setminus W$. Then a.a.s. there exists an embedding $\phi : V \to [n]$ of T in the random graph $G(n,p)$ with $p = \frac{(1-\beta_1)\log n}{n}$ such that $\phi(r') = r$ and $W \subseteq \phi(V)$.*

Before proving Proposition 2.7 we introduce some terminology that will be used in the course of our proof. Let G be a graph, let T be a tree, and let $S \subseteq V(T)$ be an arbitrary set. An S-*partial embedding* of T in G is an injective mapping $f : S \to V(G)$, such that $f(x)f(y) \in E(G)$ whenever $\{x,y\} \subseteq S$ and $xy \in E(T)$. For every vertex $v \in f(S)$ we denote $v' = f^{-1}(v)$. If $S = V(T)$, we call an S-partial embedding of T in G simply an embedding of T in G. We say that the vertices of S are *embedded*, whereas the vertices of $V(T) \setminus S$ are called *new*. An embedded vertex is called *closed* if all its neighbors in T are embedded as well. An embedded vertex that is not closed, is called *open*. The vertices of $f(S)$ are called *taken*, whereas the vertices of $V(G) \setminus f(S)$ are called *available*. With some abuse of this terminology, for a closed (respectively open) vertex $u \in S$, we will sometimes refer to $f(u)$ as being closed (respectively open) as well.

Proof of Proposition 2.7 The main idea of the proof is to embed T vertex by vertex into $G(n,p)$, giving priority to the vertices of W. We will split $G(n,p)$ into two parts G_1 and G_2, where G_2 will be used for the most part of the embedding and G_1 will be used in case of emergency.

Since the existence of the required mapping ϕ is a monotone increasing property, we can assume without loss of generality that $\beta_1 > 0$ (while $\beta_2 > 2\beta_1$ still holds).

We expose G in two rounds, that is, we split $G = G_1 \cup G_2$, where $G_1 \sim G(n,p_1)$ with $p_1 = (1-2\beta_1)\log n/n$ and $G_2 \sim G(n,p_2)$ with $(1-p_2)(1-p_1) = 1-p$. It follows that $p_2 \geq \beta_1 \log n/n$.

The basic idea of the proof is to try and embed T in G_2 and use the edges of G_1 only in emergencies. We will always try first to embed some vertex of T into some vertex of W and only if we are unable to do so embed it into some other vertex.

We begin by exposing the edges of G_1. We do so in several steps, where in each step we define some vertex sets which will be used in different parts of the embedding process. We first expose the edges of G_1 with one endpoint in W and the other in $[n] \setminus (W \cup \{r\})$. For every $w \in W$ let $Z_w = N_{G_1}(w, [n] \setminus (W \cup \{r\}))$. Let $0 < \gamma < 1/4$ be a real number satisfying

$2\gamma\log(3/(2\gamma)) = (\beta_2 - 2\beta_1)/3$. It follows by Properties **(Q1)** and **(Q2)** from Lemma 2.2 that a.a.s. $2\gamma\log n \le |Z_w| \le 10\log n$ holds for every $w \in W$. For every $w \in W$ let $X_w = \{u \in Z_w : N_{G_1}(u) \cap (W \setminus \{w\}) = \emptyset\}$. It follows by Property **(Q3)** from Lemma 2.2 that a.a.s. $|X_w| \ge 2\gamma\log n - 2/\beta_2 \ge \gamma\log n$. Denote $X = \bigcup_{w\in W} X_w$, then a.a.s. $|X| \le 10\log n \cdot n^{1-\beta_2}$. Once we embed some $w' \in V$ into $w \in W$, we will use X_w to embed the neighbors of w' in T.

Next, we expose all edges of $H_1 := G_1[[n] \setminus (W \cup X \cup \{r\})]$. Let H denote the graph obtained from H_1 by repeatedly deleting all vertices of degree less than $(1-2\beta_1)(1-b)\log n/2$; clearly $\delta(H) \ge (1-2\beta_1)(1-b)\log n/2$. We claim that a.a.s. $|V(H_1)\setminus V(H)| \le \frac{1-b}{4}\cdot n$. Indeed, otherwise there exists a set $B \subseteq V(H_1)\setminus V(H)$ of size $|B| = \frac{1-b}{4}\cdot n$ such that there are at most $\frac{1-b}{4}\cdot n\cdot(1-2\beta_1)(1-b)\log n/2$ edges of G_1 with one endpoint in B and the other in $V(H_1)\setminus B$. The expected number of such edges in G_1 is $|B||V(H_1)\setminus B|p_1 = \frac{1-b}{4}(1-2\beta_1)\left(1-\frac{1-b}{4}-o(1)\right)n\log n$. It follows by standard bounds on the tail of the binomial distribution that the probability that such a set B exists is at most

$$\binom{|V(H_1)|}{|B|}\exp\left\{-c\cdot\frac{1-b}{4}(1-2\beta_1)\left(1-\frac{1-b}{4}-o(1)\right)n\log n\right\} = o(1)\,,$$

where $c > 0$ is an appropriately chosen absolute constant.

Most of T will be embedded into H.

Next, we expose the edges of G_1 with one endpoint in X and the other in $V(H)$. It follows by Property **(Q2)** from Lemma 2.2 that a.a.s. there exists a real number $\gamma' > 0$ such that $d_{G_1}(u, V(H)) \ge \gamma'\log n$ holds for every $u \in X$ (indeed, for $N := |V(H)\cup X| \ge 3n/4$ there exists a constant $\beta_3 > 2\beta_1$ such that $|X| \le N^{1-\beta_3}$; hence we can apply Lemma 2.2 with N, X and β_3 instead of n, W and β_2).

We conclude that a.a.s. there exist positive real numbers γ and γ', a family $\{X_w : w \in W\}$ of pairwise disjoint subsets of $[n]\setminus(W\cup\{r\})$, and a subgraph $H \subseteq G_1[[n]\setminus(W\cup X\cup\{r\})]$ which satisfy the following properties:

(i) $wu \in E(G_1)$ for every $w \in W$ and every $u \in X_w$.

