Extremal problems for transversals in graphs with bounded degree

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Abstract

We introduce and discuss generalizations of the problem of independent transversals. Given a graph property $\mathcal{R}$, we investigate whether any graph of maximum degree at most d with a vertex partition into classes of size at least p admits a transversal having property $\mathcal{R}$. In this paper we study this problem for the following properties $\mathcal{R}$: "acyclic", "H-free", and "having connected components of order at most r".

We strengthen a result of [13]. We prove that if the vertex set of a d-regular graph is partitioned into classes of size $d + \lfloor d/r \rfloor$, then it is possible to select a transversal inducing vertex disjoint trees on at most r vertices. Our approach applies appropriate triangulations of the simplex and Sperner's Lemma. We also establish some limitations on the power of this topological method.

We give constructions of vertex-partitioned graphs admitting no independent transversals that partially settles an old question of Bollobás, Erdős and Szemerédi. An extension of this construction provides vertex-partitioned graphs with small degree such that every transversal contains a fixed graph H as a subgraph.

Finally, we pose several open questions.

1 Introduction

Let G be a graph and let $\mathcal{P}$ be a partition of $V(G)$ into sets $V_1, \ldots, V_n$. A *transversal* (of $\mathcal{P}$) is a subset T of $V(G)$ for which $|T \cap V_i| = 1$ for each $i = 1, \ldots, n$. The starting point of our discussion is the following theorem of Haxell.

Theorem 1.1 *[11] Let G be a graph of maximum degree d and $V_1 \cup \ldots \cup V_n = V(G)$ be a partition of its vertex set with $|V_i| \geq 2d$. Then there is a transversal T which is an independent set in G.*

This theorem seems to have appeared first explicitly in Haxell [11], although it is also a consequence of a more general result of Meshulam [16] and implicitly, even earlier, of Haxell [10]. The result has two proofs: one combinatorial [10] and another via combinatorial topology [1]; it is not clear

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how closely these two arguments are related. The statement also has several applications in different problems of graph theory, see [4, 7, 11]. In the present paper we set to study the generalizations of Theorem 1.1 in two different directions.

1.1 Acyclic transversals with bounded components

The first generalization we consider here was introduced in [13] in order to improve on a result of Alon, Ding, Oporowski and Vertigan [7]. For a fixed degree d and component size r, let us define $p(d,r)$ to be the smallest integer such that any graph with maximum degree at most d partitioned into classes of size at least $p(d,r)$ has a transversal that induces components of size at most r. Theorem 1.1 can then be rephrased as $p(d,1) \le 2d$. In [13] a generalization of the combinatorial argument of [10] implied that $p(d,r) \le d + \lfloor d/r \rfloor$ for any d and r. This was known to hold with equality only when $r = 1$ and d is a power of 2 [14, 18]. In Corollary 3.4 we establish $p(d,1) = 2d$ for every d.

Unfortunately, for $r > 1$ even the asymptotical truth escapes us. The estimate is not tight in general: $p(2,2) = 2$ as shown in [13]. In fact, the best lower bound known for $r > 1$ is $p(d,r) \ge d$ and even $p(d,2) = d$ is possible at the moment. There was hope that a generalization of the "topological" argument could provide stronger upper bounds. In the present paper we provide this missing proof via Sperner's Lemma and appropriate triangulations of the simplex. Alas, we end up with the exact same result which follows from the combinatorial counterpart. For the proof we construct triangulations which generalize the ones of Aharoni, Chudnovsky and Kotlov [1] and then finish along the lines of Aharoni and Haxell [2] applying Sperner's Lemma for our appropriately defined colored triangulation. In fact, in Corollary 2.6 we obtain a slightly stronger statement. We prove that if the class sizes are at least $d + \lfloor d/r \rfloor$, then a transversal could be selected inducing connected components which are *trees* on at most r vertices.

One of our main tools for this strengthening is a triangulation from [1]. In Corollary 2.4 we obtain a bound $p(d, forest) \le d$ on the minimum class size $p(d, forest)$ such that any graph of maximum degree at most d partitioned into classes of size $p(d, forest)$ has a transversal inducing a forest. We allow multigraphs in this definition. With Construction 3.3 we show that this bound is optimal for even d. For simple graphs the analogous value might be somewhat lower though. Our best construction here is Construction 3.7 for $H = K_3$. For an even d this construction gives a maximum degree d graph G_d whose vertices are partitioned into classes of size $\lceil \frac{3}{4} d \rceil - 1$ and no transversal of G_d is triangle-free.

In Corollary 2.9 we note that our proof implies that the *r-component complex* $\mathcal{K}_r(G)$ of a d-regular graph G with *many* (more than $(m+1)(d-1+(d+1)/r)$) vertices is m-connected. Here $\mathcal{K}_r(G)$ denotes the simplicial complex defined on the vertex set of G, where a subset forms a simplex if all connected components of the induced subgraph is of order at most r.

Unfortunately our proof does not decide the asymptotics of $p(d,r)$ for $r > 1$. Thus it is natural to investigate "how good" such proofs could become with a possibly more clever choice of colored triangulations. With Construction 3.2 of Section 3 we find that for $r = 2$, where the truth is between d and $\frac{3}{2}d$, there is an intrinsic limit of $\frac{5}{4}d$ to where such type of arguments could improve the upper bound. In particular, for $d = 2$, the combinatorial proof [13] of $p(2, 2) = 2$ *cannot* be substituted by a topological argument.

1.2 H-free transversals

The second direction we intend to generalize Theorem 1.1 is about H-free-transversals. Given a fixed graph H, let $p(d, H)$ be the smallest integer such that any graph of maximum degree at most d partitioned into classes of size at least $p(d, H)$ admits a transversal with no subgraph isomorphic to H. Theorem 1.1 can be phrased as $p(d, K_2) \leq 2d$. We find the case of $H = K_k$ particularly interesting, but at this point we are only able to provide a lower bound which we conjecture to be best possible.

For any r-regular graph H on n vertices and for any d divisible by r, in Corollary 3.8 we prove that $p(d, H) \geq \frac{n}{(n-1)r}d$. The special case of the same construction establishes $p(d, 1) = p(d, K_2) = 2d$ for every d. Earlier this was only known for powers of 2. (See Jin [14] and Yuster [18].)

