Extremal Hypergraphs for Ryser's Conjecture

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Abstract

Ryser's Conjecture states that any r-partite r-uniform hypergraph has a vertex cover of size at most $r-1$ times the size of the largest matching. For $r = 2$, the conjecture is simply König's Theorem and every bipartite graph is a witness for its tightness. The conjecture has also been proven for $r = 3$ by Aharoni using topological methods, but the proof does not give information on the extremal 3-uniform hypergraphs. Our goal in this paper is to characterize those hypergraphs which are tight for Aharoni's Theorem.

Our proof of this characterization is also based on topological machinery, particularly utilizing results on the (topological) connectedness of the independence complex of the line graph of the link graphs of 3-uniform Ryser-extremal hypergraphs. We use this information to nail down the elements of a structure we call *home-base hypergraph*. While there is a single minimal home-base hypergraph with matching number k for every positive integer $k \in \mathbb{N}$, home-base hypergraphs with matching number k are far from being unique. There are infinitely many of them and each of them is composed of k copies of two different kinds of basic structures, whose hyperedges can intersect in various restricted, but intricate ways.

Our characterization also proves an old and wide open strengthening of Ryser's Conjecture, due to Lovász, for the 3-uniform extremal case, that is, for hypergraphs with $\tau = 2\nu$.

1   Introduction

A hypergraph $\mathcal{H}$ is a pair (V, E), where $V = V(\mathcal{H})$ is the set of *vertices*, and $E = E(\mathcal{H})$ is a **multiset** of subsets of vertices called the *edges* of $\mathcal{H}$. The number of times a subset $e \subseteq V$ appears in E is called the *multiplicity* of e. If

the cardinality of every edge is r, we call $\mathcal{H}$ an *r-graph*. A 2-graph is called a *graph*. In our paper we mostly have no restriction on the multiplicity of edges; whenever we want to assume that each multiplicity is at most 1, we will explicitly say *simple hypergraph*, *simple r-graph*, or *simple graph*. An edge $e \in E$ is called *parallel* to an edge $f \in E$ if their underlying vertex subsets are the same. In particular, every edge is parallel to itself.

Let $\mathcal{H}$ be a hypergraph. A *matching* in $\mathcal{H}$ is a set of disjoint edges of $\mathcal{H}$, and the *matching number*, $\nu(\mathcal{H})$, is the size of the largest matching in $\mathcal{H}$. If $\nu(\mathcal{H}) = 1$, then $\mathcal{H}$ is called *intersecting*. A *vertex cover* of $\mathcal{H}$ is a set of vertices which intersects every edge of $\mathcal{H}$. The size of the smallest vertex cover is called the *vertex cover number* of $\mathcal{H}$ and is denoted by $\tau(\mathcal{H})$. It is immediate to see that if $\mathcal{H}$ is r-uniform, then the following bounds always hold:

$$\nu(\mathcal{H}) \leq \tau(\mathcal{H}) \leq r\nu(\mathcal{H}).$$

Both inequalities are easily seen to be tight for general hypergraphs. Ryser's Conjecture [21], which appeared first in the late 1960's, states that the upper bound can be lowered by considering only r-partite hypergraphs. An r-graph is called *r-partite* if its vertices can be partitioned into r parts, called *vertex classes*, such that every edge intersects each vertex class in exactly one vertex.

Conjecture 1 (Ryser's Conjecture)**.** *If $\mathcal{H}$ is an r-partite r-graph, then*

$$\tau(\mathcal{H}) \leq (r - 1)\nu(\mathcal{H}).$$

Around the same time a much stronger conjecture was made by Lovász [16]. The conjecture states that not only do we have a vertex cover of size $(r-1)\nu(\mathcal{H})$, but we can obtain it by repeatedly reducing the matching number by one with the removal of $r - 1$ vertices.

Conjecture 2 (Lovász Conjecture)**.** *In every r-partite r-graph there exist $r - 1$ vertices whose deletion reduces the matching number.*

These conjectures turned out to be extremely difficult to attack. Ryser's Conjecture is solved completely only for $r = 2$ and 3, and a few partial results exists for values of $r \leq 9$. The Lovász Conjecture is open even for $r \geq 3$. When $r = 2$, both conjectures are implied by the well known König's Theorem. For $r = 3$ Ryser's Conjecture was solved by Aharoni via a beautiful argument [4] that relied on a topological statement of [9]. The conjecture is wide open for $r \geq 4$. Haxell and Scott [15] have proven that for $r = 4, 5$ there is an $\epsilon > 0$ such that $\tau(\mathcal{H}) \leq (r - \epsilon)\nu(\mathcal{H})$ for any r-partite r-graph $\mathcal{H}$. The conjecture has been proven for intersecting hypergraphs when $r \leq 5$ by Tuza ([24], [25]), with $r \geq 6$ still open. Recently Francetić, Herke, McKay, and Wanless [12] proved Ryser's Conjecture for linear intersecting hypergraphs (i.e., hypergraphs where any two hyperedges intersect in exactly one vertex) when $r \leq 9$. Fractional versions of the conjecture have also been studied, and it was shown by Füredi [13] that $\tau^* \leq (r - 1)\nu$, and shown by Lovász [16] that $\tau \leq \frac{r}{2}\nu^*$, where τ^* and ν^* are the fractional vertex cover and matching numbers, respectively. Aharoni and

Berger [6] also formulated a generalization of the conjecture to matroids, which has been partially solved in a special case by Berger and Ziv [10].

A *Ryser-extremal hypergraph* is an r-partite r-graph $\mathcal{H}$ satisfying $\tau(\mathcal{H}) = (r-1)\nu(\mathcal{H})$. The set of r for which they exist is not well understood. General constructions are known only for those r for which $r-1$ is a prime power [21] or $r-2$ is a prime power [1, 5, 2, 12]. Mansour, Song, and Yuster [17] have found bounds on the minimum number of edges for an intersecting Ryser-extremal r-graph, and determined the exact numbers for the cases $r \leq 5$. Subsequently Aharoni, Barát and Wanless [5] found these values for $r = 6, 7$ (see also [2]).

Aharoni's proof of Ryser's Conjecture for 3-graphs is based on studying the bipartite graphs that arise as 2-*links*. The *link* of an r-graph $\mathcal{H}$ is the $(r-1)$-graph consisting of all $(r-1)$-subsets (with multiplicity) of the edges of $\mathcal{H}$. Correspondingly, a plausible approach to Ryser's Conjecture for r-graphs could be via understanding their $(r-1)$-uniform links. To understand better the situation for the 4-uniform Ryser's Conjecture, having structural information on the 3-uniform links would be particularly helpful. Aharoni's proof however does not provide information on the 3-graphs which are extremal for his theorem. Our aim is to give a complete characterization of them.

To this end we describe a family of hypergraphs we call *home-base hypergraphs*. The 3-uniform *loose* 3-*cycle* C is the (unique) linear intersecting hypergraph with three edges spanning six vertices (see Figure 1). Clearly $\tau(C) = 2 = 2\nu(C)$, so C is Ryser-extremal. To build a home-base hypergraph with matching number k, one first takes k disjoint copies of C. Then for some number m, $0 \leq m \leq k$, of these copies, one adds an edge containing the three degree-1 vertices of the copy (this is called a *truncated Fano plane F*). For each of the remaining $k - m$ copies of C, one can choose to add any number of additional edges e, each of which intersects C in at least two of its degree-2 vertices. (The third vertex of e can be an arbitrary element of the remaining vertex class.) The hypergraphs obtained by this procedure (after possibly adding parallel edges) are the home-base hypergraphs.
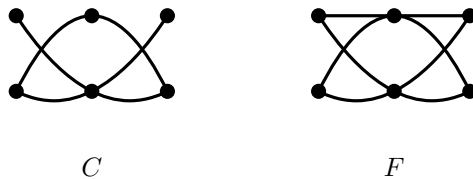

Figure 1: The loose 3-cycle C and the truncated Fano plane F

It is not very difficult to see that home-base hypergraphs are Ryser-extremal. This will be shown in Section 7. The main goal of our present paper is to prove that the converse is also true.

Theorem 1.1. *Let $\mathcal{H}$ be a 3-partite 3-graph. Then $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ if and only if $\mathcal{H}$ is a home-base hypergraph.*

With respect to the Lovász Conjecture one could speculate that among 3-partite 3-graphs with given matching number k, the hardest instances might

be those with vertex cover number as large as possible, i.e. $2k$. Indeed, for these hypergraphs the conjecture will have to be tight in every step. As it turns out the Lovász Conjecture for Ryser-extremal 3-graphs is a relatively simple consequence of our characterization in Theorem 1.1.

Corollary 1.2. *Let $\mathcal{H}$ be a 3-partite 3-graph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. Then there exists $\nu(\mathcal{H})$ pairwise disjoint pairs of vertices such that the removal of the union of any k, $1 \le k \le \nu(\mathcal{H})$, of these pairs decreases the matching number by k.*
In particular there exist two vertices, the removal of which reduces the matching number of $\mathcal{H}$.

On the proof. The proof of Theorem 1.1 is quite involved. To perhaps justify this, we observe that while home-base hypergraphs have a restricted structure, they are far from being unique: for any given positive integer $k \in \mathbb{N}$, there are infinitely many home-base hypergraphs with matching number k. In a typical extremal combinatorial problem, the greater the number of extremal configurations, the less likely that a purely combinatorial argument will lead to a solution, since a proof eventually must consider all extremal structures, at least implicitly. Our theorem establishes Aharoni's Theorem to be one of the few interesting extremal combinatorial problems with infinitely many extrema, for which the full characterization of the extremal structures is possible.

Sometimes, the difficulties posed by multiple extremal examples can be mitigated by realizing that the combinatorial problem, or rather its extremal structures, hide the features and concepts of another mathematical discipline in the background. In such cases, the simplest, or most efficient descriptions of extremal structures are not necessarily combinatorial, but might have to be formulated in another language, which could be algebraic, probabilistic, or, as in the present paper, topological.

Aharoni [4] invoked topological considerations (about the line graphs of the links of 3-partite 3-graphs) in order to prove Ryser's Conjecture for 3-graphs and it was this way that he could overcome the combinatorial difficulty of having infinitely many extremal structures. Our main task, the characterization of Ryser-extremal 3-graphs, will also start out by investigating the link graphs using topological machinery. To this end, we first consider a natural extremal graph theoretic problem of topological sort, concerning the minimization of the (topological) connectedness of the independence complex of graphs in terms of their dimension. We observe that the known lower bound $\frac{\dim(\mathcal{M}(G))}{2} - 2$ on the connectedness of the matching complex $\mathcal{M}(G)$ of bipartite graphs G is tight for the links of Ryser-extremal 3-graphs. We then characterize those bipartite graphs whose line graphs are extremal for the stated extremal graph theoretic problem, which facilitates the combinatorial description of the links and eventually of all Ryser-extremal 3-graphs.