(ii) $\gamma\log n \le |X_w| \le 10\log n$ for every $w \in W$.

(iii) $|V(H)| \ge \left(1-\frac{1-b}{4}-o(1)\right)n$.

(iv) $\delta(H) \ge (1-2\beta_1)(1-b)\log n/2$.

(v) $d_{G_1}(u, V(H)) \ge \gamma'\log n$ for every $u \in X$.

Let $M \subseteq V(H)$ be a set of size $\frac{1-b}{4}\cdot n$ and let $\gamma'' = \gamma''(\gamma', b, \beta_1) > 0$ be a real number such that $d_{G_1}(v, M) \ge \gamma''\log n$ holds for every $v \in V(H)\cup X$. Such M and γ'' exist by Lemma 2.4 (with $V = V(H)\cup X$ and $Y = V(H)$). We will usually avoid using the set M during the embedding process; it will only be used in case of emergency (see Figure 2 depicting the notions described thus far).

Finally, we expose the edges of G_1 with one endpoint in $\{r\}$ and the other in M. Standard bounds on the tail of the binomial distribution show that a.a.s. $d_{G_1}(r, M) \ge \gamma'''\log n$ holds for some positive real number $\gamma''' = \gamma'''(\beta_1, b)$. Let $\zeta = \min\{(1-2\beta_1)(1-b)/2, \gamma, \gamma', \gamma'', \gamma'''\}$.

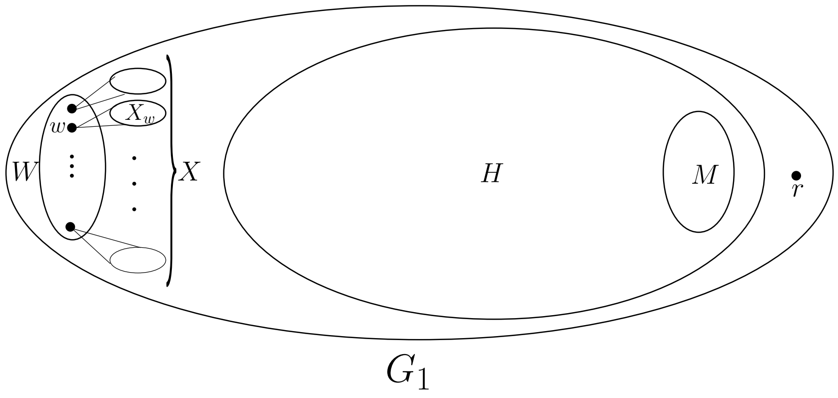

Figure 2: Embedding a tree rooted at r and covering W.

Next, we try to embed T in $G_2[V(H) \cup W \cup X \cup \{r\}]$. We embed T vertex by vertex starting with r'. At each point during the embedding process the part of T which was already embedded will be a subtree T' of T rooted at r. We will give priority to including the vertices of W in the embedding: whenever there will be a possibility to embed a path of length two from an open vertex of the current embedding to a not yet embedded vertex $w \in W$, we will do so. We will then use the available neighbors of w in X_w to embed the children of the preimage of w in T; this will close w. The edges of G_2 are exposed during the embedding process, as will be described shortly. Upon encountering certain difficulties, we will make use of the edges of G_1. At any point during the embedding of T we denote the set of embedded vertices by S and the current S-partial embedding by ϕ. Moreover, we denote $I := W \setminus \phi(S)$. Initially $S = \{r'\}$ and $\phi(r') = r$.

As long as $V \setminus S \neq \emptyset$ we proceed as follows. Let $v' \in S$ be an arbitrary open vertex and let $v = \phi(v')$. Assume first that there exist vertices $x', y' \in V \setminus S$ such that $v'x', x'y' \in E$ (that is, v' has a non-leaf neighbor which has not been embedded yet). Expose all edges of G_2 with one endpoint in $\{v\}$ and the other in X. Assume further that there exists some $w \in I$ and some $u \in X_w$ such that $vu \in E(G_2)$. Choose such w and u arbitrarily, add x' and y' to S, and update ϕ by setting $\phi(x') = u$ and $\phi(y') = w$. If y' is open, embed all of its children in T into arbitrary vertices of $X_w \setminus \{u\}$ (using distinct vertices to embed distinct children of y'). This is possible since $|X_w \setminus \{u\}| > d \geq d_T(y')$ holds by Property (ii) above (we will not use the vertices of X_w except when attempting to embed and subsequently close w). Update S, I and ϕ accordingly; note that w is now closed. Assume then that this is not the case, that is, every non-leaf neighbor of v' has already been embedded or there are no edges in G_2 with one endpoint in $\{v\}$ and the other in $\bigcup_{w \in I} X_w$. Expose all edges of G_2 with one endpoint in $\{v\}$ and the other in $V(H) \setminus (M \cup \phi(S))$. Let $\{v'_1, \ldots, v'_t\} = N_T(v') \setminus S$ be the new neighbors of v' in T. If $d_{G_2}(v, V(H) \setminus (M \cup \phi(S))) \geq t$, then, for every $1 \leq i \leq t$, we set $\phi(v'_i) = v_i$ for an arbitrary vertex $v_i \in N_{G_2}(v, V(H) \setminus (M \cup \phi(S)))$ (using distinct vertices to embed distinct v'_i) and update S accordingly. Note that v is now closed. Otherwise we declare an *emergency*. During this emergency we try to embed $v'_1, \ldots, v'_t$ into $N_{G_1}(v, M \setminus \phi(S))$. If this attempt is successful, that is, if $d_{G_1}(v, M \setminus \phi(S)) \geq t$, then v is closed and we update S and ϕ accordingly. Otherwise we declare a *failure*.