Construction 3.3 is a modified version of the above construction and provides a partial solution for a problem of Bollobás, Erdős and Szemerédi [9] studied by several researchers [4, 5, 14, 18, 6]. Let $\Delta(r, n)$ be the largest integer such that any r-partite graph $G_r(n)$ with vertex classes V_i of size n each and of maximum degree less than $\Delta(r, n)$ contains an *independent transversal*, i.e., an independent set containing one vertex from each V_i. Define $\Delta_r = \lim_{n\to\infty} \Delta(r, n)/n$, where the limit is easily seen to exist. Trivially $\Delta(2, n) = n$, thus $\Delta_2 = 1$. Graver (c.f. [9]) showed $\Delta_3 = 1$. Bollobás, Erdős and Szemerédi [9] proved that

$$\frac{2}{r} \leq \Delta_r \leq \frac{1}{2} + \frac{1}{r-2},$$

thus establishing $\mu = \lim_{r\to\infty} \Delta_r \leq 1/2$. Alon [4] showed $\Delta_r \geq 1/(2e)$ for every r. This was improved to $\Delta_r \geq 1/2$ by Haxell [11] thus eventually settling a conjecture of [9] and establishing $\mu = 1/2$. Exact values of Δ_r were known only when $r = 3, 5$ or a power of 2. Alon [6] observed that a theorem of Aharoni and Haxell [3] also gives $\Delta_r \geq \left\lceil \frac{r}{2(r-1)} \right\rceil$. This can be paired with the constructions of Jin [14] to provide $\Delta_r = \frac{r}{2r-1}$ for $r = 2^p$. For other integers r, the upper bounds of Jin were somewhat improved by Alon, but exact results were not known.

Here we extend the above for every even r. More precisely, in Corollary 3.6 we prove that for every $r \geq 2$ even and for every n,

$$\Delta(r, n) = \left\lceil \frac{rn}{2(r-1)} \right\rceil.$$

2 Transversals spanning bounded connected components

Given a graph G, a simplicial complex $\mathcal{K}$, and a mapping $l : V(\mathcal{K}) \to V(G)$, the pair $(\mathcal{K}, l)$ is called a *G-labeled simplicial complex.* If l is clear from the context we simply use $\mathcal{K}$ to denote the G-labeled complex. A 1-dimensional simplex $\{x, y\}$ of $\mathcal{K}$ is called *ruined* if $l(x)$ and $l(y)$ are adjacent in G. An r-dimensional simplex S of $\mathcal{K}$ is called ruined if the graph of ruined 1-dimensional faces of S is connected and spans all $r + 1$ vertices of S. The m-dimensional solid ball is denoted by B^m; its boundary, the $(m - 1)$-dimensional sphere is S^{m-1}. S^{-1} is just the empty set.

The link of a simplex σ in a simplicial complex is defined as $lk_{\mathcal{K}}(\sigma) := \{\tau \in \mathcal{K} : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \mathcal{K}\}$. The join of two simplicial complexes with disjoint vertex sets is defined as $\mathcal{K} * \mathcal{K}' := \{\tau \cup \tau' : \tau \in \mathcal{K}, \tau' \in \mathcal{K}'\}$. For a more detailed discussion of the topological concepts we refer the reader to the excellent survey of Björner [8].

Throughout this paper when talking about a *subdivision* of a complex, we mean *PL-subdivision*, when we speak of a *triangulation* of the sphere S^m or ball B^m we mean PL-triangulation, where *PL* stands for *piecewise linear.* This technical property is needed to assure key properties like the following: the link of a simplex of a PL-triangulated sphere or ball is itself a PL-triangulated sphere unless the simplex is in the boundary of the ball [8]. We mention here that one can avoid PL-triangulations and triangulated balls and spheres by using an alternative *homologic* approach, and speaking about boundaries of chains. This alternative treatment is somewhat less intuitive, and requires a homologic version of Sperner's theorem, but it avoids most technical difficulties.

A triangulation $\mathcal{T}'$ of B^m is called a *filling* of the triangulation $\mathcal{T}$ of S^{m-1} if $\mathcal{T}$ is the boundary of $\mathcal{T}'$. A G-labeled triangulation $(\mathcal{T}', l')$ of B^m is called a *filling* of the G-labeled triangulation $(\mathcal{T}, l)$ of S^{m-1}, if $\mathcal{T}'$ is a filling of $\mathcal{T}$ and $l'|_{V(\mathcal{T})} = l$.

We say that a simplex σ is *multi-colored* if all its vertices are assigned distinct colors.

We use Sperner's Lemma [17]. It states that an appropriately colored subdivision of a multi-colored simplex contains a multi-colored simplex.

Lemma 2.1 *[17] Let $\mathcal{T}$ be a triangulation of the n dimensional simplex σ. Suppose that the vertices of $\mathcal{T}$ are colored by $n + 1$ colors, such that*

(1) each vertex of σ receives a different color (i.e., σ is multi-colored) and

(2) the vertices of $\mathcal{T}$ on any face τ of σ are colored by the colors of the vertices of τ.

Then there exists a multi-colored n-dimensional simplex in $\mathcal{T}$.

As a warm-up let us discuss forest transversals. The next theorem is our tool to construct forest transversals in graphs.

Theorem 2.2 *Let $m \geq 0$ and $d \geq 0$ be arbitrary integers. Let G be a graph of maximum degree d and W be a designated subset of the vertices $V(G)$, $|W| > md$. Every G-labeled triangulation $(\mathcal{T}, l)$ of S^{m-1} has a G-labeled filling $(\mathcal{T}', l')$, such that* *(i)* $l'(v) \in W$ *for every* $v \in V(\mathcal{T}') \setminus V(\mathcal{T})$.

(ii) Every cycle of ruined edges is contained in $\mathcal{T}$.

(iii) Every path of ruined edges with both endpoints in $\mathcal{T}$ *is fully contained in* $\mathcal{T}$.

For the proof we use the following triangulation constructed by Aharoni, Chudnovsky and Kotlov [1].

Lemma 2.3 *[1, Lemma 1.2] Given a triangulation* $\mathcal{T}$ *of* S^{m-1}*, there is a filling* $\mathcal{T}'$ *of* $\mathcal{T}$ *and an ordering of the new vertices* $V(\mathcal{T}') \setminus V(\mathcal{T}) = \{v_1, \ldots, v_s\}$ *such that for all* i*, the vertex* v_i *is connected to at most* $2m$ *vertices of* $V(\mathcal{T}) \cup \{v_1, \ldots, v_{i-1}\}$. □

Proof of Theorem 2.2. The triangulation of Lemma 2.3 provides a filling of $\mathcal{T}$ such that we add the vertices one by one and each new vertex is connected to at most $2m$ older ones. Since $|W| > md$ we can ensure that the label of each new vertex is chosen such that there is only *at most one* ruined edge from that vertex going to an older vertex. Thus we avoid the creation of cycles of ruined edges and also paths of ruined edges connecting vertices of $\mathcal{T}$. □

An immediate application of Theorem 2.2 is for forest-transversals. Let us recall that $p(d, forest)$ is the smallest integer, such that any d-regular graph partitioned into classes of size at least $p(d, forest)$ has a cycle-free transversal.