Structure of the paper. Our paper is divided into two parts, the first focused on graphs, and the second on hypergraphs. In Part I we develop the necessary knowledge about the structure of the link graphs of Ryser-extremal 3-graphs.

Part II contains all our arguments involving 3-graphs. This includes in Section 7 the proof of the easy direction of Theorem 1.1 and the proof of Corollary 1.2. The rest of Part II is concerned with the structural hypergraph-theoretic part of the difficult direction of Theorem 1.1, where we use the information on the line graph of the link graphs to nail down the elements of the home-base hypergraph structure. In Section 13 we make a few comments and state several open questions.

Part I
Line Graphs of Bipartite Graphs

2   An Extremal Problem for Bipartite Graphs

In this section we define a graph parameter of topological nature which will be our main tool to describe the line graphs of the link graphs of Ryser-extremal 3-graphs.

A *(geometric) simplicial complex* $\mathcal{K}$ is a family of simplices in $\mathbb{R}^N$ such that (1) if τ is a face of a simplex $\sigma \in \mathcal{K}$ then $\tau \in \mathcal{K}$ and (2) if $\sigma, \sigma' \in \mathcal{K}$ then $\sigma \cap \sigma'$ is a face of both σ and σ'. We write $\|\mathcal{K}\|$ for the body of $\mathcal{K}$, i.e., the union of its simplices.

An *(abstract) simplicial complex* $\mathcal{C}$ is a simple hypergraph that is closed under taking subsets. The simple hypergraph consisting of the vertex sets of simplices of a geometric simplicial complex $\mathcal{K}$ (called the vertex scheme of $\mathcal{K}$) is an abstract simplicial complex. A function $\ell : V(\mathcal{C}_1) \to V(\mathcal{C}_2)$ between two simplicial complexes $\mathcal{C}_1, \mathcal{C}_2$ is called a *simplicial map* if for every $\sigma \in \mathcal{C}_1$ the set $\ell(\sigma) \in \mathcal{C}_2$. The pair $(\mathcal{C}_1, \ell)$ is called a $\mathcal{C}_2$-*labeled simplicial complex.*

The k-dimensional solid ball is denoted by B^k and its boundary, the $(k-1)$-dimensional sphere, by S^{k-1}. The boundary S^{-1} of B^0 is just the empty set.

A *triangulation* of a topological space X is a geometric simplicial complex whose body is homeomorphic to X. Throughout this paper when talking about a triangulation of the sphere S^{k-1} or B^k we mean PL-triangulation, where PL stands for piecewise linear. This technical property is needed to assure certain key properties of triangulations (c.f. the excellent survey of Björner [11]).

Given a $\mathcal{C}$-labeled triangulation $(\mathcal{T}, \ell)$ of S^{k-1}, a *filling* of $(\mathcal{T}, \ell)$ is a $\mathcal{C}$-labeled triangulation $(\mathcal{T}', \ell')$ of B^k whose boundary is $\mathcal{T}$ and $\ell' : V(\mathcal{T}') \to V(\mathcal{C})$ is a simplicial map with $\ell'|_{V(\mathcal{T})} = \ell$.

Our main parameter is defined as follows.

Definition 2.1. For a simplicial complex $\mathcal{C}$ we define the function $\eta(\mathcal{C})$ as the maximum d such that, for every $0 \leq k \leq d-1$, every $\mathcal{C}$-labeled triangulation of S^{k-1} has a filling.

Note that $\eta(\mathcal{C}) \geq 1$ means that $\mathcal{C}$ is non-empty. We remark that $\eta(\mathcal{C})$ is a a sort of "shifted connectedness" of the complex $\mathcal{C}$, that is $\mathcal{C}$ as a topological

space is $(\eta(\mathcal{C}) - 2)$-connected, but not $(\eta(\mathcal{C}) - 1)$-connected (c.f. Proposition 2.8 in [23]). This parameter was introduced by Aharoni and Berger [6].

For a graph H, we define the *independence complex* $\mathcal{I}(H)$ to be the abstract simplicial complex on the vertices of H whose simplices are the independent sets of H. For example the independence complex of the complete bipartite graph $K_{m,n}$ is the disjoint union of an $(m-1)$-dimensional and an $(n-1)$-dimensional simplex. Then $\eta(\mathcal{I}(K_{m,n})) = 1$, since the labeling of the two points of S^0 with vertices from opposite sides of $K_{m,n}$ does not have a filling.

One of the basic parameters of a simplicial complex, be it abstract or geometric, is its *dimension* $\max\{|V(\sigma)| - 1 : \sigma \in \mathcal{C}\}$. The value of η for an arbitrary simplicial complex, or even for an arbitrary graph's independence complex can be arbitrarily small while its dimension is large: just consider the complete bipartite graph $K_{d+1,d+1}$, having an independence complex with dimension d and $\eta(\mathcal{I}(K_{d+1,d+1})) = 1$.

Comparing dimension and η becomes more interesting if we introduce restrictions on the graphs we consider. For *line graphs* for example, a lower bound on η in terms of the dimension is implicit in the work of Aharoni and Haxell [9]. The *line graph* $L(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the simple graph $L(\mathcal{H})$ on the vertex set $E(\mathcal{H})$ with $e, f \in V(L(\mathcal{H}))$ adjacent if $e \cap f \neq \emptyset$. The independence complex of the line graph of a hypergraph $\mathcal{H}$ is called the *matching complex* and is denoted by $\mathcal{M}(\mathcal{H})$. Note that the dimension of $\mathcal{M}(\mathcal{H})$ is $\nu(\mathcal{H}) - 1$.

Theorem 2.2. *For every r-graph $\mathcal{G}$ we have*

$$\eta(\mathcal{M}(\mathcal{G})) \geq \frac{|\nu(\mathcal{G})|}{r}.$$

Though not stated explicitly there, a proof of Theorem 2.2 (for simplices) can be read out of [9]. In fact we will use the same idea to prove Lemma 4.2, and in Remark 4.3 we also include the proof of Theorem 2.2.

We begin our study of Ryser-extremal 3-graphs with their link graphs.

Definition 2.3. Let $\mathcal{H}$ be a 3-partite 3-graph with parts V_1, V_2, and V_3. Let $S \subseteq V_i$ for some $= 1, 2, 3$. Then the *link graph* $\mathrm{lk}_{\mathcal{H}}(S)$ is the bipartite graph with vertex classes V_j and V_k (where $\{i, j, k\} = \{1, 2, 3\}$) whose edge multiset is $\{e \setminus V_i : e \in E(\mathcal{H}), e \cap V_i \subseteq S\}$.

Note that a pair of vertices appears as an edge in $\mathrm{lk}_{\mathcal{H}}(S)$ with the same multiplicity as the number of edges in $\mathcal{H}$ that contain it together with a vertex from S.

The link graphs of Ryser-extremal 3-graphs attest that Theorem 2.2 is optimal for $r = 2$, that is, among bipartite graphs they minimize η of the matching complex.

Theorem 2.4. *If $\mathcal{H}$ is a 3-partite 3-graph with vertex classes V_1, V_2, and V_3, such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, then for each i we have*

  (i) $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_i)\right)\right) = \nu(\mathcal{H})$.

(ii) $\nu(\mathrm{lk}_{\mathcal{H}}(V_i)) = \tau(\mathcal{H})$.

In particular

$$\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_i)\right)\right) = \frac{\nu(\mathrm{lk}_{\mathcal{H}}(V_i))}{2}. \tag{2.1}$$

We prove Theorem 2.4 in Section 3. On the way, we give a reformulated proof of Aharoni's Theorem [4]. A sort of converse of Theorem 2.4 is also true: *every* bipartite graph which is optimal for Theorem 2.2 is the link of some Ryser-extremal 3-graph. Since this direction is not needed for the proof of Theorem 1.1 we relegate its proof to Section 13.1, the conclusions.

In the main theorem of Part I, proven in Section 4, we characterize those bipartite graphs which are extremal for Theorem 2.2 and hence we also obtain valuable structural information about the link graphs of Ryser-extremal 3-graphs.

Theorem 2.5. *Let G be a bipartite graph. Then $\eta\left(\mathcal{M}\left(G\right)\right) = \frac{\nu(G)}{2}$ if and only if G has a collection of $\nu(G)/2$ pairwise vertex-disjoint subgraphs, each of them a C_4 or a P_4, such that every edge of G is parallel to an edge of one of the C_4's or is incident to an interior vertex of one of the P_4's.*

To be precise, in Part I we will in fact only prove the "only if" direction of this theorem. The other direction will be proven in Section 13.1, as again it is not necessary for Theorem 1.1.

Structure of Part I. In Section 3 we give the proof of Theorem 2.4. In Section 4 we prove Theorem 2.5 and also comment on the proof of Theorem 2.2. In Section 5 we define the notion of good sets, which will turn out to be very useful to have in one of the link graphs of a Ryser-extremal 3-graph. In the main theorem of Section 5 we show that the lack of good sets in a bipartite graph imposes very strong restrictions on its structure. This will be crucial for our proof of Theorem 1.1 in Part II.

3   Link Graphs

In this section we prove Theorem 2.4 and derive a reformulated proof of Aharoni's Theorem.

The proof of Theorem 2.4 and hence that of 2.5 needs a special case of a theorem of Aharoni and Berger [6], which is a deficiency version of a statement implicit in [9]. It makes a direct connection between the size of the largest hypergraph matching and η of the link. Aharoni's original proof in [4] uses a weaker combinatorial deficiency theorem that does not directly imply the upper bound on η that we will require.

Theorem 3.1 ([6]). *Let $d \geq 0$ be an integer and let $\mathcal{H}$ be a 3-uniform 3-graph with vertex classes V_1, V_2, and V_3. If we have that $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(S)\right)\right) \geq |S| - d$ for every $S \subseteq V_i$, then $\nu(\mathcal{H}) \geq |V_i| - d$.*

Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3. We aim to show that $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$. To do this, we will consider the link graph. We will use the vertex cover number of $\mathcal{H}$ to find a lower bound on η of the matching complex of the link graphs, and we will use the matching number of $\mathcal{H}$ to find an upper bound for at least one link. Combining these bounds will yield the desired inequality $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$.