In order to complete the proof of the proposition it suffices to prove that a.a.s. there are no failures and that a.a.s. $W \subseteq \phi(V)$. Starting with the former assume there is a failure at v. It follows that $d_{G_1}(v, M \setminus \phi(S)) < t \leq d$. Denote $k := d_{G_1}(v, M)$; recall that $k \geq \zeta \log n$ holds by the choice of M and since $v \notin W$ as it is open. Denote $A_v := N_{G_1}(v, M)$ and $B_v := N_{G_1}(A_v, V(H) \cup X \cup \{r\})$. It follows by Property **(Q1)** of Lemma 2.2 that a.a.s. $|B_v| \leq 100 \log^2 n$. Since we use the vertices of M only during emergencies, it follows that we already treated at least $(k-d)/d$ emergencies at vertices of B_v. Let $w \in B_v$ be an arbitrary vertex at which we have treated an emergency. Observe that when the state of emergency was declared at w there were $m \geq |V(H)| - |\phi(S)| - |M| - |W| \geq \frac{1-b}{3} \cdot n$ vertices available to embed the new neighbors of the preimage of w. Since the degree in G_2 of w into this set was distributed as $Bin(m, p_2)$, it follows by standard bounds on the tail of the binomial distribution that the probability of declaring an emergency at w is at most

$$Pr[Bin(m, p_2) < d] \leq e^{-\frac{1}{3} \cdot \frac{1-b}{3} \beta_1 \log n} \leq n^{-c},$$

where $c = c(b, \beta_1) > 0$ is an appropriately chosen real number. Note that this bound is independent of the occurrence of any other emergency.

Hence, the probability that there are at least $(k-d)/d$ emergencies at vertices of B_v is at most

$$\binom{|B_v|}{\frac{k-d}{d}} n^{-c \cdot \frac{k-d}{d}} < \left(100 \log^2 n \cdot n^{-c}\right)^{c' \log n} = o(1/n),$$

where $c' = c'(c, d, \zeta) > 0$ is an appropriate constant.

Applying the union bound over all vertices of $\phi(V)$ proves that a.a.s. there are no failures.

Next we prove that a.a.s. $W \subseteq \phi(V)$. Fix some $w \in W$ and assume that $w \notin \phi(V)$. It follows that $w \in I$ holds throughout the embedding process. Consider an arbitrary point during the embedding process. At this given point let $v' \in S$ be an open vertex and let $x', y' \in V \setminus S$ be vertices for which $v'x', x'y' \in E$. Since $\phi(y') \neq w$, it follows that either $d_{G_2}(\phi(v'), X_w) = 0$ or $d_{G_2}(\phi(v'), X_w) > 0$ but also $d_{G_2}(\phi(v'), X_z) > 0$ for some $z \in I \setminus \{w\}$. The probability of this happening is at most

$$(1-p_2)^{|X_w|} + p_2|X_w| \cdot \sum_{z \in I \setminus \{w\}} p_2|X_z| \leq e^{-p_2|X_w|} + \Theta(n^{-1-\beta_2} \log^4 n) \leq e^{-K \log^2 n/n},$$

where $K = K(\gamma, \beta_1) > 0$ is an appropriate constant.

Since $\Delta(T) \leq d$ and $|V| \geq an$, it follows that there exists some constant $c'' = c''(d, a) > 0$ such that there are at least $c''n$ vertices $v' \in V$ which are neither leaves nor parents of leaves. For every such vertex v' there are vertices $x', y' \in V$ such that $v'x', x'y' \in E$. In each embedding step we embed at most $d+1$ vertices. Hence there are at least $\frac{c''n}{d+1}$ attempts to embed vertices of V into w, where all attempts are mutually independent. It follows that

$$Pr[w \notin \phi(V)] \leq \left(e^{-K \log^2 n/n}\right)^{c''n/(d+1)} = o(1/n).$$

Applying the union bound over all vertices of W proves that a.a.s. $W \subseteq \phi(V)$. This concludes the proof of the proposition. □

Our proof of Proposition 2.7 does not apply to pseudo-random graphs. For such graphs we prove the following embedding criterion.

Proposition 2.8 *Let* $\gamma, \varepsilon > 0$ *be real numbers. Let* $n_0 = n_0(\gamma, \varepsilon)$ *be a sufficiently large positive integer, let* $n \geq n_0$ *be an integer and let* $d = \sqrt{\log n}$. *Let* T *be a tree on* $n' \leq (1-\varepsilon)n$ *vertices with maximum degree at most* d. *Let* $H = (V, E)$ *be a graph on* n *vertices such that* $\delta(H) \geq \gamma \log n$. *If, moreover,* H *satisfies Properties* **(P3)**, **(P4)** *and* **(P5)** *from Lemma 2.1, then* H *contains a copy of* T.

The proof of Proposition 2.8 is via a simple application of the following corollary of a theorem of Haxell [12].

Theorem 2.9 *[4, Theorem 3] Let* d, m *and* M *be positive integers, and let* $0 \leq \ell \leq 2dm$. *Assume that* H *is a non-empty graph satisfying the following two conditions.*

(*i*) *For every* $X \subseteq V(H)$ *with* $0 < |X| \leq m$, $|N_H(X)| \geq d|X| + 1$.

(*ii*) *For every* $X \subseteq V(H)$ *with* $m < |X| \leq 2m$, $|N_H(X)| \geq d|X| + M$.

Then H *contains every tree* T *with* $M + \ell$ *vertices and maximum degree at most* d, *provided that* T *has at least* ℓ *leaves.*