Corollary 2.4 $p(d, forest) \leq d$.

Proof. Suppose G is a graph of maximum degree d and $V(G) = V_1 \cup \ldots \cup V_n$, $|V_i| \geq d$. We consider the $(n-1)$-dimensional simplex σ with vertex set $\{v_1, \ldots, v_n\}$. We create a G-labeled triangulation $(\mathcal{T}, l)$ of σ, such that

- for every vertex x of the triangulation and face τ of σ containing x, $l(x) \in \cup_{i:v_i \in \tau} V_i$ and
- the graph of ruined edges induces a forest.

We proceed by cell-induction, i.e., subdivide and label the faces of σ in an arbitrary nondecreasing order of their dimension. We start by labeling each vertex v_i by an arbitrary element $l(v_i)$ of V_i. Let τ be an m-dimensional face of σ, $m > 0$, whose boundary is subdivided and G-labeled. Let $W = \cup_{i:v_i \in \tau} V_i$. As $|W| \geq (m+1)d > md$, we can apply Theorem 2.2 to obtain an appropriate labeled subdivision of τ. Notice that we do not create a cycle of ruined edges disjoint from the boundary, because of condition *(ii)*. Cycles of ruined edges intersecting the boundary could not be created either because of condition *(iii)*.

Eventually the whole simplex σ is subdivided without creating a cycle of ruined edges. If each vertex of this triangulation is colored with the index of the class of its label, then the assumptions of Sperner's Lemma are satisfied and the existence of a full-dimensional multi-colored simplex is guaranteed. The labels of this multi-colored simplex determine a transversal with no cycle. □

For multigraphs the bound in Corollary 2.4 is tight for even d, as it is witnessed by doubling the edges of any family of graphs which provide $p(d/2, 1) = d$. These graphs are given in Construction 3.3.

Our best example for *simple graphs* is weaker. Construction 3.7 for $H = K_3$ shows that the analogous value $p_s(d, forest)$ for simple graphs satisfies $p_s(d, forest) \geq \frac{3}{4}d$ for even d.

Next we prove a strengthening of the bounded component transversal result from [13]. Our main tool is the following theorem about the existence of labeled fillings with certain properties.

Theorem 2.5 *Let $m \geq 0$ and $r \geq 1$ be arbitrary integers. Let G be a graph of maximum degree $d \geq r - 1$ and designated subset $W \subseteq V(G)$ of size $|W| > m(d - 1 + (d + 1)/r)$. Then a G-labeled triangulation $(\mathcal{T}, l)$ of S^{m-1} admits a G-labeled filling $(\mathcal{T}', l')$ satisfying the following properties:*
(a) $l'(v) \in W$ for every $v \in V(\mathcal{T}') \setminus V(\mathcal{T})$.
(b) Every cycle of ruined edges of $(\mathcal{T}', l')$ is contained in $\mathcal{T}$.
(c) There is no ruined edge between $V(\mathcal{T})$ and $V(\mathcal{T}') \setminus V(\mathcal{T})$.
(d) The ruined r-simplices of $(\mathcal{T}', l')$ are contained in $\mathcal{T}$.

For a vertex w of a graph G, $N(w)$ denotes the set of vertices adjacent to w.

Proof. We prove the theorem by induction on m. For $m = 0$ the statement is trivial.

Suppose $m > 0$. We construct $\mathcal{T}'$ in three phases.

First we apply the "excising technique" of [1] to create an inner "crust" which contains no ruined edges going to the boundary. We excise the vertices of $\mathcal{T}$ one by one from the inner boundary and use the induction hypothesis for $(m - 1)$ in each step.

In the second phase we fill (the inner boundary of) the crust constructed in the first phase. We use Theorem 2.2 here. We obtain a filling $\mathcal{S}_0$ of $\mathcal{T}$ satisfying properties (a)-(c). But we may create a number of ruined r-simplices in this phase.

Finally, in the third phase we remove the ruined r-simplices constructed in the second phase. We remove them one by one and fill up the resulting "holes". We use the induction hypothesis for $(m - 1)$ in each step.

Let us start with the first phase. We take the vertices $u_1, \ldots, u_t$ of $V(\mathcal{T})$ and excise them one by one. We do this by creating an increasing sequence $\mathcal{T} = \mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_t$ of labeled complexes. The complex $\mathcal{T}_i$ satisfies properties (a)-(d), furthermore it has an "inner boundary" $\mathcal{T}_i'$ with $V(\mathcal{T}_i') \cap V(\mathcal{T}) = \{u_{i+1}, \ldots, u_t\}$, such that $\mathcal{T}_i'$ is a triangulation of S^{m-1} and the union of $\mathcal{T}_i$ with any filling of $\mathcal{T}_i'$ is a filling of $\mathcal{T}$. We start with $\mathcal{T}_0 = \mathcal{T}_0' = \mathcal{T}$. For $i > 0$ consider the vertex u_i and its link $lk_{\mathcal{T}_{i-1}'}(\{u_i\})$, which is an $(m - 2)$-sphere. (Note that we talk about PL-triangulations.) By induction there is a G-labeled filling $\tilde{\mathcal{T}}_i$ of this link, with subset $\tilde{W}_i = W \setminus N(l(u_i))$ satisfying properties (a)-(d). (We naturally assume here, and later in this proof, that the set of new vertices introduced in a filling is disjoint from the set of old vertices, that is we have $(V(\tilde{\mathcal{T}}_i) \setminus V(lk_{\mathcal{T}_{i-1}'}(\{u_i\}))) \cap V(\mathcal{T}_{i-1}) = \emptyset$.) Observe that $|\tilde{W}_i| \geq |W| - d > (m - 1)(d - 1 + (d + 1)/r)$. We then create $\mathcal{T}_i$ by adding the join of $\tilde{\mathcal{T}}_i$ with u_i to $\mathcal{T}_{i-1}$. This operation excises u_i from the interior boundary of $\mathcal{T}_i$. More formally, let $\mathcal{T}_i = \mathcal{T}_{i-1} \cup (\tilde{\mathcal{T}}_i * \{\emptyset, \{u_i\}\})$ and we obtain the inner boundary $\mathcal{T}_i' = (\mathcal{T}_{i-1}' \setminus (\tilde{\mathcal{T}}_i * \{\emptyset, \{u_i\}\})) \cup \tilde{\mathcal{T}}_i$. By the choice of $\tilde{W}_i$, we do not add a ruined edge going to u_i. By property (c) of the induction hypothesis there are no ruined edges added going to other vertices of $V(\mathcal{T})$. In conclusion, the newly introduced ruined edges are "separated" from the old ones, that is there are no ruined edges between $V(\mathcal{T}_{i-1})$ and $V(\mathcal{T}_i) \setminus V(\mathcal{T}_{i-1})$. Thus properties (a)-(d) hold for $\mathcal{T}_i$ by the induction hypothesis.