Proposition 3.2. *Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3. Then for each $i \in \{1, 2, 3\}$ we have the following:*

(i) For all $S \subseteq V_i$ we have

$$\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(S)\right)\right) \geq \frac{\tau(\mathcal{H}) - (|V_i| - |S|)}{2}.$$

(ii) If $\nu(\mathcal{H}) < |V_i|$, then there is some $S \subseteq V_i$ such that

$$\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(S)\right)\right) \leq \nu(\mathcal{H}) - (|V_i| - |S|).$$

(iii) For every $S \subseteq V_i$ for which the inequality in (ii) holds we have

$$|S| \geq |V_i| - (2\nu(\mathcal{H}) - \tau(\mathcal{H})).$$

Proof. Let $S \subseteq V_i$. We construct a vertex cover T_S of $\mathcal{H}$ by taking the vertices in $V_i \setminus S$ and a minimum vertex cover of $\mathrm{lk}_{\mathcal{H}}(S)$. This is clearly a vertex cover of $\mathcal{H}$ because any edge not incident to S intersects $V_i \setminus S$ and any edge incident to S induces an edge in the link of S, and hence intersects the vertex cover of the link. We have $|T_S| = |V_i| - |S| + \tau(\mathrm{lk}_{\mathcal{H}}(S))$, and since this is a vertex cover, we thus have

$$|V_i| - |S| + \tau(\mathrm{lk}_{\mathcal{H}}(S)) \geq \tau(\mathcal{H}) \tag{3.1}$$

for all subsets $S \subseteq V_i$. By König's Theorem, we have $\tau(\mathrm{lk}_{\mathcal{H}}(S)) = \nu(\mathrm{lk}_{\mathcal{H}}(S))$. We therefore have a lower bound on the matching number of the link graph, and so by Theorem 2.2, we have

$$\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(S)\right)\right) \geq \frac{\nu(\mathrm{lk}_{\mathcal{H}}(S))}{2} \geq \frac{\tau(\mathcal{H}) - (|V_i| - |S|)}{2},$$

which is the inequality in statement (i).

Now we want to show that the inequality in statement (ii) holds for some S. Suppose to the contrary that for every $S \subseteq V_i$ we had $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(S)\right)\right) \geq \nu(\mathcal{H}) - (|V_i| - |S|) + 1 = |S| - (|V_i| - \nu(\mathcal{H}) - 1)$. Then we can apply Theorem 3.1 with $d = |V_i| - \nu(\mathcal{H}) - 1 \geq 0$ to get that $\nu(\mathcal{H}) \geq |V_i| - (|V_i| - \nu(\mathcal{H}) - 1) = \nu(\mathcal{H}) + 1$, which is a contradiction. Thus some $S \subseteq V_i$ must indeed satisfy the inequality in (ii).

Now consider such an S. Combining the inequalities in (i) and (ii), we get

$$\frac{\tau(\mathcal{H}) - (|V_i| - |S|)}{2} \leq \nu(\mathcal{H}) - (|V_i| - |S|),$$

from which the inequality in (iii) follows after some rearranging. $\qquad\square$

Note that Aharoni's Theorem follows in one line from the above proposition: there is an $S \subseteq V_i$ such that $|S| \geq |V_i| - (2\nu(\mathcal{H}) - \tau(\mathcal{H}))$, and hence

$$\tau(\mathcal{H}) + |V_i| - |S| \leq 2\nu(\mathcal{H}).$$

Since $|V_i| \geq |S|$, we thus have $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$ as desired.

We use Proposition 3.2 to derive the main theorem of this section.

Proof of Theorem 2.4. Applying Proposition 3.2 to $\mathcal{H}$, we see by (iii) that in (ii) equality holds if and only if $S = V_i$ for some i. Combining the inequalities in (i) and (ii) for $S = V_i$ with the fact that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ immediately gives that $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_i)\right)\right) = \nu(\mathcal{H})$, showing part (i) of Theorem 2.4. This gives the following chain of inequalities:

$$\frac{\tau(\mathcal{H})}{2} = \nu(\mathcal{H}) = \eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_i)\right)\right)$$
$$\geq \frac{\nu(\mathrm{lk}_{\mathcal{H}}(V_i))}{2} = \frac{\tau(\mathrm{lk}_{\mathcal{H}}(V_i))}{2}$$
$$\geq \frac{\tau(\mathcal{H})}{2},$$

where the first inequality is valid because of Theorem 2.2, the equality following it is König's Theorem, and the last inequality is just equation (3.1) for $S = V_i$. It follows that every inequality is actually an equality, from which part (ii) of Theorem 2.4 follows.

From parts (i), (ii), and the fact that $\nu(\mathcal{H}) = \frac{\tau(\mathcal{H})}{2}$, it follows that the link graphs $\mathrm{lk}_{\mathcal{H}}(V_i)$ of a Ryser-extremal 3-graph $\mathcal{H}$ must be extremal for Theorem 2.2:

$$\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_i)\right)\right) = \frac{\nu(\mathrm{lk}_{\mathcal{H}}(V_i))}{2}.$$

$\square$

4 The CP-Decomposition Theorem

In this section we prove the direction of Theorem 2.5 that is relevant to the proof of Theorem 1.1. The proof of the converse will be addressed in Section 13.

Before beginning the proof we need a couple of preliminary definitions and statements about triangulations of the sphere and the ball. Given a triangulation $\mathcal{T}$ of S^{k-1} and a point v in the interior of a $(k-1)$-dimensional simplex $\sigma \in \mathcal{T}$, we define the triangulation $\mathcal{T}_v^*$ of S^{k-1} by removing σ from $\mathcal{T}$ and adding the simplices spanned by $\{v\} \cup \tau$ for all proper faces τ of σ. (This is just the stellar subdivision of $\mathcal{T}$ with center at v.)

Proposition 4.1. *Let $\mathcal{T}$ be a triangulation of S^{k-1} and let H be a graph. Suppose ℓ is an $\mathcal{I}(H)$-labeling of $\mathcal{T}_v^*$, such that the restriction $\ell|_{V(\mathcal{T})}$ is an $\mathcal{I}(H)$-labeling of $\mathcal{T}$. If there is a filling of $(\mathcal{T}_v^*, \ell)$, then there exists a filling of the restriction $(\mathcal{T}, \ell|_{V(\mathcal{T})})$.*

Proof. Note that a map into $V(\mathcal{I}(H))$ is an $\mathcal{I}(H)$-labeling if and only if the labels of any 1-simplex are not adjacent in H. Let $(\mathcal{U}, \tilde{\ell})$ be a filling of $(\mathcal{T}_v^*, \ell)$. Move the point v slightly into the interior of $||\mathcal{T}_v^*||$, forming a new k-simplex with the vertices of σ. This is a filling of $(\mathcal{T}, \ell|_{V(\mathcal{T})})$, since its 1-skeleton is identical (as a graph) to that of $(\mathcal{U}, \tilde{\ell})$. $\square$

Note that the assumption that the restriction $\ell|_{V(\mathcal{T})}$ is an $\mathcal{I}(H)$-labeling of $\mathcal{T}$ is superfluous if $k \geq 3$.

The next lemma combines the essence of the proof of Theorem 2.2 (implicit in [9]), with a construction of Aharoni, Chudnovsky and Kotlov [8]. For a graph H and subset $I \subseteq V(H)$, we say that a simplex σ with vertex labels from $V(H)$ is *I-blocking* if each element of I is adjacent (in H) to some label of σ.

Lemma 4.2. *Let H be a (loopless) graph and $I \subseteq V(H)$ an independent set. Any $\mathcal{I}(H)$-labeled triangulation of S^{k-1} with no I-blocking simplices has a filling.*

Proof. Let $(\mathcal{T}, \ell)$ be an $\mathcal{I}(H)$-labeled triangulation of S^{k-1} with no I-blocking simplices. By Lemma 1.1 of [8] (see also Lemma 2.2 of [9]) there is a triangulation $\mathcal{T}'$ of B^k with boundary $\mathcal{T}$, such that

(1) for every $x \in V(\mathcal{T}') \setminus V(\mathcal{T})$, its neighbors in the 1-skeleton of $\mathcal{T}'$ that lie on S^{k-1} form a (possibly empty) simplex σ_x of $\mathcal{T}$

(2) Any 1-simplex of $\mathcal{T}'$ with endpoints in $V(\mathcal{T})$ is also a 1-simplex in $\mathcal{T}$.

We give an $\mathcal{I}(H)$-labeling ℓ' of $\mathcal{T}'$, which extends ℓ. For each interior point $x \in V(\mathcal{T}') \setminus V(\mathcal{T})$ we label x with an arbitrary vertex of I that is not adjacent to any label of σ_x. (Such a vertex exists, since there are no I-blocking simplices in $\mathcal{T}$.) To check that this gives an $\mathcal{I}(H)$-labeling, consider an arbitrary 1-simplex τ of $\mathcal{T}'$. If the vertices x and y of τ are both in $V(\mathcal{T})$ then τ is also a 1-simplex of $\mathcal{T}$ by (2) and hence $\ell'(x)\ell'(y) = \ell(x)\ell(y) \notin E(H)$. If $x \in V(\mathcal{T}') \setminus V(\mathcal{T})$ and $y \in V(\mathcal{T})$, then $\ell'(x)\ell'(y) \notin E(H)$ by our choice of $\ell'(x)$. If both vertices x and y are in the interior $V(\mathcal{T}') \setminus V(\mathcal{T})$, then $\ell'(x)$ and $\ell'(y)$ are both in I, which is an independent set. $\square$

Remark 4.3. To derive the proof of Theorem 2.2, fix a maximum matching M of $\mathcal{G}$. Note that if $k \leq \left\lceil \frac{\nu(\mathcal{G})}{r} \right\rceil - 1$, then $rk < \nu(\mathcal{G}) = |M|$. Therefore no simplex of a triangulation $\mathcal{T}$ of S^{k-1} is big enough to be M-blocking. Indeed, the union of the edges of $\mathcal{G}$ labeling a simplex of $\mathcal{T}$ contains at most rk vertices and hence can intersect at most that many edges of M. Therefore by Lemma 4.2 applied with $H = L(\mathcal{G})$ and $I = M$, for each $k \leq \left\lceil \frac{\nu(\mathcal{G})}{r} \right\rceil - 1$, every $\mathcal{I}(L(\mathcal{G}))$-labeled triangulation of S^{k-1} has a filling, so $\eta(\mathcal{M}(\mathcal{G})) \geq \left\lceil \frac{\nu(\mathcal{G})}{r} \right\rceil$.

Before stating the main theorem of this section, we precisely describe the characterizing structure appearing in Theorem 2.5.

Definition 4.4. Let $k \in \mathbb{N}$, let G be a bipartite graph, and let $M \subseteq E(G)$ be a matching in G of size $2k$. A *CP-decomposition of G with respect to M* is a set $\mathcal{Q} = \mathcal{C} \cup \mathcal{P}$ of k vertex-disjoint subgraphs of G such that

(1) Each $C \in \mathcal{C}$ is isomorphic to C_4 and contains two edges of M.

(2) Each $P \in \mathcal{P}$ is isomorphic to P_4 and contains two edges of M.

(3) Every edge of G is equal to or parallel to an edge of some $C \in \mathcal{C}$, or is incident to an interior vertex of some $P \in \mathcal{P}$.

Note that it follows that $k = |M| / 2$.