Proof of Proposition 2.8 Let ℓ denote the number of leaves of T and let $M = n' - \ell$. Let $m = \max\left\{\frac{n(\log\log n)^{3/2}}{\log n}, \frac{\ell}{2d}\right\}$; note that $0 \leq \ell \leq 2dm$. Hence, by Theorem 2.9, it suffices to prove that H satisfies Properties (i) and (ii) above. First, assume for the sake of contradiction that there exists a set $X \subseteq V$ of size $0 < |X| \leq \min\left\{m, \frac{n(\log\log n)^2}{\log n}\right\}$ such that $|N_H(X)| \leq d|X|$. Let $Y \supseteq N_H(X) \setminus X$ be an arbitrary subset of $V \setminus X$ of size $|Y| = d|X|$. Since H satisfies Property **(P4)** from Lemma 2.1, it follows that $e_H(X, Y) \leq \frac{|X| \log n}{\log\log n}$. On the other hand, since H satisfies Property **(P3)** from Lemma 2.1, $\delta(H) \geq \gamma \log n$ and $Y \supseteq N_H(X)$ is disjoint from X, it follows that $e_H(X, Y) \geq \gamma|X| \log n - \frac{2|X| \log n}{\log\log n} > \frac{|X| \log n}{\log\log n}$. This is clearly a contradiction. Next, assume for the sake of contradiction that there exists a set $X \subseteq V$ of size $\frac{n(\log\log n)^2}{\log n} < |X| \leq m$ such that $|N_H(X)| \leq d|X|$; note that $m = \frac{\ell}{2d}$ holds in this case. Let $Y = V \setminus (X \cup N_H(X))$. Note that $|Y| \geq n - (d+1)m \geq n - \ell \geq \varepsilon n \geq \frac{n(\log\log n)^{3/2}}{\log n}$ and $e_H(X, Y) = 0$. Since $|X| \geq \frac{n(\log\log n)^{3/2}}{\log n}$ and H satisfies Property **(P5)** from Lemma 2.1, it follows that $e_H(X, Y) > 0$. This is clearly a contradiction. Finally, assume for the sake of contradiction that there exists a set $X \subseteq V$ of size $m < |X| \leq 2m$ such that $|N_H(X)| < d|X| + M$. Let $Y = V \setminus (X \cup N_H(X))$. Note that $|Y| \geq n - 2m(d+1) - M \geq \varepsilon n + \ell - \frac{d+1}{d}\ell \geq \varepsilon n/2 \geq \frac{n(\log\log n)^{3/2}}{\log n}$ and $e_H(X, Y) = 0$. Since $|X| \geq m \geq \frac{n(\log\log n)^{3/2}}{\log n}$, this contradicts Property **(P5)** from Lemma 2.1. $\square$

Finally, we cite a known criterion for embedding not too large trees into sparse random graphs.

Theorem 2.10 *[2, Theorem 1.1] Let* $d \geq 2$, *let* $0 < \varepsilon < 1/2$ *and let*

$$c \geq \varepsilon^{-1} 10^6 d^3 \log d \log^2(2\varepsilon^{-1}).$$

Then a.a.s. the random graph $G(n, c/n)$ contains every tree of maximum degree at most d on $(1-\varepsilon)n$ vertices.

2.4 Embedding star forests in bipartite graphs

Assume we have embedded all vertices of some tree T, except for a set L of some of its leaves, into a graph G. Let $U \subseteq V(G)$ denote the image of $V(T) \setminus L$ under this embedding. In order to embed the vertices of L as well, we will need to connect certain vertices of U with vertices of $V(G) \setminus U$. The following lemma asserts that this is indeed possible, given that T and G satisfy certain conditions.

Lemma 2.11 *Let $F = (A_F \cup B_F, E_F)$ be a bipartite graph and let d be a positive integer such that $1 \leq d_F(u, B_F) \leq d$ holds for every $u \in A_F$ and $d_F(u, A_F) = 1$ holds for every $u \in B_F$. Let $G = (A \cup B, E)$ be a bipartite graph and assume that there exist positive integers δ_A, δ_B and s such that the following properties hold:*

(i) $|A| = |A_F|$ *and* $|B| = |B_F|$.

(ii) $d_G(u, B) \geq \delta_A$ *for every* $u \in A$ *and* $d_G(u, A) \geq \delta_B$ *for every* $u \in B$.

(iii) $e_G(X, Y) < \min\{\delta_A|X|, \delta_B|Y|\}$ *for every* $X \subseteq A$ *of size* $|X| \leq s$ *and every* $Y \subseteq B$ *of size* $|Y| \leq d|X|$.

(iv) $e_G(X, Y) > 0$ *for every* $X \subseteq A$ *and* $Y \subseteq B$ *of sizes* $|X|, |Y| > s$.

Then, for any bijection $f : A_F \to A$, there exists an embedding of F in G which maps every $u \in A_F$ to $f(u)$.

In the proof of Lemma 2.11 we will make use of the following polygamous version of Hall's Theorem (see e.g. [7]).

Proposition 2.12 *Let $G = (A \cup B, E)$ be a bipartite graph, where $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_r\}$, and let $d_1, \ldots, d_k$ be positive integers. Then, there exists a spanning subgraph H of G such that $deg_H(a_i) = d_i$ for every $1 \leq i \leq k$ and $deg_H(b_j) = 1$ for every $1 \leq j \leq r$ if and only if $|N_G(S)| \geq \sum_{i: a_i \in S} d_i$ for every $S \subseteq A$.*

Proof of Lemma 2.11 Assume for the sake of contradiction that there exists a set $X \subseteq A$ which does not satisfy Hall's condition, that is, $|N_G(X)| < \sum_{u \in X} d_F(f^{-1}(u))$. Note that $\sum_{u \in X} d_F(f^{-1}(u)) \leq d|X|$ holds by our assumption that $d_F(u, B_F) \leq d$ for every $u \in A_F$. First, assume that $|X| \leq s$. It follows that $\delta_A|X| > e_G(X, N_G(X)) = \sum_{u \in X} d_G(u, B) \geq \delta_A|X|$, where the first inequality holds by Property (*iii*) and the last by Property (*ii*). This is clearly a contradiction. Assume then that $|X| > s$. Let $Y \subseteq B \setminus N_G(X)$ be an arbitrary set of size $|Y| = \sum_{u \in A \setminus X} d_F(f^{-1}(u))$ (such a set exists since $\sum_{u \in A} d_F(f^{-1}(u)) = \sum_{v \in B_F} d_F(v) = |B_F| = |B|$ and $|N_G(X)| < \sum_{u \in X} d_F(f^{-1}(u))$). Since, by assumption $1 \leq d_F(u, B_F) \leq d$ holds for every $u \in A_F$, it follows that $|A \setminus X| \leq |Y| \leq d|A \setminus X|$. If $|X| < |A| - s$, then $|Y| \geq |A \setminus X| > s$ which contradicts Property (*iv*) above since $|X| > s$ but $e_G(X, Y) = 0$ (as

$Y \cap N_G(X) = \emptyset$). Assume then that $|X| \geq |A| - s$. Since $N_G(Y) \subseteq A \setminus X$, it follows that $e_G(A \setminus X, Y) = e_G(N_G(Y), Y) = \sum_{u \in Y} d_G(u, A) \geq \delta_B |Y|$. This contradicts Property *(iii)* above since $|A \setminus X| \leq s$ and $|Y| \leq d|A \setminus X|$. □

2.5 Hamilton connectivity

In this subsection we prove a sufficient condition for a graph to be Hamilton connected.