Eventually all vertices of $V(\mathcal{T})$ excised from the inner boundary. Hence $\mathcal{T}_t$ is a crust having properties (a)-(d) and its inner boundary $\mathcal{T}_t'$ is disjoint from $\mathcal{T}$.

As the second phase of the construction, we apply Theorem 2.2 to fill $\mathcal{T}_t'$ such that the new labels are from W. This is possible, because our extra condition on the maximum degree ensures that $m(d-1+(d+1)/r) \geq md$. Let $\mathcal{S}_0$ be the union of $\mathcal{T}_t$ and this filling of $\mathcal{T}_t'$. Clearly, $\mathcal{S}_0$ is a filling of $\mathcal{T}$ satisfying properties (a), (b), and (c).

In the third phase of our construction we get rid of any ruined r-simplices in $\mathcal{S}_0 \setminus \mathcal{T}$ that we may have created in the second phase. For $m < r$ no such simplices are created, so property (d) is automatically satisfied. For $m \geq r$ our plan is to modify $\mathcal{S}_0$ to get rid of all ruined r-simplices one by one. We will change our triangulation locally and be careful not to spoil properties (a), (b), and (c). In one step we remove a ruined r-simplex $\sigma \notin \mathcal{T}$, together with all the simplices containing it, thus creating an m-dimensional "hole" in B^m. Then we fill up this hole differently, such that we do not create new ruined r-simplices, properties (a)-(c) are still satisfied, while σ is gone. Since the number of ruined simplices was finite to begin with, after applying this operation finitely many times we will have a triangulation $\mathcal{T}'$ with no ruined r-simplices outside of $\mathcal{T}$.

Suppose that $\mathcal{S}_i$ is the filling of $\mathcal{T}$ we obtained after getting rid of the i^{th} ruined r-simplex in $\mathcal{S}_0$. Fix an arbitrary ruined r-simplex σ_{i+1} of $\mathcal{S}_i \setminus \mathcal{T}$. In fact $\sigma_{i+1} \in \mathcal{S}_0 \setminus \mathcal{T}_t$. The link $lk_{\mathcal{S}_i}(\sigma_{i+1})$ of the r-simplex σ_{i+1} in $\mathcal{S}_i$ is a triangulated S^{m-r-1}. We find a filling $\tilde{\mathcal{S}}_i$ of $lk_{\mathcal{S}_i}(\sigma_{i+1})$ using the induction hypothesis for the designated subset $W_i \subseteq W$ containing the non-neighbors of the labels of the vertices of σ_{i+1}. Formally, let $N_{\sigma_{i+1}}$ be the set of vertices of G that are neighbors to the label of some vertex of σ_{i+1} and let $W_i = W \setminus N_{\sigma_{i+1}}$. Then $|N_{\sigma_{i+1}}| \leq d(r+1)-(r-1)$, since σ_{i+1} is a ruined r-simplex. Hence

$$|W_i| \quad \geq \quad |W| - |N_{\sigma_{i+1}}| > m(d-1+(d+1)/r) - dr - d + r - 1 = (m-r)(d-1+(d+1)/r),$$

i.e., we can indeed use the induction hypothesis.

Now we are ready to define $\mathcal{S}_{i+1}$. First we remove all simplices from $\mathcal{S}_i$ which contain σ_{i+1}. This of course creates an m-dimensional "hole" in B^m. Then in order to fill it, we add all simplices of the form $\sigma' \cup \tilde{\sigma}$, where $\sigma' \subsetneq \sigma_{i+1}$ and $\tilde{\sigma} \in \tilde{\mathcal{S}}_i$. That is, we add $\tilde{\mathcal{S}}_i * \delta\sigma_{i+1}$, where $\delta\sigma_{i+1}$ is the boundary complex of σ_{i+1}.

Starting from the filling $\mathcal{S}_i$ of $\mathcal{T}$ we replaced a subcomplex with another, both of them triangulated B^m and having $lk_{\mathcal{S}_i}(\sigma_{i+1}) * \delta\sigma_{i+1}$ as their boundary, so the resulting complex $\mathcal{S}_{i+1}$ is also a filling of $\mathcal{T}$. There are no ruined edges between $V(\sigma_{i+1})$ and $V(\mathcal{S}_{i+1}) \setminus V(\mathcal{S}_i)$ because of the choice of W_i. All other edges between $V(\mathcal{S}_i)$ and $V(\mathcal{S}_{i+1}) \setminus V(\mathcal{S}_i)$ are also edges between $lk_{\mathcal{S}_i}(\sigma_{i+1})$ and $V(\tilde{\mathcal{S}}_i) \setminus lk_{\mathcal{S}_i}(\sigma_{i+1})$, hence not ruined by property (c) of the induction hypothesis. In conclusion, there are *no ruined edges* between the newly introduced vertices and the "old" vertices. Thus, we did not spoil properties (b) or (c), and did not create any new ruined r-simplices. Clearly, property (a) is also maintained.

The ruined simplex σ_{i+1} is gone, so the number of ruined r-simplices decreased by one. After finitely many steps we obtain a filling $\mathcal{T}'$ of $\mathcal{T}$ satisfying properties (a)-(d). □

The following corollary is immediate.

Corollary 2.6 *Let r be an arbitrary positive integer. Let G be a graph of maximum degree d, and let $\mathcal{P}$ be a partition $V_1 \cup \ldots \cup V_m = V(G)$ such that $|V_i| \geq d + \lfloor d/r \rfloor$ for $i = 1, \ldots, m$. Then there exists a transversal T of $\mathcal{P}$ such that the connected components of the induced subgraph $G|T$ are trees on at most r vertices.*

Proof. It is enough to prove the statement for $r \leq d+1$ (for higher values of r the statement of the Corollary is *weaker* than the one for $r = d+1$).