We are now ready to state the theorem, which directly implies the "only if" direction of Theorem 2.5 by taking M to be a maximum matching of G.

Theorem 4.5 (CP-Decomposition Theorem). *Let G be a bipartite graph and let $M \subseteq E(G)$ be a matching in G. If $\eta(G) \leq \frac{|M|}{2}$, then G has a CP-decomposition with respect to M.*

Proof. Note that by Theorem 2.2 the matching M must be of maximum size and $\eta(\mathcal{M}(G)) = k$, where $|M| = \nu(G) = 2k$. We plan to apply Lemma 4.2 with $H = L(G)$ and $I = M$.

By the definition of η, there exists a non-fillable $\mathcal{M}(G)$-labeled triangulation of S^{k-1}. Consider the set $\mathcal{S}$ of all such triangulations having the smallest number $q(M) = q$ of M-blocking simplices. By Lemma 4.2, $q \geq 1$.

For $(\mathcal{T}, \ell) \in \mathcal{S}$ each M-blocking simplex σ in $\mathcal{T}$ defines a collection $\mathcal{Q}_\sigma$ of k pairwise disjoint P_4's and C_4's in G as follows. Let $L_\sigma \subseteq E(G)$ be the set of labels of σ. Since any edge of G is adjacent to at most two edges of M, σ must have k distinct labels and hence its dimension is $k-1$. Then $L_\sigma \cup M$ forms k pairwise disjoint P_4's in G. Let $\mathcal{P}_\sigma$ denote the set of those that are an induced P_4 in G and let $\mathcal{C}_\sigma$ denote the set of those C_4's that contain the remaining P_4's. Let $\mathcal{Q}_\sigma = \mathcal{P}_\sigma \cup \mathcal{C}_\sigma$.

We say an edge of G *spoils* $\mathcal{Q}_\sigma$ if it is not induced by $V(C)$ for any $C \in \mathcal{C}_\sigma$ and it is not incident to an inner vertex of P for any $P \in \mathcal{P}_\sigma$. Observe that if no edge spoils $\mathcal{P}_\sigma$ then $\mathcal{Q}_\sigma$ is a CP-decomposition. Hence we may assume that each $\mathcal{Q}_\sigma$ is spoiled by some edge.

Among all $(\mathcal{T}, \ell)$ in $\mathcal{S}$, all M-blocking simplices $\sigma \in \mathcal{T}$, and all edges f spoiling $\mathcal{Q}_\sigma$, we choose $(\mathcal{T}, \ell, \sigma, f)$ to minimize the number r of endpoints of f in $V(L_\sigma)$. Clearly $r \in \{0, 1, 2\}$.

Case 1. $r = 0$.

We will arrive to a contradiction with $(\mathcal{T}, \ell)$ being non-fillable as follows. Let v be an interior point of σ and consider the triangulation $\mathcal{T}_v^*$ of S^{k-1} and extend ℓ by labeling v with $f =: \ell(v)$. Note that this extension is a $\mathcal{M}(G)$-labeling of $\mathcal{T}_v^*$, since $r = 0$, and all new 1-simplices are labeled with f and an element of L_σ.

We claim that the number of M-blocking $(k-1)$-simplices in $(\mathcal{T}_v^*, \ell)$ is $q-1$. For this observe that the M-blocking simplex σ is not in $\mathcal{T}_v^*$, while none of the new $(k-1)$-simplices are M-blocking. Indeed, that could only happen if f blocks the very same two edges of M that are not blocked by τ for some $(k-2)$-face τ of σ. Since f is not induced by $V(C)$ for any $C \in \mathcal{C}_\sigma$, this is not possible.

Since the $\mathcal{M}(G)$-labeled triangulation $(\mathcal{T}_v^*, \ell)$ of S^{k-1} has only $q-1$ M-blocking simplices, it must have a filling. Then by Proposition 4.1 the triangulation $(\mathcal{T}, \ell)$ also has a filling, which is the promised contradiction.

Case 2. $r \geq 1$.

Let $e \in L_\sigma$ be such that there exists a vertex $x \in e \cap f$. Since f spoils $\mathcal{Q}_\sigma$ we know $x \in V(C)$ for some $C \in \mathcal{C}_\sigma$ and the other endpoint of f is not in $V(C)$.

Again, let v be an interior point of σ and consider the triangulation $\mathcal{T}_v^*$ of S^{k-1}, but this time extend ℓ by labeling v with the edge $e' =: \ell(v)$ that is opposite to the L_σ-edge e in C. Note that the extension is an $\mathcal{M}(G)$-labeling since all new 1-simplices are labeled with e' and an element of L_σ. Then $(\mathcal{T}_v^*, \ell)$ has exactly q M-blocking $(k-1)$-simplices (since σ is no longer in the triangulation and among the k newly introduced $(k-1)$-simplices the only M-blocking one is σ' spanned by the vertex set $V(\sigma) \setminus \{v'\} \cup \{v\}$, where $\ell(v') = e$.) By the contrapositive of Proposition 4.1 the triangulation $(\mathcal{T}_v^*, \ell)$ has no filling.

Thus $(\mathcal{T}_v^*, \ell) \in \mathcal{S}$. Observe that $\mathcal{Q}_\sigma = \mathcal{Q}_{\sigma'}$ since e and e' define the same C_4. Therefore f spoils $\mathcal{Q}_{\sigma'}$. But $L_{\sigma'} = L_\sigma \setminus \{e\} \cup \{e'\}$ shares fewer than r vertices with f, since $x \notin V(L_{\sigma'})$ and the other endpoint of f is not in C and hence not a vertex of e'. This contradicts our choice of $(\mathcal{T}, \ell, \sigma, f)$. □

5 Good Sets

This section introduces the concept of good sets, which (as we will later see in Part II) will help us find the substructure we need in our Ryser-extremal hypergraph in order to prove our characterization theorem by induction. The main result of this section implies that we can find good sets inside our link graphs in several cases, and hence if there are no good sets, we will know that the link graphs must have a certain form.

We start with a graph-theoretic definition, which will form the backbone of the definition of a good set.

Definition 5.1. Let G be a bipartite graph with vertex classes A and B. A subset $X \subseteq B$ is called *decent* if it satisfies the following conditions:

(1) $\nu(G) = |N(X)| + |B \setminus X|$,

(2) For every $x \in X$ and $y \in N(x)$ the edge xy participates in a maximum matching of G.

Note that by condition (1) it follows that $|N(X)| \leq |X|$.

Lemma 5.2. *Let G be a bipartite graph with vertex classes A and B, and let M be a maximum matching in G. Let $X_0 \subseteq B$ be the set of M-unsaturated vertices in B, and let X be the set of vertices in B reachable on an M-alternating path from X_0 (including X_0). Then X is decent, and $|N(X)| = |X| - |X_0|$.*

Proof. Let $Y = N(X)$. Then Y is the set of vertices in A reachable on an M-alternating path from X_0. To see this, consider a vertex $x \in X$ and a neighbor

$y \in N(x)$. Either x is unsaturated, in which case $x \in X_0$, so xy is an M-alternating path from X_0 to y, or there is an M-alternating path from X_0 to x, which must end with a matching edge. If y is on this path, we are done. Otherwise, xy is not a matching edge, and hence we can extend our path by the edge xy.

We claim that M saturates Y with (X, Y)-edges. This is because M is maximum, and thus every M-alternating path starting from an unsaturated vertex must end in a saturated vertex, and therefore every vertex of Y is incident to an edge of M. Extending the path by such a matching edge must land us in X by definition. Thus this matching edge is an (X, Y)-edge. Therefore $|N(X)| = |X| - |X_0|$. Since X contains all M-unsaturated vertices, M saturates Y and $B \setminus X$ with distinct edges, and these are clearly all the edges of M. Thus $\nu(G) = |Y| + |B \setminus X|$, so we have (1).

We now show that X satisfies (2). Take an edge $e \in E(G)$ between X and Y. If $e \in M$, then we are done. If it has an M-unsaturated vertex, then it is only adjacent to one matching edge $m \in M$, and so $M \cup \{e\} \setminus \{m\}$ is a maximum matching containing e.

Otherwise, e is adjacent to two matching edges $m, m' \in M$. Since e goes between X and Y, the vertices of m and m' are reachable by an M-alternating path starting from X_0. Without loss of generality, the vertex in $m \cap e$ is in X. So consider an M-alternating path from X_0 which ends at that vertex. Note that its last edge is m. If m' is not in this path, then we can extend the path by e and m'. Switching along this extended path will create a maximum matching containing e (since the path ends at an M-unsaturated vertex). If, however, m' was in the original path, then adding e to the path forms an M-alternating cycle. Switching the matching along the cycle produces the desired matching. Therefore X is decent, as desired. □

Definition 5.3. Let G be a bipartite graph. A subset X of a vertex class of G is called *equineighbored* if X is nonempty and $|N(X)| = |X|$.

Note that if G has a perfect matching, then each vertex class is an equineighbored set (unless G is the empty graph).

Lemma 5.4. *Let G be a bipartite graph with vertex classes A and B and let M be a perfect matching in G. Let $X_0 \subseteq B$, and let X be the set of vertices in B reachable on an M-alternating path from X_0 (including X_0) starting with a non-matching edge. Then X is equineighbored.*

Proof. Let $Y = N(X)$. Since M is a perfect matching, every $y \in Y$ has a partner $x \in B$ matched to it by M. If there is an M-alternating path from X_0 to y starting with an edge not in M, then $x \in X$ because either $x \in X_0 \subseteq X$ or the path can be extended by the matching edge xy. If this holds for every $y \in Y$, then there is a matching from Y to X, so that $|Y| \leq |X|$, from which $|Y| = |X|$ follows by Hall's Theorem.

Therefore, we need to show that every $y \in Y$ can be reached from X_0 by an M-alternating path starting with a non-matching edge. Since $y \in N(X)$, it has

a neighbor $x \in X$. By the definition of X, there is such an M-alternating path ending in x. If y is on that path, we are done. Otherwise, xy is not an edge of M (because the path to x ends with the matching edge incident to x), and so the path could be extended by xy, and thus y is on such a path. This concludes the proof. $\square$

Lemma 5.5. *Let G be a bipartite graph with vertex classes A and B, and let M be a perfect matching in G. Let $X \subseteq B$ be a minimal equineighbored set in B. Then X is decent.*

Proof. Since G has a perfect matching, there is a matching saturating B, and since $|X| = |N(X)|$, we have $\nu(G) = |B| = |N(X)| + |B \setminus X|$, which is (1).