Lemma 2.13 *Let $\beta, c > 0$ be real numbers, let $n_0 = n_0(\beta, c)$ be a sufficiently large positive integer and let $n \geq n_0$. Let G be a graph on n vertices which satisfies Properties* **(P3)**, **(P4)**, *and* **(P5)**. *Let $H = (V, E)$ be an induced subgraph of G on cn vertices. If $\delta(H) \geq \beta \log n$, then H is Hamilton connected.*

In our proof of Lemma 2.13 we will make use of the following sufficient condition for a graph to be Hamilton connected.

Theorem 2.14 *[13, Theorem 1.2] There exists an integer n_0 such that for every integers $n \geq n_0$ and $12 \leq d \leq e^{\sqrt[3]{\log n}}$ the following holds. If $G = (V, E)$ is a graph on n vertices which satisfies the following two properties:*

(H1) *For every $S \subseteq V$, if $|S| \leq \frac{n \log\log n \log d}{d \log n \log\log\log n}$, then $|N(S)| \geq d|S|$;*

(H2) *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ with $|A|, |B| \geq \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$;*

then G is Hamilton connected.

Proof of Lemma 2.13 Note that H satisfies Properties **(P3)**, **(P4)**, and **(P5)** from Lemma 2.1. It follows from Theorem 2.14 that it suffices to prove that H satisfies Properties **(H1)** and **(H2)** for some $12 \leq d \leq e^{\sqrt[3]{\log n}}$. Fix $d = \sqrt{\log n}$. Starting with **(H1)**, we claim that $|N_H(A)| \geq |A|\sqrt{\log n}$ holds for every $A \subseteq V$ of size $a \leq \frac{n}{\log n}$. Indeed, assume for the sake of contradiction that there exists a set $A \subseteq V$ of size $a \leq \frac{n}{\log n}$ such that $|N_H(A)| < a\sqrt{\log n}$. Since H is an induced subgraph of G, $\delta(H) \geq \beta \log n$, and H satisfies Property **(P3)** from Lemma 2.1, it follows that $e_H(A, V \setminus A) = e_G(A, V \setminus A) \geq a\beta \log n - 2\frac{a \log n}{\log\log n} \geq a\beta \log n/2$. Let $B \subseteq V \setminus A$ be an arbitrary set of size $a\sqrt{\log n}$ which contains $N_H(A) \setminus A$. It follows from the discussion above that $e_H(A, B) \geq a\beta \log n/2 > \frac{a \log n}{\log\log n}$. This contradicts the fact that H satisfies Property **(P4)** from Lemma 2.1. Since $\frac{n}{\log n} \geq \frac{cn \log\log(cn) \log(\sqrt{\log n})}{\sqrt{\log n} \log(cn) \log\log\log(cn)}$, it follows that H satisfies Property **(H1)**.

Next, we claim that H satisfies Property **(H2)**. Indeed, since H satisfies Property **(P5)** from Lemma 2.1 and since

$$\frac{n(\log\log n)^{3/2}}{\log n} \leq \frac{cn \log\log(cn) \log(\sqrt{\log n})}{4130 \log(cn) \log\log\log(cn)},$$

it follows that H satisfies Property **(H2)**. This concludes the proof of the lemma. □

3 Proof of the main results

Proof of Theorem 1.2 The main idea of the proof is as follows. Using Lemma 2.3 we split T into two trees sharing one vertex – a small tree T_1 with many leaves and the remaining large tree T_2. We remove all leaves of T_1 to obtain a tree T_1'. We split $G(n,p)$ into three random graphs G_1, G_2 and G_3. We embed T_1' into G_1 using Theorem 2.10. We would like to embed T_2 and then the leaves of T_1 into $G_2 \cup G_3$; for the latter we use Lemma 2.11. In order to make sure that the conditions of this lemma are satisfied, we use G_2 to identify a set W of vertices which are problematic in this respect and, when embedding T_2, we use Proposition 2.7 to make sure we cover W. Figure 3 depicts the different stages of the embedding of T in $G(n,p)$ as well as the notions introduced during the proof.

We expose G in three rounds, that is, we split $G = G_1 \cup G_2 \cup G_3$, where $G_1, G_2 \sim G(n, \frac{\varepsilon \log n}{4n})$ and $G_3 \sim G(n, \frac{(1+\varepsilon/2)\log n}{n})$. Note that we thus indeed have $G \sim G(n,p)$ with $p \leq \frac{(1+\varepsilon)\log n}{n}$.

Let $r' \in V(T)$ be a vertex at which T can be split into two rooted subtrees T_1 and T_2 (both rooted at r') such that the subtree T_1 has γn vertices and $\gamma_1 n$ leaves for some $\beta \leq \gamma_1 < \gamma \leq \varepsilon/14$, where $\beta = \beta(\varepsilon/14, \alpha) > 0$ is the real number whose existence is guaranteed by Lemma 2.3. Let L_1 be the set of leaves of T_1 and let $T_1' = T_1 \setminus L_1$.

We first expose the edges of G_1. Let $f_1 : V(T_1') \to [n]$ be an embedding of T_1' in G_1. Such an embedding exists a.a.s. by Theorem 2.10 since $|V(T_1')| \leq n/2$. Let $U = f_1(V(T_1'))$ denote the image of the embedding f_1, let $L_0 = f_1(N_{T_1}(L_1))$ denote the images of the parents of the leaves of T_1 under this embedding and let $r = f_1(r')$ denote the image of the root of T_1. Note that $\gamma_1 n/d \leq |L_0| \leq |L_1| = \gamma_1 n$.