We construct a transversal T of $\mathcal{P}$ such that the connected components of the induced subgraph $G|T$ are trees on at most r vertices.

Let us denote the vertices of the $(n-1)$-dimensional simplex σ by $v_1, \ldots, v_n$.

Our goal is to define a G-labeled subdivision of the complex consisting of the faces of σ such that

- for every vertex x of the subdivision of a face τ of σ we have $l(x) \in \cup_{i:v_i \in \tau} V_i$
- the subdivision has no cycle of ruined edges and
- has no ruined r-simplex.

We, again, proceed by cell-induction. As a start, we label each vertex v_i of σ by an arbitrary vertex $l(v_i) \in V_i$. Suppose we are given an $m \geq 1$-dimensional face τ of σ with a labeled subdivision of its boundary. Then, by the previous theorem, it is possible to extend this triangulation to the interior of τ without creating ruined r-simplices and cycles, such that the labels are from the set $\cup_{i:v_i \in \tau} V_i$. We just note that $|\cup_{i:v_i \in supp(\tau)} V_i| \geq (d + \lfloor d/r \rfloor)(m+1) \geq (d + d/r - (r-1)/r)(m+1) > (d-1+(d+1)/r)m$.

Eventually, the whole simplex σ has such a labeled triangulation. Assigning color i to the vertices with label from V_i we obtain a *colored* triangulation respecting the assumptions of Sperner's Lemma. Thus, a multi-colored simplex could be found. The labels of the vertices of this multi-colored simplex form a transversal having the desired property. □

We also obtained a new proof of the following statement on finding transversals with bounded connected components (which are not necessarily acyclic).

Corollary 2.7 *[13, Theorem 4.1] For arbitrary positive integers r and d,*

$$p(d, r) \leq d + \left\lfloor \frac{d}{r} \right\rfloor .$$

For a graph G let $\mathcal{K}_r(G)$ denote the simplicial complex defined on the vertices of G, which contains all simplices inducing connected components of size at most r in G. In particular $\mathcal{K}_1(G)$ is called the *independent set complex* of G, it consists of the the independent sets of G. We refer to $\mathcal{K}_2(G)$ as the *induced matching complex* of G. Using this notation Theorem 2.5 could be stated in the language of topology.

A simplicial complex $\mathcal{K}$ is said to be m-connected if its body $||\mathcal{K}||$ (the corresponding topological space) is m-connected, i.e., every continuous $f : S^i \to ||\mathcal{K}||$ can be extended to a continuous map $B^{i+1} \to ||\mathcal{K}||$ for $-1 \leq i \leq m$ (in other words f is nullhomotopic). The m-connectedness of $\mathcal{K}_r(G)$ can be described using fillings of G-labeled triangulations.

In the remainder of this section a G-labeled simplex is called *multi-labeled* if the labels of its vertices are all distinct.

Proposition 2.8 *For a graph G, and $m, r \geq 0$ the complex $\mathcal{K}_r(G)$ is m-connected if and only if the following holds for all $-1 \leq i \leq m$: Every G-labeled triangulation of S^i without a ruined, multi-labeled r-simplex has a filling without a ruined, multi-labeled r-simplex.*

Proof. Notice that for any complex $\mathcal{K}$ the map $l : V(\mathcal{K}) \to V(G)$ is a simplicial map $l : \mathcal{K} \to \mathcal{K}_r(G)$ if and only if the G-labeled complex $(\mathcal{K}, l)$ has no ruined, multi-labeled r-simplex. Indeed, the image under l of a ruined, multi-labeled r-simplex is a set of $r+1$ distinct vertices spanning a connected subgraph in G, and such a set is not a simplex of $\mathcal{K}_r(G)$. To see the reverse direction assume S is a simplex of $\mathcal{K}$ but its image under l is not a simplex of $\mathcal{K}_r(G)$. Then $l(S)$ contains $r+1$ distinct vertices spanning a connected subgraph, and taking inverse images of these we find a ruined, multi-labeled, r-dimensional face of S.

Both directions of the proposition is a simple consequence of the above observation and the simplicial approximation theorem.

Assume first that the filling property is satisfied. We need to show that every continuous map $f : S^i \to ||\mathcal{K}_r(G)||$ is nullhomotopic for $i \leq m$. By the simplicial approximation theorem, there exist a triangulation $\mathcal{T}$ of S^i and a simplicial map $l : \mathcal{T} \to \mathcal{K}_r(G)$ such that its affine extension $||l|| : S^i \to ||\mathcal{K}_r(G)||$ is homotopic to f. (In fact, any fine enough triangulation $\mathcal{T}$ will do here.) Therefore, it is enough to to show that $||l||$ is nullhomotopic by finding a continuous extension to B^i. As $(\mathcal{T}, l)$ has no ruined, multi-labeled r-simplex it has a filling $(\mathcal{T}', l')$ that has no ruined, multi-labeled r-simplex. Thus $l' : \mathcal{T}' \to \mathcal{K}_r(G)$ is a simplicial map and its affine extension $||l'||$ is a continuous function extending $||l||$ to the ball B^{i+1}.

For the reverse implication assume $\mathcal{K}_r(G)$ is m-connected. Let $i \leq m$, and let $(\mathcal{T}, l)$ be a G-labeled triangulation of S^i without ruined, multi-labeled r-simplices. Now l is simplicial map from $\mathcal{T}$ to $\mathcal{K}_r(G)$ and its affine extension $||l||$ is a continuous map from $||\mathcal{T}|| \cong S^i$ to $||\mathcal{K}_r(G)||$. Therefore it can be extended to a continuous map $f : B^{i+1} \to ||\mathcal{K}_r(G)||$. We use simplicial approximation for f and find a suitable triangulation $\mathcal{T}'$ of B^{i+1} and get a simplicial map $l' : \mathcal{T}' \to \mathcal{K}_r(G)$ approximating f. As $f|S^i = ||l||$ we can make sure that the boundary of the complex $\mathcal{T}'$ is $\mathcal{T}$ and l and l' agree on $\mathcal{T}$. This means that l' is a filling of l with no ruined, multi-labeled r-simplices. □

Corollary 2.9 *Let $m \geq 0$ and $r \geq 1$ be arbitrary integers. If G is a graph on more than $m(d-1+(d+1)/r)$ vertices with maximum degree $d \geq r-1$, then $\mathcal{K}_r(G)$ is $(m-1)$-connected.*

Proof. By Proposition 2.8 we need to show that a G-labeled triangulation of S^i without ruined, multi-labeled simplices has a filling still without ruined, multi-labeled simplices for $i < m$. Theorem 2.5 applies here with $W = V(G)$ and states the existence of a filling without any new ruined r-simplices, much less ruined, multi-labeled simplices. □

3 Constructions

3.1 Non-fillable labeled triangulations

First we give bounds on the connectedness of $\mathcal{K}_r(G)$ for $r = 1$ and 2. The examples are very similar to each other. The graphs constructed are disjoint unions of smaller graphs and we use the simple observation that for the disjoint union of two graphs G and G' the complex $\mathcal{K}_r(G \cup G')$ is the join of the complexes $\mathcal{K}_r(G)$ and $\mathcal{K}_r(G')$.