We now show that X satisfies (2). Let $Y = N(X)$. Let $x \in X$, $y \in Y$, and let $xy \in E(G)$. Fix a perfect matching M. Because $N(X) = Y$, it must match X to Y. If $xy \in M$, we are done. Otherwise there exist edges $xy', x'y \in M$ adjacent to xy. We claim that these edges participate in an M-alternating cycle with xy, and thus by switching along the cycle we get a new perfect matching which does include xy. To show that this happens, consider all M-alternating paths starting at x' with a non-matching edge. If there is such a path which hits y', then we can extend the path by $y'x$ and xy to give an M-alternating cycle in which xy participates. So assume that no such path hits y'. Let X' be the set of X-vertices which we can hit on such a path. Then X' is a proper ($x \notin X'$) non-empty ($x' \in X'$) equineighbored subset of X by Lemma 5.4 applied with $X_0 = \{x'\}$. This is a contradiction because X was chosen to be minimal. $\square$

Definition 5.6. Let G be a bipartite graph with vertex classes A and B. A subset $X \subseteq B$ is called *good* if it is decent, and if for all $y \in N(X)$ we have $\eta\left(\mathcal{M}\left(G_y\right)\right) > \eta\left(\mathcal{M}\left(G\right)\right)$, where $G_y = G - \{yz \in E(G) : z \in B \setminus X\}$.

Note in particular that if X is good, then $\{yz \in E(G) : z \in B \setminus X\} \neq \emptyset$ for all $y \in N(X)$.

Lemma 5.7. *Let G be a bipartite graph with vertex classes A and B. Suppose $\nu(G) = 2k$ for some integer k and $\eta\left(\mathcal{M}\left(G\right)\right) = k$. If G has no good set in A nor in B, then the following hold:*

(i) G has a perfect matching

(ii) For every minimal equineighbored subset $X \subseteq A$ or $X \subseteq B$ we have $|X| = 2$ and $G[X \cup N(X)]$ is a C_4 (possibly with parallel edges).

Note that the minimality requirement in (ii) is well-defined because by (i) both A and B are equineighbored.

Proof. First, we show that (i) holds. Suppose G does not have a perfect matching. Let M be a maximum matching in G. By assumption, there are some M-unsaturated vertices in $A \cup B$. Without loss of generality assume that at least one of them is in B. Let X_0 be the set of M-unsaturated vertices in B. Consider all the M-alternating paths in G starting from X_0. Let X be the set of

vertices in B reachable on an M-alternating path from X_0 (including X_0), and let $Y = N(X)$. We claim that X is a good subset, which will be a contradiction.

By Lemma 5.2 X is decent, so we must simply check that for all $y \in Y$ we have $\eta\left(\mathcal{M}\left(G_y\right)\right) > \eta\left(\mathcal{M}\left(G\right)\right)$. Let $y \in Y$. Clearly M is still a maximum matching in G_y and X_0 remains the set of M-unsaturated vertices. All of the (X, Y)-edges have been preserved in G_y, so X and Y are still the sets of vertices reachable by an M-alternating path from X_0. Suppose for the sake of contradiction that we had $\eta\left(\mathcal{M}\left(G_y\right)\right) = k$. Then by Theorem 4.5 there is a CP-decomposition $\mathcal{Q}$ of G_y with respect to M. Let I and O be the set of interior and end vertices of the P_4's in $\mathcal{Q}$, respectively. We claim that $Y \subseteq I$ and $X \setminus X_0 \subseteq O$. This leads to a contradiction, as $y \in Y$, being a vertex of I, should be adjacent to another vertex of I (which must be M-saturated by definition), but all M-saturated neighbors of y in G_y are in $X \setminus X_0 \subseteq O$.

To prove our claim let $P = x_0 x_1 \ldots$ be an M-alternating path in G_y starting in X_0. Then by part (3) of the definition of CP-decomposition $x_1 \in I$, and hence its M-neighbor $x_2 \in O$. Given that $x_{2j-1} \in I$ and $x_{2j} \in O$, the edge $x_{2j} x_{2j+1}$ shows that $x_{2j+1} \in I$ by part (3) of the definition of CP-decomposition. This then implies that the M-neighbor $x_{2j+2} \in O$, verifying our claim and completing the proof of part (i).

Now we will show that (ii) holds. Let $X \subseteq B$ be a minimal equineighbored set. We want to show that $|X| = 2$, from which easily follows that the edges incident to X form a C_4 (possibly with parallel edges). Indeed, if X is a minimal equineighbored set of size 2, then its vertices must both have two neighbors (a vertex with only one neighbor would be a proper equineighbored subset, a vertex with more than two neighbors is ruled out by $|N(X)| = 2$, and an isolated vertex is ruled out by the fact that we have a perfect matching), which means they both connect to both neighbors of X, which forms a C_4.

So suppose that $|X| \neq 2$. We will show that X is good. By Lemma 5.5, X is decent, so we must simply check that for all $y \in N(X)$, the graph G_y formed by erasing from G all edges incident to y and not incident to X has the property that $\eta\left(\mathcal{M}\left(G_y\right)\right) \geq k+1$.

Indeed, suppose it did not. We could then apply Theorem 4.5 to get a CP-decomposition of G_y. Note that X is still a minimal equineighbored subset of B in G_y.

Claim. *X does not contain any interior vertex of a P_4 in any CP-decomposition of G_y with respect to any perfect matching.*

Proof. Fix a perfect matching M of G_y, and fix a CP-decomposition $\mathcal{C} \cup \mathcal{P}$ of G_y with respect to M. Let X_0 be the set of interior vertices of the paths $P \in \mathcal{P}$ in X. We will show $X \setminus X_0$ is equineighbored and non-empty, which implies X_0 is empty. Let Y_0 denote the set of endpoints of the paths $P \in \mathcal{P}$ which are partnered with the vertices of X_0 in the matching M. There are no edges between Y_0 and $X \setminus X_0$ since each edge incident to Y_0 must connect to some interior vertex of a P_4 in $\mathcal{P}$. Therefore $|N(X \setminus X_0)| = |N(X)| - |Y_0| = |X \setminus X_0|$. Hence $X \setminus X_0$ is equineighbored. Note that $X \setminus X_0 \neq \emptyset$, as for any path $P \in \mathcal{P}$

having an interior vertex in X_0 does have an end vertex in $X \setminus X_0$ (since the other interior vertex is in $N(X)$, which is fully matched to X via M). □

Claim. *X does not contain any vertices of a C_4 in any CP-decomposition of G_y with respect to any perfect matching.*

Proof. Fix a perfect matching M of G_y, and fix a CP-decomposition $\mathcal{C} \cup \mathcal{P}$ of G_y with respect to M. Let X_0 be the vertices of some 4-cycle $C \in \mathcal{C}$ which are contained in X. Then $X \setminus X_0$ is also equineighbored because the two vertices of C which are adjacent to X_0 are not in the neighborhood of $X \setminus X_0$ as X does not contain any interior vertices of any $P \in \mathcal{P}$ by the previous claim, and the only neighbors of the vertices of C are other vertices of C and interior vertices of paths $P \in \mathcal{P}$ by the definition of a CP-decomposition. Therefore we would remove at least as many vertices from the neighborhood of X as we would remove from X. It follows that if X_0 is nonempty, then $|X_0| = 2$, because if $|X_0| = 1$, then we would have $|N(X \setminus X_0)| < |X \setminus X_0|$, which contradicts the fact that G_y has a perfect matching. Since $|X| \neq 2$, we cannot have $X \setminus X_0 = \emptyset$, so $X \setminus X_0$ is a proper equineighbored subset of X, which is a contradiction to the minimality of X. □

Thus we have shown that X consists entirely of endpoints of P_4's (there are no other types of vertices, since we have a perfect matching). Then y is an interior vertex of some P_4. However, y only has neighbors in X, so this cannot be the case (since every interior vertex of a path is adjacent to another interior vertex). Since we have reached a contradiction, it follows that we must have $\eta(\mathcal{M}(G_y)) \geq k + 1$. Thus X is a good set, which is a contradiction to the conditions of the lemma. Therefore, we must have $|X| = 2$. This proves part (ii) of the lemma. □

Part II
Home-Base Hypergraphs

6   Overview of Part II

The main aim of this part is to prove Theorem 1.1. In Section 7 we define home-base partitions, which is a formulation of the definition of home-base hypergraph more suited to our proof than that given in the introduction. In this section, we also derive Corollary 1.2, and prove the easy direction of Theorem 1.1: that home-base hypergraphs are Ryser-extremal.

The proof of the reverse implication will be done by induction on $\nu(\mathcal{H})$. The case $\nu(\mathcal{H}) = 0$ is trivial, and even the case $\nu(\mathcal{H}) = 1$ is not difficult to check. Much of the work involved in proving the cases $\nu(\mathcal{H}) \geq 2$ consists of finding an appropriate structure to which we can apply induction. That means a subhypergraph $\mathcal{H}_0 \subseteq \mathcal{H}$ which also satisfies $\tau(\mathcal{H}_0) = 2\nu(\mathcal{H}_0)$ and has

$\nu(\mathcal{H}_0) < \nu(\mathcal{H})$. By induction, this will have a home-base partition, but in order to be able to extend this partition to a home-base partition of the whole of $\mathcal{H}$ we will also need the edges outside of $\mathcal{H}_0$ to behave nicely.

A more precise description of the structure of the proof is given by the flow chart in Figure 2. Please note that it is intended as a guide to be referred to throughout the proof, and many of the terms will only be introduced in later sections.

In Section 8, we prove some important properties of home-base hypergraphs, which will be essential for several parts of the rest of the proof.

In Section 9, we define and study cromulent and perfectly cromulent triples. A perfectly cromulent triple is a set of vertices such that the rest is a home-base hypergraph that interacts with the rest of the edges in a controlled fashion. This turns out to be precisely the substructure we need so that we can extend the home-base partition given by induction to a home-base partition of the whole hypergraph. Cromulent triples are apparently weaker versions of perfectly cromulent triples, but careful considerations will show that no cromulent triple can actually fail to be perfectly cromulent under the assumption that $\tau = 2\nu$. Therefore, it will be enough to find just a cromulent triple in order to show that we have a home-base hypergraph.

In Section 10, we show how to use a good set to find a perfectly cromulent triple and hence conclude that we are dealing with a home-base hypergraph. The rest of Section 10 is devoted to exploring how the edges of the link graphs extend to hyperedges under the assumption that there are no good sets and no cromulent triples.

In Section 11, we use the information on how the links extend, together with the fact that the links have CP-decompositions to show that the hypergraph must contain a truncated multi-Fano plane that interacts minimally with the rest of the hypergraph, which by induction will have a home-base partition. It is then easy to show that adding the lone F results in a home-base partition of the whole hypergraph.

The proof of Theorem 1.1 is assembled from all of the theorems and lemmas of the preceeding four sections in Section 12.