Now we expose the edges of G_2 with one endpoint in L_0 and the other in $[n] \setminus U$. Let $W = \{w \in [n] \setminus U : d_{G_2}(w, L_0) < \frac{\varepsilon \log n}{8n}|L_0|\}$. We claim that a.a.s. W is a "small" set. Indeed, for every $w \in [n] \setminus U$, let A_w denote the event "$w \in W$". For any $w \in [n] \setminus U$ we have $d_{G_2}(w, L_0) \sim Bin(|L_0|, \varepsilon \log n/(4n))$. Hence, it follows by standard bounds on the tail of the binomial distribution (see e.g. [14]) that

$$
\begin{aligned}
Pr[A_w] &= Pr\left[d_{G_2}(w, L_0) < \frac{\varepsilon \log n}{8n}|L_0|\right] = Pr\left[d_{G_2}(w, L_0) < \mathbb{E}(d_{G_2}(w, L_0))/2\right] \\
&\leq \exp\left\{-\frac{1}{8}|L_0|\frac{\varepsilon \log n}{4n}\right\} \leq n^{-\frac{\gamma_1 \varepsilon}{32d}},
\end{aligned}
$$

where the last inequality follows by the aforementioned lower bound on $|L_0|$. By Markov's inequality we conclude that a.a.s. $|W| \leq n^{1-\frac{\gamma_1 \varepsilon}{33d}}$.

Note that, by the definition of W, we have $d_{G_2}(v, L_0) \geq \frac{\varepsilon \log n}{8n}|L_0|$ for every $v \in [n] \setminus (U \cup W)$.

Next we expose the edges of G_3; we do so in two stages. In the first stage we expose only those edges which have one endpoint in L_0 and the other in $[n] \setminus U$. It follows from Property **(P2)** of Lemma 2.1 that there exists a real number $\mu = \mu(\varepsilon, \gamma) > 0$ such that a.a.s. $d_{G_3}(v, [n] \setminus (U \cup W)) \geq \mu \log n$ holds for every $v \in L_0$. Moreover, it follows from Property **(P5)** of Lemma 2.1 that a.a.s. $e_{G_3}(A, B) > 0$ holds for every $A \subseteq L_0$ and every $B \subseteq [n] \setminus (U \cup W)$ of size $|A| = |B| = \frac{n(\log\log n)^{3/2}}{\log n}$.

Let $X \subseteq [n] \setminus (U \cup W)$ be a set of size $\gamma_1 n/2$ such that $d_{G_2 \cup G_3}(u, X) \geq \zeta \log n$ holds for every vertex $u \in L_0$, where $\zeta = \zeta(\mu, \gamma_1) > 0$ is an appropriately chosen real number. Such a set X

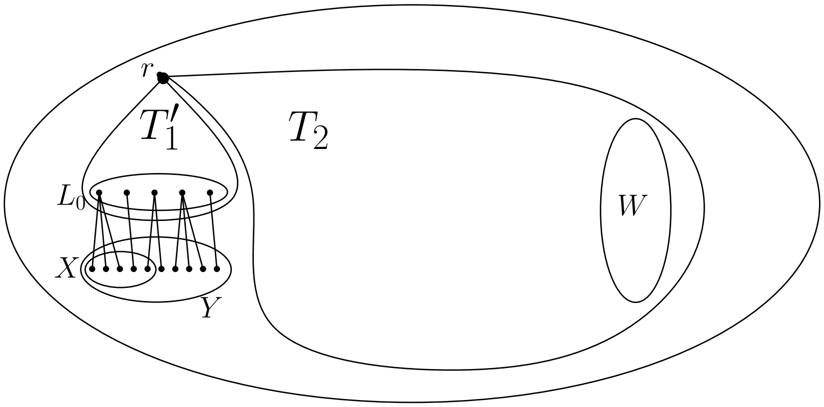

Figure 3: Embedding a tree with linearly many leaves.

exists by Lemma 2.4. Note that $d_{G_2 \cup G_3}(x, L_0) \geq \frac{\varepsilon \log n}{8n}|L_0| \geq \frac{\gamma_1 \varepsilon}{8d} \log n$ holds for every vertex $x \in X$ since $X \cap W = \emptyset$ by construction.

Now comes the second stage of exposure of the edges of G_3, where we expose all edges of G_3 which were not exposed in the first stage. Let $f_2 : V(T_2) \to ([n] \setminus (U \cup X)) \cup \{r\}$ be an embedding of T_2 in $G_3[([n] \setminus (U \cup X)) \cup \{r\}]$ such that $f_2(r') = r$ and $W \subseteq f_2(V(T_2))$. Such an embedding exists a.a.s. by Proposition 2.7 since $|V(T_2)| \leq (1-\gamma)n + 1 \leq (1-\gamma_1/3)(1-\gamma+\gamma_1/2)n \leq (1-\gamma_1/3)|[n] \setminus (U \cup X)|$ (this gives $b < 1$ in Proposition 2.7) and since $|[n] \setminus (U \cup X)| \geq (1-\varepsilon/9)n$ (this gives $\beta_1 < 0$ in Proposition 2.7 which ensures that $\beta_2 > 2\beta_1$ holds).