Our first example shows that Corollary 2.9 (and thus also Theorem 2.5) is best possible when $r = 1$. Note that independent set complexes are widely studied and several lower bound on their connectedness and acyclicity is known. The one closest to our result is Proposition 3.1 in [16] a special case of which claims that $\mathcal{K}_1(G)$ is $(\lceil\tilde{\gamma}(G)/2\rceil - 2)$-acyclic over the reals for any graph G. Here $\tilde{\gamma}(G)$ is the *total domination number*, the cardinality of the smallest set S of vertices in G such that every vertex of G is a neighbor of a vertex in S. Using the obvious bound $\tilde{\gamma}(G) \geq n/d$, where n is the number of vertices of G and d is the maximum degree we obtain the same bound on the acyclicity of $\mathcal{K}_1(G)$ as Corollary 2.9 gives for its connectedness. We remark that in the $r = 1$ case our argument also naturally generalizes to show that $\mathcal{K}_1(G)$ is $(\lceil\tilde{\gamma}(G)/2\rceil - 2)$-connected.

Construction 3.1

Take G_1 to be the disjoint union of m copies of $K_{d,d}$. It has $2md$ vertices, one too few for Corollary 2.9 to show that the independent set complex $\mathcal{K}_1(G_1)$ is $(m-1)$-connected. We show that $\mathcal{K}_1(G)$ is homotopy equivalent to S^{m-1} and therefore it is not $(m-1)$-connected.

The complex $\mathcal{K}_1(K_{d,d})$ consists of two disjoint $(d-1)$-simplices, so it is homotopy equivalent to S^0. As G_1 consists of m copies of $K_{d,d}$, the complex $\mathcal{K}_1(G_1)$ is the m-fold join of $\mathcal{K}_1(K_{d,d})$ and therefore homotopy equivalent to S^{m-1} as claimed. □

Although Corollary 2.9 is best possible in general for $r = 1$, the independence complex $\mathcal{K}_1(G)$ can be arbitrarily more connected for particular graphs G, than what is guaranteed by this result. For the cycle Corollary 2.9 gives that $\mathcal{K}_1(C_n)$ is $(\lceil n/4\rceil - 2)$-connected. Kozlov [15] determined the homotopy type of of the independent set complex of cycles and his result implies the stronger statement that $\mathcal{K}_1(C_n)$ is $\lfloor(n+1)/3\rfloor - 2$ connected.

For $r > 1$ we don't know whether Corollary 2.9 is tight. The following construction for $r = 2$ shows that in this case the lower bound of $\frac{3}{2}md - \frac{1}{2}m$ on the size of the graph in Corollary 2.9 cannot be lowered below $\frac{5}{4}md$. For $r = d = 2$ the construction gives $\frac{5}{2}m$, which is tight. Clearly, this construction bounds also how far the lower bound on the size of W in Theorem 2.5 can be lowered.

In [13] a combinatorial argument establishes that $p(2,2) = 2$. Notice that the standard topological proof of the same fact (similar to the proofs of Corollaries 2.4 and 2.6) through the method of Aharoni and Haxell [2] is *impossible*. It would require a strengthening of Theorem 2.5 for the $r = d = 2$ case, which is impossible by the example below.

Construction 3.2

Let d be even. A blown-up five-cycle $H_{d/2}$ is a graph on the vertex set $\cup_{j=0}^{4} A_j$, $|A_j| = d/2$, where $x \in A_j$ and $y \in A_l$ are connected if and only if $j - l \equiv \pm 1$ modulo 5. Let G_2 be the disjoint union of k copies of the blown-up five-cycle $H_{d/2}$. We claim that $\mathcal{K}_2(G_2)$ is homotopy equivalent to S^{2k-1} and therefore it is not $(2k-1)$-connected.

As in the previous construction it is enough to prove that for a single blown-up five cycle $H_{d/2}$ the complex $\mathcal{K}_2(H_{d/2})$ is homotopy equivalent to the circle S^1. This implies that $\mathcal{K}_2(G_2)$ is homotopy equivalent to the k-fold join of S^1, which is S^{2k-1}.

The maximal simplices of the independent set complex $\mathcal{K}_1(H_{d/2})$ are the sets $A_j \cup A_{j+2}$ for $0 \le j \le 4$. Here (and later in this construction) the indices are understood modulo 5. This complex is easily seen to be homotopy equivalent with the cycle S^1. The maximal simplices of $\mathcal{K}_2(H_{d/2})$ are the same simplices together with the simplices $\{x, y\} \cup A_j$, where $x \in A_{j-2}$, $y \in A_{j+2}$ and $0 \le j \le 4$. Any 1-simplex spanning an edge in $H_{d/2}$ is contained in a unique maximal simplex of $\mathcal{K}_2(H_{d/2})$. If a non-maximal simplex of a simplicial complex is contained in a unique maximal simplex then one can *collapse* this face, i.e., remove all simplices containing it and the remaining complex is homotopy equivalent to the one before the collapse. Therefore we can collapse any 1-simplex of $\mathcal{K}_2(H_{d/2})$ which spans an edge in $H_{d/2,d/2}$ and the remaining complex is homotopy equivalent to $\mathcal{K}_2(H_{d/2})$. As the maximal simplices containing these 1-simplices are distinct we can collapse all the 1-simplices spanning an edge simultaneously and the remaining complex is still homotopy equivalent to (in fact a strong deformation retract of) $\mathcal{K}_2(H_{d/2})$. Notice, that the complex remaining after collapsing all the 1-simplices corresponding to edges of $H_{d/2}$ is exactly $\mathcal{K}_1(H_{d/2})$. Therefore $\mathcal{K}_2(H_{d/2})$ is homotopy equivalent to $\mathcal{K}_1(H_{d/2})$ and to S^1. □

This example shows that with parameters $m = 2k$, $r = 2$ an d even, $|V(G)| = 5(d/2)k = \frac{5}{4}dm$, the statement of Corollary 2.9 is not true. The question remains open, whether the topological proof for $r = 2$ could be strengthened from $\frac{3}{2}dm$ or the counterexample improved from $\frac{5}{4}dm$.