7   Home-Base Partitions

There are essentially two types of intersecting Ryser-extremal 3-graphs. One of them is the truncated Fano plane F mentioned in the Introduction, with vertex set $\{a, b, c, x, y, z\}$ and edges abc, axy, xbz, and xyc. It is pictured in Figure 3. Adding parallel edges to any hypergraph does not affect the vertex cover number or the matching number. We call any 3-graph a *truncated multi-Fano plane*, if it is obtained from the truncated Fano-plane by adding an arbitrary number of parallel edges. Note that it is in a sense maximal, as one cannot add any edge not parallel to an existing edge and obtain an intersecting 3-partite 3-graph.
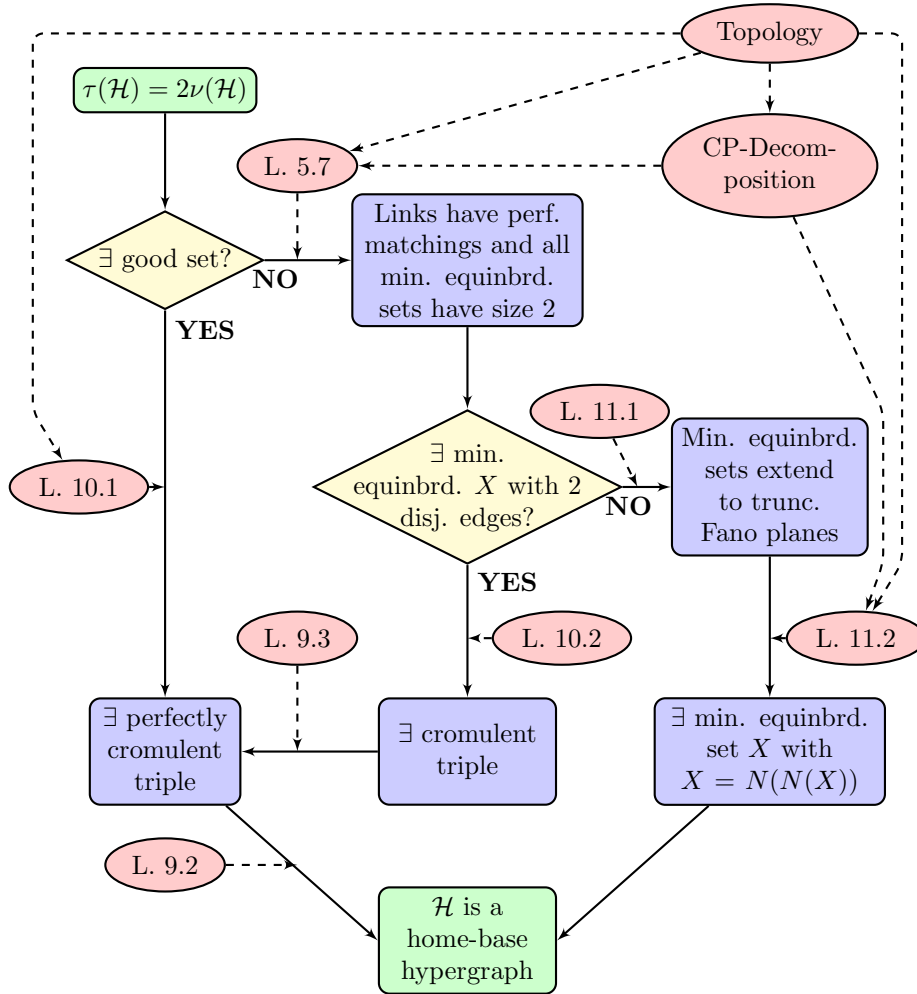
Figure 2: A flow-chart describing the logic of the proof with relevant lemmas shown.
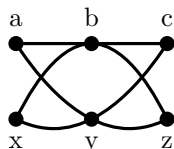
Figure 3: The truncated Fano plane.

The other type of intersecting Ryser-extremal 3-graph is the loose 3-cycle, which one may obtain by removing the edge abc (or really any edge) from the truncated Fano plane. It has three degree-2 vertices, x, y, and z, of which every edge contains two. One can extend C by adding edges (perhaps containing new vertices) which contain two of the degree 2 vertices and still obtain an intersecting hypergraph (and obviously the vertex cover number does not decrease). This creates a family of edges which is intersecting simply because they all contain two of the vertices x, y, and z. Thus this family is determined by the set $R = \{x, y, z\}$.
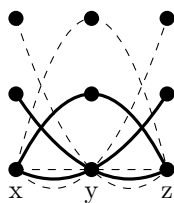

Figure 4: The loose 3-cycle, with possible additional edges drawn in dashed lines.

As stated in the Introduction, the simplest way to describe a home-base hypergraph $\mathcal{H}$ with $\nu(\mathcal{H}) = k$ is as follows. Start with a set of k disjoint hypergraphs, each of which is a copy of F or a loose 3-cycle C as above. Now add any number of additional edges e, each of which is parallel to an edge in a copy of F, or intersects some C in at least two of its degree-2 vertices. This does capture the notion of home-base hypergraph, however for our inductive proof the following series of more technical definitions will be needed.

Definition 7.1. Let $\mathcal{H}$ be a 3-partite 3-graph. An *FR-partition* of $\mathcal{H}$ is a triple $(\mathcal{F}, \mathcal{R}, W)$ with $\mathcal{F}, \mathcal{R} \subseteq 2^{V(\mathcal{H})}$ and $W \subseteq V(\mathcal{H})$ which satisfies the following conditions:

(1) $\mathcal{F} \cup \mathcal{R} \cup \{W\}$ is a partition of the vertices of $\mathcal{H}$,

(2) For each $F \in \mathcal{F}$, the induced hypergraph $\mathcal{H}|_F$ is isomorphic to a truncated multi-Fano plane,

(3) Each $R \in \mathcal{R}$ is a three-vertex set with one vertex from each vertex class of $\mathcal{H}$,

(4) $|\mathcal{F} \cup \mathcal{R}| = \nu(\mathcal{H})$.

Note that $\mathcal{F}$ is a 6-graph and $\mathcal{R}$ is a 3-graph.

Definition 7.2. $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3, and let $(\mathcal{F}, \mathcal{R}, W)$ be an FR-partition of $\mathcal{H}$. For each vertex class V_i, we define a bipartite graph B_i with vertex classes $\mathcal{R}$ and $W \cap V_i$ and with an edge between $R \in \mathcal{R}$ and $w \in W \cap V_i$ precisely when there is an edge of $\mathcal{H}$ containing w and two vertices of R. The partition $(\mathcal{F}, \mathcal{R}, W)$ is called *matchable* if each B_i has a matching saturating $\mathcal{R}$.

An example of a non-matchable FR-partition is given in the following picture, where the boxes correspond to two R's and the unboxed vertices are in W:
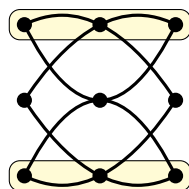

Figure 5: An unmatchable FR-partition.

Definition 7.3. An FR-partition $(\mathcal{F}, \mathcal{R}, W)$ of $\mathcal{H}$ is said to have the *edge-home* property if every edge of $\mathcal{H}$ is either in $\mathcal{H}|_F$ for some $F \in \mathcal{F}$ or contains two vertices from some $R \in \mathcal{R}$.

Definition 7.4. A matchable FR-partition with the edge-home property is called a *home-base partition*. $\mathcal{H}$ is called a *home-base hypergraph* if it has a home-base partition.

Notation. For each $F \in \mathcal{F}$, we call an edge an F-edge if it is in $\mathcal{H}|_F$. For each $R \in \mathcal{R}$, we call an edge an R-edge if it contains two vertices from R. We call an edge an $\mathcal{F}$-edge if it is an F-edge for some $F \in \mathcal{F}$, and call an edge an $\mathcal{R}$-edge if it is an R-edge for some $R \in \mathcal{R}$.

Here follows an example of a home-base hypergraph. The boxes correspond to members of $\mathcal{F}$ or $\mathcal{R}$, and the unboxed vertices are in W. The bolded edges are the edges of $\mathcal{H}|_F$ for some $F \in \mathcal{F}$ or the edges corresponding to the edges of arbitrarily chosen matchings saturating $\mathcal{R}$ in the auxiliary bipartite graphs B_i.
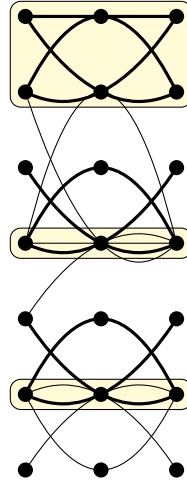
Figure 6: A home-base hypergraph with its home-base partition.

We can easily see one direction of Theorem 1.1:

Proposition 7.5. *If $\mathcal{H}$ has a home-base partition $(\mathcal{F}, \mathcal{R}, W)$, then $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$.*

Proof. Let $T \subseteq V(\mathcal{H})$ be a vertex cover. We aim to show that it has size at least $2\nu(\mathcal{H}) = 2|\mathcal{F} \cup \mathcal{R}|$. Since the partition is matchable, each of the auxiliary bipartite graphs B_1, B_2, and B_3 have matchings saturating $\mathcal{R}$, say M_1, M_2, and M_3, respectively. Then each $R = \{r_1, r_2, r_3\} \in \mathcal{R}$ has three W-vertices, $w_i^R \in V_i$ assigned to it, so that $Rw_i^R \in M_i$, which means that $w_i^R r_j r_k$ are edges for each choice of $\{i, j, k\} = \{1, 2, 3\}$. So consider only the edges of this form together with the edges of $\mathcal{H}|_F$ for each $F \in \mathcal{F}$. Each set of edges for each $R \in \mathcal{R}$ and $F \in \mathcal{F}$ is disjoint from the other sets, so any vertex cover must cover each set with different vertices. Since each such set forms an intersecting 3-partite 3-graph with vertex cover number 2, T must have at least two vertices for each $R \in \mathcal{R}$ and each $F \in \mathcal{F}$, giving a total of at least $2|\mathcal{R} \cup \mathcal{F}| = 2\nu(\mathcal{H})$ vertices as required. This shows $\tau(\mathcal{H}) \geq 2\nu(\mathcal{H})$. Since Ryser's Conjecture is true for 3-partite 3-graphs, we have $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. $\square$

Note that we did not make use of the edge-home property in this proof. This property is necessary however to ensure that if a home-base partition exists, then it is unique. Uniqueness is not necessary for our proof of the main theorem, for its proof we refer to [20].

The definition of home-base hypergraphs together with Theorem 1.1 allows us to prove the Lovász Conjecture for Ryser-extremal 3-graphs.