Let $\nu = \min\{\frac{\gamma_1 \varepsilon}{8d}, \zeta\}$ and let $Y = [n] \setminus (f_1(V(T_1')) \cup f_2(V(T_2)))$. At this point of the embedding process all that is left to embed are the leaves of T_1, to be embedded bijectively into Y. Note that $Y \supseteq X$ and that $Y \cap W = \emptyset$ (the sets W, X and Y, their interrelations and their use in the proof are depicted in Figure 3). It follows from the former and from our choice of the set X that $d_{G_2 \cup G_3}(u, Y) \geq \nu \log n$ holds for every $u \in L_0$. It follows from the latter and from the definition of W that $d_{G_2 \cup G_3}(u, L_0) \geq \nu \log n$ holds for every $u \in Y$. Let $F = T_1[L_1 \cup N_{T_1}(L_1)]$ and let $H = (L_0 \cup Y, E_{G_2 \cup G_3}(L_0, Y))$. Since, by the aforementioned properties and by Properties **(P4)** and **(P5)** of Lemma 2.1, F and H satisfy the conditions of Lemma 2.11 (with $\delta_A = \delta_B = \nu \log n$, $d = \Delta(T)$ and $s = \frac{n(\log \log n)^2}{\log n}$), it follows that there exists an embedding f_3 of F in H such that $f_3(u) = f_1(u)$ for every $u \in N_{T_1}(L_1)$. It is clear that the mapping $\phi : V(T) \to [n]$ defined by

$$\phi(u) := \begin{cases} f_1(u) & \text{if} \quad u \in V(T_1') \\ f_2(u) & \text{if} \quad u \in V(T_2) \\ f_3(u) & \text{if} \quad u \in L_1 \end{cases}$$

is an embedding of T in G. This concludes the proof of the theorem. □

Proof of Theorem 1.4 The main idea of the proof is to delete a long bare path from T, embed the remainder into $G(n,p)$ using results on embedding non-spanning trees and then embed the missing path between the two embedded endpoints.

It follows by Lemma 2.1 that a.a.s. $G(n, (1+\varepsilon)\log n/n)$ satisfies Properties **(P1)**–**(P5)**. Hence, in order to prove that $G(n,(1+\varepsilon)\log n/n)$ is a.a.s. $\mathcal{L}$-universal, it suffices to prove the any graph which satisfies Properties **(P1)**–**(P5)** is $\mathcal{L}$-universal.

Let $G=(V,E)$ be a graph on n vertices which satisfies Properties **(P1)**–**(P5)**. Let T be a tree on n vertices with maximum degree at most d which admits a bare path of length αn. We will prove that G contains T as a subgraph.

Let $t=\alpha n+1$ and let $P=(v_1,v_2,\ldots,v_t)$ be a bare path of length αn in T. Let T_P be the tree obtained from T by contracting the path P to a single edge between v_1 and v_t. Let $V=V_1\cup V_2$ be a partition of V which satisfies the following properties:

(i) $|V_1|=\alpha n/2$ and $|V_2|=(1-\alpha/2)n$;

(ii) $d_G(v,V_1)\geq \mu\log n$ for every $v\in V$;

(iii) $d_G(v,V_2)\geq \mu\log n$ for every $v\in V$;

where $\mu=\mu(\alpha,\varepsilon)>0$ is an appropriately chosen real number. Such a partition exists by Lemma 2.4 using the upper bound on $\Delta(G)$ and the lower bound on $\delta(G)$ ensured by Properties **(P1)** and **(P2)** respectively.

Let $\phi: V(T_P)\to V_2$ be an embedding of T_P in $G[V_2]$. Such an embedding exists by Proposition 2.8 since $|V(T_P)|\leq(1-\alpha/2)|V_2|$ and since $G[V_2]$ satisfies Property *(iii)* above and Properties **(P3)**–**(P5)** from Lemma 2.1. Let $U_2:=\phi(V(T_P))$ and let $U_1:=(V\setminus U_2)\cup\{\phi(v_1),\phi(v_t)\}$. In order to complete the embedding of T in G, it suffices to prove that there is a Hamilton path in $G[U_1]$ whose endpoints are precisely $\phi(v_1)$ and $\phi(v_t)$. In order to do so it suffices to prove the stronger result asserting that $G[U_1]$ is Hamilton connected. This however readily follows from Lemma 2.13 since $G[U_1]$ satisfies Properties **(P3)**–**(P5)** from Lemma 2.1 and since $\delta(G[U_1])\geq\mu\log n$ holds by Property *(ii)* above. □

4 Concluding remarks

We have proven that a bounded degree tree T on n vertices is contained asymptotically almost surely in a random graph $G(n,(1+\varepsilon)\log n/n)$, where $\varepsilon>0$ is arbitrarily small but fixed, provided that T has linearly many leaves or alternatively in the case where T contains a bare path of linear length (in which case $G(n,(1+\varepsilon)\log n/n)$ a.a.s. contains all such trees simultaneously). These results are optimal as for $p(n)=(1-\varepsilon)\log n/n$ the random graph $G(n,p)$ is asymptotically almost surely disconnected and thus does not contain spanning trees at all.

Our proofs can be tightened rather easily to show that the embedding results we obtained hold already for $p(n)=(1+\varepsilon(n))\log n/n$, where $\varepsilon(n)$ is some concrete function tending to 0 with n. Since we are rather doubtful the bounds on the error term $\varepsilon(n)$ obtained in this fashion would be close to being optimal, we chose not to pursue this goal.

Our proof of Theorem 1.4 uses randomness in a rather limited way and thus applies to pseudo-random graphs as well. Namely, we in fact proved that any graph G on n vertices satisfying Properties **(P1)**–**(P5)** contains any given bounded degree tree which admits a bare path on

$\Theta(n)$ vertices. Hence the random graph $G(n, (1+\varepsilon)\log n/n)$ is a.a.s. *universal* for this class of trees. Note that Properties **(P1)**–**(P5)** were tailored for the random graphs in question; they could be somewhat weakened so as to include a wider class of pseudo-random graphs. Our proof of Theorem 1.2 relies on the multiple rounds of exposure in an essential way and thus cannot be applied in a pseudo-random setting. It would be interesting to prove universality-type results for the trees covered by Theorem 1.2.