3.2 Partitioned graphs without independent transversals

Let $n, d, k \geq 1$ be integers such that $d \geq kn/(2k-1)$. In this section we construct a graph $G_{k,n,d}$ of maximum degree at most d, together with a vertex set partition into $2k$ disjoint subsets $V_1, \ldots, V_{2k}$ of size $|V_i| = n$, $i = 1, \ldots, 2k$, such that there exists no independent transversal with respect to this partition, i.e., every subset $T \subseteq V(G)$ with the property $|T \cap V_i| = 1$, $i = 1, \ldots, 2k$, spans at least one edge.

Construction 3.3

If $n \leq d$, then $G_{k,n,d}$ could be chosen to be the disjoint union of $k \geq 1$ complete bipartite graphs $K_{n,n}$, the bipartite classes forming the vertex partition into $2k$ parts.

Thus we can assume $d < n$ and by our condition $n \leq 2d - \frac{d}{k} < 2d$. Let $i = 2d - n$, $q = \lceil \frac{d-i}{i} \rceil$ and $r = d - qi$. We have $1 \leq r \leq i \leq d-1$ and $1 \leq q \leq k-1$.

The graph $G_{k,n,d}$ is the disjoint union of $2q+1$ complete bipartite graphs H_i with vertex sets $A_i \cup B_i$, $i = 1, \ldots, 2q+1$ and an independent set W of $2(k-q-1)n$ points. The graph H_{q+1} is isomorphic to $K_{d-i+r,d-i+r}$ and all other graphs H_i are isomorphic to $K_{d,d}$.

The partition classes are defined as follows. For $i = 1, \ldots q$, $V_i = A_i \cup B'_{i+1}$, where $B'_j \subseteq B_j$ is an arbitrary $(d-i)$-element subset of B_j. Symmetrically, for $i = 1, \ldots q$, $V_{q+1+i} = B_{q+1+i} \cup A'_{q+i}$, where $A'_j \subseteq A_j$ is an arbitrary $(d-i)$-element subset of A_j. The leftover elements are divided into two classes: $V_{q+1} = B_1 \cup (\cup_{j=2}^{q+1}(B_j \setminus B'_j))$ and $V_{2q+2} = A_{2q+1} \cup (\cup_{j=q+1}^{2q}(A_j \setminus A'_j))$. This way all the classes are of size $2d - i = n$. In case $q < k-1$, then $W \neq \emptyset$ and we create the required $2k$ classes by arbitrarily partitioning the independent set W.

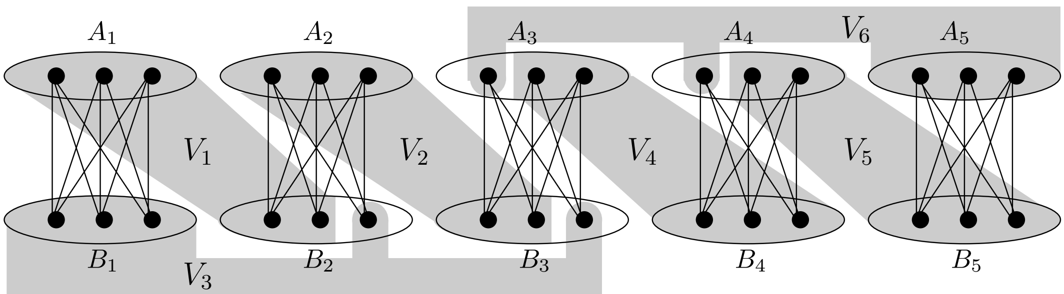

Figure 1: The partitioned graph $G_{3,5,3}$

Suppose for a contradiction that there exists an independent transversal T of $G_{k,n,d}$. If $T \cap B_i \neq \emptyset$ for some index $i \leq q$, then $T \cap A_i = \emptyset$ because T is independent. Therefore $T \cap B_{i+1} \neq \emptyset$ as well, since T is a transversal. Thus, eventually, $T \cap B_{q+1} \neq \emptyset$, since $V_{q+1} \subseteq \cup_{j=1}^{q+1} B_j$ ensures that *there is* at least one index $i \leq q+1$ with $T \cap B_i \neq \emptyset$. For symmetric reasons $T \cap A_{q+1} \neq \emptyset$, which provides the contradiction sought after. $\square$

Generalization of this construction will be presented in the next subsection. We prefer to discuss the important special case of independent transversals in this formulation, because we find it more transparent, than the (more intuitive) way of Construction 3.7. We remark that if $k = d$ is a power of 2 and $n = 2d - 1$, then our graphs are the same as the one used by Yuster [18], but our vertex partitions are different.

Taking the parameters $d = k$, and $n = 2d - 1$ our construction establishes $p(d, 1) = 2d$ for every d. Previously this only was known for powers of 2 [14, 18].

Corollary 3.4 *For every integer $d \geq 1$,*

$$p(d, 1) = 2d.$$

□

The same construction partially answers a question of Bollobás, Erdős, and Szemerédi [9], which was studied extensively by a number of researchers. Let us recall that $\Delta(r, n)$ denotes the largest integer such that any r-partite graph $G_r(n)$ with vertex classes V_i of size n each and of maximum degree less than $\Delta(r, n)$ contains an *independent transversal*, i.e., an independent set containing one vertex from each V_i. The limit $\Delta_r = \lim_{n\to\infty} \Delta(r, n)/n$ is easily seen to exist.

Haxell [11] showed $\mu = \lim_{r\to\infty} \Delta_r = 1/2$, but until very recently the exact values of Δ_r were known only for $r = 2, 3$, [9], and $r = 4, 5$ [14]. Alon [6] observed that the method of [11] actually implies $\Delta_r \geq \left\lceil \frac{r}{2(r-1)} \right\rceil$. Thus Jin's construction [14] is optimal and for powers of 2 one has $\Delta_r = \frac{r}{2(r-1)}$.

Here we extend the above result for all even r and determine not only Δ_r, but all the values $\Delta(r, n)$ in this case. The following Proposition appears in [6] and is an immediate consequence of a theorem of Aharoni and Haxell [3].