Proof of Corollary 1.2. By Theorem 1.1 we have that $\mathcal{H}$ has a home-base partition $(\mathcal{F}, \mathcal{R}, W)$. Then by definition $|\mathcal{F}| + |\mathcal{R}| = \nu(\mathcal{H})$. Let us now define a pair of vertices from each element of $\mathcal{F} \cup \mathcal{R}$. For each $F \in \mathcal{F}$ we take any of the partition classes V_i and for each $R \in \mathcal{R}$ we take an arbitrary 2-element subset

$R' \subseteq R$. We claim that this system of $\nu(\mathcal{H})$ pairwise disjoint pairs of vertices satisfies the statement of the theorem. To check this, suppose we delete from $\mathcal{H}$ the union of an arbitrary k-set of these pairs, say the ones corresponding to some subfamilies $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{R}' \subseteq \mathcal{R}$. Consider a maximum matching M in the remaining hypergraph. By the edge-home property each edge e of M has a "home": e is either contained in some $F \in \mathcal{F}$ or it has two common vertices with some $R \in \mathcal{R}$. In either case the home of e cannot be from $\mathcal{F}' \cup \mathcal{R}'$ since each pair of vertices deleted from these sets had a non-empty intersection with any edge of $\mathcal{H}$ that had its home there. Hence the edges of M must have their home among the sets of $(\mathcal{F} \setminus \mathcal{F}') \cup (\mathcal{R} \setminus \mathcal{R}')$. Since any two edges having the same home intersect, any set from $(\mathcal{F} \setminus \mathcal{F}') \cup (\mathcal{R} \setminus \mathcal{R}')$ can be home to at most one edge of M. Hence $|M| \leq |\mathcal{F} \cup \mathcal{R}| - |\mathcal{F}' \cup \mathcal{R}'| = \nu(\mathcal{H}) - k$ and the claim is proved. $\qquad \square$

8 Properties of Home-Base Hypergraphs

The next couple of sections will establish some basic properties of home-base hypergraphs that we will need in the proof of Theorem 1.1.

First is what we call the "monster lemma", which states under which conditions a monster can eat some vertices of a home-base hypergraph without reducing the matching number.

But before we can prove it, we shall need some definitions.

8.1 Essential and Superfluous Vertices

Definition 8.1. Let G be a bipartite graph with vertex classes X_1 and X_2. A subset $C \subseteq X_i$ is called *essential* if there is a subset $U \subseteq X_{3-i}$ with $|U| = |C|$ and $C = N(U)$.

We remark briefly that non-empty essential subsets are precisely the neighborhoods of equineighbored subsets. We will of course apply this concept to the bipartite graphs B_i from the matchability criterion of FR-partitions.

Let $\mathcal{H}$ be a 3-partite 3-graph on vertex classes V_1, V_2, and V_3 with a matchable FR-partition $(\mathcal{F}, \mathcal{R}, W)$. We call a vertex v in V_i *essential* if $v \in W$ and $\{v\} \subseteq W \cap V_i$ is essential in B_i. If $R \in \mathcal{R}$ has only $v \in W \cap V_i$ as its neighbor in B_i, then we say v is essential for R.

Lemma 8.2. *Let B be a bipartite graph with vertex classes $\mathcal{R}$ and W, which has a matching saturating $\mathcal{R}$. Then W contains a unique maximal essential subset, which is the union of all essential subsets of W.*

Proof. Let $C_1, C_2 \subseteq W$ be essential. Then we claim $C_1 \cup C_2$ is also essential. Consider $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{R}$ such that $C_1 = N_B(\mathcal{U}_1)$, $C_2 = N_B(\mathcal{U}_2)$, $|\mathcal{U}_1| = |C_1|$ and $|\mathcal{U}_2| = |C_2|$. Then $N_B(\mathcal{U}_1 \cup \mathcal{U}_2) = C_1 \cup C_2$ and by Hall's Theorem, $|C_1 \cup C_2| \geq |\mathcal{U}_1 \cup \mathcal{U}_2|$. But of course $N_B(\mathcal{U}_1 \cap \mathcal{U}_2) \subseteq C_1 \cap C_2$ and thus again by Hall's Theorem, $|C_1 \cap C_2| \geq |\mathcal{U}_1 \cap \mathcal{U}_2|$. By the inclusion-exclusion principle, we thus

have $|C_1| + |C_2| - |C_1 \cup C_2| \geq |\mathcal{U}_1| + |\mathcal{U}_2| - |\mathcal{U}_1 \cup \mathcal{U}_2|$, and since $|\mathcal{U}_1| = |C_1|$ and $|\mathcal{U}_2| = |C_2|$, we find that $|C_1 \cup C_2| \leq |\mathcal{U}_1 \cup \mathcal{U}_2|$, so that in fact there is equality. This proves that $C_1 \cup C_2$ is essential. Therefore the union over all essential subsets of W gives the unique maximal essential set. $\qquad\square$

A vertex of W that is not in the maximal essential set is called *superfluous*. Note that any one superfluous vertex w can be removed, and the rest of the bipartite graph will still have a matching saturating $\mathcal{R}$. (Indeed, otherwise by Hall's Theorem $|N(\mathcal{U}) \setminus \{w\}| < |\mathcal{U}|$ for some $\mathcal{U} \subseteq \mathcal{R}$ with $w \in N(\mathcal{U})$. But then $\mathcal{U}$ is equineighbored and hence $N(\mathcal{U})$ is essential, implying that w is in the union of all essential subsets, contradicting Lemma 8.2.) Again, we will apply this fact to the bipartite graphs B_i from the matchability criterion of FR-partitions.

Let $\mathcal{H}$ be a home-base hypergraph on vertex classes V_1, V_2, and V_3 with a home-base partition $(\mathcal{F}, \mathcal{R}, W)$. Then the auxiliary bipartite graphs B_i have vertex classes $\mathcal{R}$ and $W \cap V_i$ and a matching saturating $\mathcal{R}$. Therefore, each $W \cap V_i$ contains a unique maximum essential subset C_i, and we may call a vertex of V_i superfluous if it is in $W \cap V_i \setminus C_i$. Clearly superfluous vertices are non-essential W-vertices in a stronger form. We can make the following observation:

Observation 8.3. *Let $\mathcal{H}$ be a 3-partite 3-graph with a matchable FR-partition $(\mathcal{F}, \mathcal{R}, W)$, and let $S \subseteq W$ be a set of superfluous vertices with at most one vertex in each vertex class. Then $(\mathcal{F}, \mathcal{R}, W \setminus S)$ is a matchable FR-partition of $\mathcal{H} - S$.*

Proof. Since removing any single superfluous vertex s from any of the bipartite graphs B_i leaves a matching saturating $\mathcal{R}$, $(\mathcal{F}, \mathcal{R}, W \setminus \{s\})$ is a matchable FR-partition. Since removing s from one does not change the other graphs B_j at all, we can do this for each vertex class independently. $\qquad\square$

We will need the following simple lemma about removing superfluous vertices later in Section 10.

Lemma 8.4. *Let B be a bipartite graph with vertex classes $\mathcal{R}$ and W that has a matching saturating $\mathcal{R}$, and let $C \subseteq W$ be the maximal essential subset. If $p \in C$ and $s \in W \setminus C$, then p is essential in B if and only if it is essential in $B - s$.*

Proof. If p is essential in B, then it clearly is essential in $B - s$.

Conversely, assume p is essential in $B - s$. Let $\mathcal{U} \subseteq \mathcal{R}$ be such that $N_B(\mathcal{U}) = C$ and $|\mathcal{U}| = |C|$, which exists by the definition of essential subsets. Since p is essential, there is a unique $R \in \mathcal{R}$ such that $N_{B-s}(R) = \{p\}$. We claim that $R \in \mathcal{U}$. Suppose not. Then $N_B(R) \subseteq \{s, p\}$, and hence $N_B(\mathcal{U} \cup \{R\}) \subseteq C \cup \{s\}$. Since $|\mathcal{U} \cup \{R\}| = |\mathcal{U}| + 1 = |C \cup \{s\}|$, this would make $C \cup \{s\}$ an essential set in B, a contradiction, since C is maximal. Hence $R \in \mathcal{U}$, from which follows that $s \notin N_B(R)$, and thus $N_B(R) = \{p\}$, so p is essential in B. $\qquad\square$

8.2 The Monster Lemma

Lemma 8.5. *Let $\mathcal{H}$ be a 3-partite 3-graph that has a matchable FR-partition $(\mathcal{F}, \mathcal{R}, W)$. Let $a, b, c \in V(\mathcal{H})$ be in different vertex classes. Suppose that the following two conditions hold:*

(1) For every $F \in \mathcal{F}$, there is an F-edge avoiding $\{a, b, c\}$,

(2) For every $R \in \mathcal{R}$, there is an R-edge avoiding $\{a, b, c\}$.

Then $\nu(\mathcal{H} - \{a, b, c\}) = \nu(\mathcal{H})$.

Proof. Let V_1, V_2, and V_3 be the vertex classes of $\mathcal{H}$, where $a \in V_1$, $b \in V_2$, and $c \in V_3$. We will select a matching $\mathcal{M} \subseteq E(\mathcal{H})$ of size $\nu(\mathcal{H})$ avoiding $\{a, b, c\}$.

First, for each $F \in \mathcal{F}$ we choose an arbitrary edge from $\mathcal{H}|_F$ avoiding $\{a, b, c\}$ and include it in $\mathcal{M}$. This can be done by condition (1). These edges are all pairwise disjoint, since the members of $\mathcal{F}$ are pairwise disjoint. Furthermore, we will describe a procedure that selects pairwise disjoint $\mathcal{R}$-edges, one for each $R \in \mathcal{R}$, each containing a W-vertex and avoiding $\{a, b, c\}$. Because they contain a W-vertex, these $\mathcal{R}$-edges will all be disjoint from the $\mathcal{F}$-edges we already put into $\mathcal{M}$ (since both W and $V(\mathcal{R})$ are disjoint from $V(\mathcal{F})$). If successful, we will have constructed the required matching $\mathcal{M}$, since $|\mathcal{M}| = |\mathcal{F}| + |\mathcal{R}| = \nu(\mathcal{H})$.

How we choose the $\mathcal{R}$-edges will fall into several cases. We introduce the following convenient notation for talking about $\mathcal{R}$-edges. An $\mathcal{R}$-edge xyz of $\mathcal{H}$ is called a *WRR-edge* if $x \in W \cap V_1$. Analogously, xyz is called an *RWR-edge* or an *RRW-edge* if $y \in W \cap V_2$ or $z \in W \cap V_3$, respectively.

Case 1. At least one of the vertices a, b, or c is in $V(\mathcal{R})$.