It was proved in [6] that, if $p = c\log n/n$ for any constant $c > 2e^2$, then a.a.s. $G(n,p)$ contains almost every tree on n vertices (that is, if one draws a labeled tree T uniformly at random from the class of all labeled trees on n vertices and then draws a random graph $G \sim G(n, c\log n/n)$ for some $c > 2e^2$, then a.a.s. T is a subgraph of G. Note that the order of exposure is not important here, that is, we can first draw the graph G and only then the tree T; this leads to an almost universality type statement in the spirit of [11]). Since a typical tree on n vertices has $\Omega(n)$ leaves, it seems plausible that one could adapt the proof of Theorem 1.2 to strengthen the aforementioned result of [6] by replacing the constant c with $1+\varepsilon$ for an arbitrarily small $\varepsilon > 0$. Since the maximum degree of a random tree is a.a.s. $(1+o(1))\log n/\log\log n$ (see e.g. [17]), this is not immediate. However, it is indeed doable; we can prove the following result.

Theorem 4.1 *Let $\varepsilon > 0$ be a real number, let $G \sim G(n, (1+\varepsilon)\log n/n)$ and let T be drawn uniformly at random from the class of all labeled trees on n vertices. Then a.a.s. T is a subgraph of G.*

Theorem 4.1 is a corollary of the following result.

Theorem 4.2 *For every positive real numbers α, β and ε there exists an integer n_0 such that if $T = (V, E)$ is a tree on $n \geq n_0$ vertices which satisfies the following properties:*

(1) $\Delta(T) = o(\log n)$;

(2) *There exist a vertex $u \in V$ and subtrees T_1 and T_2 of T such that $V(T_1) \cup V(T_2) = V$, $V(T_1) \cap V(T_2) = \{u\}$ and*

  (2.1) $|V(T_1)| \leq \varepsilon n/14$ *and the number of vertices of T_1 which are parents of leaves is at least αn;*

  (2.2) *The number of vertices of T_2 which are neither leaves nor parents of leaves is at least βn;*

then a.a.s. T is a subgraph of $G(n, (1+\varepsilon)\log n/n)$.

Using known properties of random trees and the fact that we did not assume any restrictions on the maximum degree of the tree in Lemma 2.3 one can see that a random tree satisfies a.a.s. all the properties required in Theorem 4.2 (with appropriate values of α and β). Hence, Theorem 4.1 follows from Theorem 4.2 as claimed.

As suggested above, the proof of Theorem 4.2 is very similar to the proof of Theorem 1.2. We sketch the main required changes below.

Proposition 2.7 needs to be adapted to handle vertices of degree as high as $o(\log n)$. One can easily verify that most parts of the proof can indeed be adapted using very few and straightforward changes. On the other hand, the last part, where we prove that a.a.s. $W \subseteq \phi(V)$, does not work with degree greater than $c(\log n)^{1/3}$ for some appropriate constant $c > 0$. However, we only use the bound on the maximum degree in this part in order to show that T_2 has $\Omega(n)$ vertices which are neither leaves nor parents of leaves; instead this is now assumed in Theorem 4.2. Using our assumption that T_1 has $\Omega(n)$ parents of leaves, the proof of Theorem 1.2 can also be easily adapted to the setting of Theorem 4.2. The only substantial difference is that we cannot use Theorem 2.10 to embed T_1' (since its maximum degree is unbounded). Instead we do the following. We split G into two graphs only, $G_1 \sim G(n, \frac{(1+\varepsilon/2)\log n}{n})$ and $G_2 \sim G(n, \frac{\varepsilon \log n}{2n})$, where G_1 will be used to embed T_1' and subsequently T_2 and the remainder of T will be embedded in $G_1 \cup G_2$. We expose G_1 vertex by vertex and embed T_1' greedily (while reserving enough edges of G_1 which are incident with the image of the joint root of T_1' and T_2 for its neighbors in T_2). We then embed T_2 and subsequently the remainder of T as in the proof of Theorem 1.2.

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References

[1] M. Ajtai, J. Komlós and E. Szemerédi, The longest path in a random graph, *Combinatorica* 1 (1981), 1–12.

[2] N. Alon, M. Krivelevich and B. Sudakov, Embedding nearly-spanning bounded degree trees, *Combinatorica* 27 (2007), 629–644.

[3] N. Alon and J. H. Spencer, **The Probabilistic method**, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, 3rd edition, 2008.

[4] J. Balogh, B. Csaba, M. Pei, and W. Samotij, Large bounded degree trees in expanding graphs, *Electronic J. Combinatorics* 17 (2010), R6.

[5] J. Balogh, B. Csaba and W. Samotij, Local resilience of almost spanning trees in random graphs, *Random Structures and Algorithms* 38 (2011), 121–139.

[6] E. A. Bender and N. C. Wormald, Random trees in random graphs, *Proc. Amer. Math. Soc.* 103 (1988), 314–320.

[7] R. Diestel, **Graph Theory**, 4th edition, Springer, 2010.

[8] W. Fernandez de la Vega, Long paths in random graphs, *Studia Sci. Math. Hungar.* 14 (1979), 335–340.

[9] W. Fernandez de la Vega, Trees in sparse random graphs, *Journal of Combinatorial Theory Ser. B* 45 (1988), 77–85.

[10] J. Friedman and N. Pippenger, Expanding graphs contain all small trees, *Combinatorica* 7 (1987), 71–76.

[11] A. Frieze and M. Krivelevich, Almost universal graphs, *Random Structures and Algorithms* 28 (2006), 499–510.

[12] P. Haxell, Tree embeddings, *Journal of Graph Theory* 36 (2001), 121–130.

[13] D. Hefetz, M. Krivelevich and T. Szabó, Hamilton cycles in highly connected and expanding graphs, *Combinatorica* 29 (2009), 547–568.

[14] S. Janson, T. Łuczak and A. Ruciński, **Random graphs**, Wiley, 2000.

[15] M. Krivelevich, Embedding spanning trees in random graphs, *SIAM J. Discrete Math.* 24 (2010), 1495–1500.

[16] C. J. H. McDiarmid, Concentration, Probabilistic methods for algorithmic discrete mathematics, 195-248, Algorithms Combin., 16, Springer, Berlin, 1998.

[17] J. W. Moon, On the maximum degree in a random tree, Michigan Math. J. 15 (1968), 429–432.

[18] D. B. West, **Introduction to Graph Theory**, Prentice Hall, 2^{nd} edition, 2001.