Proposition 3.5 *[6, Proposition 5.2]*

$$\Delta(r, n) \geq \left\lceil \frac{rn}{2(r-1)} \right\rceil.$$

For each r even and for arbitrary n our construction provides graphs of maximum degree $d = \lceil \frac{rn}{2(r-1)} \rceil$ with no independent transversal. Hence we have

Corollary 3.6 *For every integer $n \geq 1$ and $r \geq 2$ even,*

$$\Delta(r, n) = \left\lceil \frac{rn}{2(r-1)} \right\rceil.$$

Therefore for every r even we have

$$\Delta_r = \frac{r}{2(r-1)}.$$

3.3 Partitioned graphs without H-free transversals

Construction 3.7

Let H be an r-regular graph on n vertices and let d be a multiple of r. We prove a lower bound on $p(d,H)$ (see definition in Section 1.2) by giving an inductive construction. For every $c < \frac{n}{n-1} \cdot \frac{d}{r}$ we construct a graph of maximum degree at most d with a vertex partition into classes of size c which does not admit an H-free transversal. We proceed by induction on c. For a positive integer j, let $H(j)$ be a blow-up of H, such that each vertex is replaced with j independent vertices and each edge is replaced with a copy of $K_{j,j}$.

For $c \le d/r$, one can take the blow-up $H(c)$ of H, with the independent sets being the classes of its vertex partition.

Now let $\frac{d}{r} < c < \frac{n}{n-1} \cdot \frac{d}{r}$ and suppose we have a graph $\tilde{G}$ with vertex partition $\tilde{V}_1 \cup \ldots \cup \tilde{V}_m$, and class size $|\tilde{V}_i| = nc - nd/r$ containing no H-free transversal. Such a partitioned graph exists by our induction hypothesis since our assumption on c guarantees that $nc - nd/r < c$.

Our graph G will contain a copy of $\tilde{G}$ and m copies $H_1, \ldots, H_m$ of $H(d/r)$. The vertex partition of G will consist of mn classes of size c. For each i and j, $1 \le i \le m$, $1 \le j \le n$, we define a class $V_i^j = W_i^j \cup \tilde{W}_i^j$ of G, where W_i^j is the set of d/r independent vertices corresponding to vertex j of H in H_i and $\tilde{W}_i^1 \cup \ldots \cup \tilde{W}_i^n$ is an arbitrary partition of $\tilde{V}_i$ into parts of size $|\tilde{W}_i^j| = c - d/r$. Note that the size of $\tilde{V}_i$ is $n(c - d/r)$ and that the size of each class V_i^j is c.

It remains to see why there is no H-free transversal. Suppose there is one, denoted by T. Since T is H-free, for each copy H_i of $H(d/r)$ there is a j, $1 \le j \le n$, such that $T \cap W_i^j = \emptyset$. That is $T \cap \tilde{W}_i^j \ne \emptyset$, since T is a transversal with respect to the V_i^j. Thus for each i, $1 \le i \le m$, $T \cap \tilde{V}_i \ne \emptyset$. So T contains a transversal of $\tilde{G}$ with respect to the sets $\tilde{V}_i$, which cannot be H-free, a contradiction. □

Corollary 3.8 *Let H be an r-regular graph on n vertices and d be a multiple of r. Then we have*

$$p(d,H) \ge \frac{n}{(n-1)r} d.$$

Note that for $H = K_3$ this corollary gives $p(d, K_3) \ge \frac{3}{4} d$.

4 Remarks and open problems

Given an arbitrary graph property $\mathcal{R}$, define $p(d,\mathcal{R})$ to be the smallest integer p such that any graph G of maximum degree d with a vertex partition into classes of size p admits a transversal spanning a subgraph having property $\mathcal{R}$. We propose the general question of determining $p(d,\mathcal{R})$ for various graph properties $\mathcal{R}$. In this paper we investigated this function when $\mathcal{R}$ is "H-free", "acyclic", or "having connected components of order at most r".

The most interesting open question regarding H-free transversals is the case of cliques, in particular triangle-free transversals. Currently we only know $\frac{3}{4}d \leq p(d, K_3) \leq d$. For regular H we conjecture that our construction is optimal. For non-regular H we don't even have a conjecture.

Our other most important problem is the asymptotic determination of $p(d, r)$ for any fixed r, when d tends to infinity. This problem is already open for $r = 2$. The lone existing lower bound [13] makes even $p(d, 2) = d$ a possibility. This in fact was shown to be true for $d = 2$ ([13]). The smallest unknown case is $p(3, 2)$ which is either 3 or 4. The question is whether every partition of the vertex set of a 3-regular graph into subsets of size 3 allows for a transversal inducing only a matching.

An interesting line of research is to investigate the limits of the triangulation-method of Aharoni and Haxell more thoroughly, i.e., to decide whether Corollary 2.9 (or Theorem 2.5) is optimal. Let us formulate a special case of this problem more precisely. Suppose C is a constant, $5/4 \leq C < 3/2$. Given any G-labeled triangulation of S^{m-1} containing no ruined 2-simplex, does there exist an extension into a B^m with no ruined 2-simplex, provided $|V(G)| > Cmd$? In other words, what is the smallest number of vertices in a d-regular graph, which guarantees the m-connectedness of the induced matching complex.

We know that $p(d, forest) = d$ but the graphs showing the lower bound have parallel edges. It would be interesting to find (at least asymptotically) the minimum class size for a vertex partition of *simple* d-regular graphs that ensures the existence of a cycle-free transversal. This value is between $\lceil \frac{3}{4}d \rceil$ and d.

The numbers Δ_r for odd $r \geq 7$ are extremely intriguing. Currently it is known that

$$\frac{r}{2(r-1)} \leq \Delta_r \leq \Delta_{r-1} = \frac{r-1}{2(r-2)}.$$

The fact that $\Delta_2 = \Delta_3$ (or $\Delta_4 = \Delta_5$) means that the freedom of an extra class of size n besides the first two (or the first four) does not help to prevent an independent transversal. It would be very interesting to decide whether this phenomenon is just an artifact of the parameters being too small or there is something deeper going on implying $\Delta_{2l} = \Delta_{2l+1}$ for every l. We vote for the latter.

Acknowledgement. We would like to thank Péter Csorba and Gábor Moussong for fruitful discussions. Péter Csorba greatly simplified our first proof of the bound on the connectedness of the complexes discussed in Constructions 3.1 and 3.2.

Note added in proof. The very last conjecture of the last section, namely that $\Delta_{2l} = \Delta_{2l+1}$ for every l, was proved by P. Haxell and the first author [12].

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