We may assume without loss of generality that $a \in V(\mathcal{R})$. First we choose a matching M_1 saturating $\mathcal{R}$ in the auxiliary bipartite graph B_1. Such a matching exists by the matchability of the FR-partition. Each edge $Rw \in M_1$, with $R \in \mathcal{R}$ and $w \in W \cap V_1$ corresponds to a WRR-edge of $\mathcal{H}$ consisting of w and two vertices of R. These edges form a matching $\mathcal{M}'$ of $\mathcal{R}$-edges in $\mathcal{H}$. Each edge in $\mathcal{M}'$ contains a W-vertex in V_1 and hence avoids $a \in V(\mathcal{R}) \cap V_1$. The only problem might be that b or c appear in some of these edges, rendering those edges unsuitable. If b is contained in the R-edge $e_1 \in \mathcal{M}'$ for some $R \in \mathcal{R}$, then replace e_1 in $\mathcal{M}'$ with an arbitrary RWR-edge e_2 for R. Such an edge exists because B_2 has a matching saturating $\mathcal{R}$, and it is disjoint from all other edges in $\mathcal{M}'$ because these are WRR-edges. The vertex of e_2 in V_1 cannot be a, since then all R-edges would intersect $\{a, b\}$, contradicting condition (2). Similarly, the vertex of e_2 in V_3 cannot be c, since then all R-edges would intersect $\{b, c\}$. Finally, if c is contained in the R'-edge $e_3 \in \mathcal{M}'$ for some $R' \in \mathcal{R}$, then replace e_3 in $\mathcal{M}'$ with an arbitrary RRW-edge e_4 for R'. Such an edge exists because B_3 has a matching saturating $\mathcal{R}$, and it is disjoint from all other edges of $\mathcal{M}'$ because they are all WRR- and RWR-edges. The edge e_4 cannot contain a, otherwise all R'-edges would intersect $\{a, c\}$, contradicting (2). The edge e_4 also does not contain b, since otherwise every R'-edge would intersect $\{b, c\}$, again contradicting (2).

Now the vertices of the matching $\mathcal{M}'$ avoid $\{a, b, c\}$ and Case 1 is complete.

Let us assume from now on that none of the vertices a, b, and c are in $V(\mathcal{R})$.

Case 2. None of the vertices a, b, and c are essential.

First we choose a matching M_1 in B_1 saturating $\mathcal{R}$, which exists by the matchability of the FR-partition. This corresponds to a matching $\mathcal{M}'$ in $\mathcal{H}$ consisting of WRR-edges. Clearly, b and c are avoided by the edges of $\mathcal{M}'$ because $b, c \notin V(\mathcal{R})$. If a is contained in an R-edge $e_1 \in \mathcal{M}'$ for some $R \in \mathcal{R}$, then replace e_1 in $\mathcal{M}'$ by an arbitrary RWR-edge e_2 for R that avoids b. This can be done, since b is not essential. The edge e_2 also avoids a and c because $a, c \notin V(\mathcal{R})$, and it is disjoint from all other edges of $\mathcal{M}'$ because they are all WRR-edges.

Hence we have the required matching $\mathcal{M}'$ avoiding $\{a, b, c\}$ and Case 2 is complete.

Case 3. Not all of the vertices a, b, and c are essential W-vertices for the same $R \in \mathcal{R}$.

We may assume without loss of generality that a is essential for $R \in \mathcal{R}$ (If no vertex is essential, we are in Case 2). By assumption, not both b and c are essential for R as well, so assume without loss of generality that b is not essential for R. We choose a matching $M_1 \subseteq E(B_1)$ saturating $\mathcal{R}$. This corresponds to a matching $\mathcal{M}'$ in $\mathcal{H}$ consisting of WRR-edges. Clearly, b and c are avoided by the edges of $\mathcal{M}'$ because $b, c \notin V(\mathcal{R})$. Since a is essential for R, it must be that $Ra \in M_1$ because a is the only neighbor of R in $W \cap V_1$. Let $e_1 \in \mathcal{M}'$ be the edge corresponding to $Ra \in M_1$. We replace e_1 in $\mathcal{M}'$ by an arbitrary RWR-edge e_2 for R that avoids b. This can be done, since b is not essential for R. The edge e_2 also avoids a and c because $a, c \notin V(\mathcal{R})$, and it is disjoint from all other edges of $\mathcal{M}'$ because they are all WRR-edges.

This means that $\mathcal{M}'$ avoids $\{a, b, c\}$, and so Case 3 is complete.

Case 4. The vertices a, b, and c are all essential W-vertices for $R \in \mathcal{R}$.

By condition (2), there must be an R-edge e avoiding a, b, and c. At least two of its vertices must be in R, so assume without loss of generality that $e \cap V_2, e \cap V_3 \subseteq R$. We choose a matching M_1 in B_1 saturating $\mathcal{R}$. It corresponds to a matching $\mathcal{M}'$ of WRR-edges in $\mathcal{H}$. Because a is essential for R, it follows that there is an edge of $\mathcal{M}'$ containing a and two vertices of R. Replace it by e, which avoids a, b, and c and is disjoint from the other edges of $\mathcal{M}'$ because its V_1-vertex is not in W (because a is the only W-vertex in a WRR-edge of R) and its other vertices are in R. The rest of the edges of $\mathcal{M}'$ clearly avoid a, b, and c, since the one edge of $\mathcal{M}'$ containing a has already been replaced, and $b, c \notin V(\mathcal{R})$.

We must be careful because in this case, one of the edges of $\mathcal{M}'$, namely e, is not necessarily contained in $V(\mathcal{R}) \cup W$, as has been true in all other cases. Thus, the V_1-vertex of e may be in some $F \in \mathcal{F}$, and hence could potentially intersect the F-edge which we added to $\mathcal{M}$ in the beginning. However, since $\mathcal{H}|_F$ is a truncated multi-Fano plane, it cannot be covered by one vertex, so there is an F-edge disjoint from e with which we can replace our original choice of edge for $\mathcal{M}$. Note that we do not need to worry about avoiding $\{a, b, c\}$ with this edge, as these are all in W.

Adding the edges in $\mathcal{M}'$ to $\mathcal{M}$ gives us our desired matching avoiding $\{a, b, c\}$. This concludes Case 4.

These cases exhaust all possibilities, so the proof is complete. $\qquad\square$

In order to facilitate the use of this lemma, we prove in some specific cases that the conditions are fulfilled.

Corollary 8.6. *Let $\mathcal{H}$ be a 3-partite 3-graph with a matchable FR-partition $(\mathcal{F}, \mathcal{R}, W)$. Let $a, b, c \in V(\mathcal{H})$ be in different vertex classes, and let $S \subseteq W$ be a set of superfluous vertices with at most one vertex in each vertex class. Then in any of the following cases we have $\nu(\mathcal{H} - (\{a, b, c\} \cup S)) = \nu(\mathcal{H})$:*

(1) $a \in V(\mathcal{F})$, $b \in W$, and c is arbitrary,

(2) $a \in R \in \mathcal{R}$, $b \notin R$, and $c \notin V(\mathcal{R})$,

(3) $a \in W$ is essential for $R \in \mathcal{R}$ in $\mathcal{H}$, b is not essential for R in $\mathcal{H} - S$, and $c \notin V(\mathcal{R})$,

(4) $a \in W$ is not essential in $\mathcal{H} - S$, $b \notin V(\mathcal{R})$, and c is arbitrary.

Proof. Let V_1, V_2, and V_3 be the vertex classes of $\mathcal{H}$, where $a \in V_1$, $b \in V_2$, and $c \in V_3$. Let $S' = S \setminus \{a, b, c\}$. By Observation 8.3, the hypergraph $\mathcal{H}' = \mathcal{H} - S'$ has the matchable FR-partition $(\mathcal{F}, \mathcal{R}, W \setminus S')$, and hence $\nu(\mathcal{H}') = \nu(\mathcal{H})$. We will apply Lemma 8.5 to $\mathcal{H}'$ to find a matching in $\mathcal{H}'$ of size $\nu(\mathcal{H}')$ avoiding $\{a, b, c\}$. This constitutes a matching in $\mathcal{H} - (\{a, b, c\} \cup S)$ of size $\nu(\mathcal{H})$, as desired. We must simply check that the two conditions of Lemma 8.5 hold.
Case 1. $a \in V(\mathcal{F})$, $b \in W$, and c is arbitrary.

For any $F \in \mathcal{F}$, there is an F-edge avoiding $\{a, b, c\}$, because $b \in W$, and a and c, being in different vertex classes, do not cover every edge of $\mathcal{H}'|_F$ (a truncated multi-Fano plane).

Let $R = \{r_1, r_2, r_3\} \in \mathcal{R}$ (where $r_i \in V_i$). We will find an R-edge avoiding $\{a, b, c\}$. If $c \in R$, then there is an R-edge avoiding $\{a, b, c\}$ because the matchability of B_3 ensures that there is an R-edge $r_1 r_2 w$ with $w \in W \cap V_3$, which clearly avoids $\{a, b, c\}$, because $a, b \notin V(\mathcal{R})$, and $c \in R$. Suppose $c \notin R$. By the matchability of B_1, there is an R-edge $w' r_2 r_3$, where $w' \in W \cap V_1$, and this edge avoids $\{a, b, c\}$ because $a \in V(\mathcal{F})$, $b \in W$, and $c \notin R$.

Therefore Lemma 8.5 applies, and we have $\nu(\mathcal{H}' - \{a, b, c\}) = \nu(\mathcal{H})$.
Case 2. $a \in R \in \mathcal{R}$, $b \notin R$, and $c \notin V(\mathcal{R})$.

For any $F \in \mathcal{F}$, there is an F-edge avoiding $\{a, b, c\}$, because $a \in V(\mathcal{R})$, and b and c do not cover every edge of $\mathcal{H}'|_F$ (a truncated multi-Fano plane).

Let $R' = \{r_1, r_2, r_3\} \in \mathcal{R}$ (where $r_i \in V_i$). We will find an R'-edge avoiding $\{a, b, c\}$. If $b \in R'$, then $R' \neq R$, so $a \notin R'$. There is an R'-edge $r_1 w r_3$ with $w \in W \cap V_2$ by matchability applied to B_2. This edge avoids $\{a, b, c\}$ because $a \notin R'$, $b \in R'$, and $c \notin V(\mathcal{R})$. Suppose $b \notin R'$. By the matchability of B_1, there is an R'-edge $w' r_2 r_3$, where $w' \in W \cap V_1$, and this edge avoids $\{a, b, c\}$ because $a \in V(\mathcal{R})$, $b \notin R'$, and $c \notin V(\mathcal{R})$.

Therefore Lemma 8.5 applies, and we have $\nu(\mathcal{H}' - \{a, b, c\}) = \nu(\mathcal{H})$.

For Cases 3 and 4, let us observe that since $\mathcal{H} - S \subseteq \mathcal{H}' \subseteq \mathcal{H}$, if a is essential in $\mathcal{H}$ then it is essential in $\mathcal{H}'$, which in turn implies it is essential in $\mathcal{H} - S$. Conversely if b (or a) is not essential in $\mathcal{H} - S$ then it is not essential in $\mathcal{H}'$, implying it is not essential in $\mathcal{H}$.

Case 3. $a \in W$ is essential for $R \in \mathcal{R}$ in $\mathcal{H}$, b is not essential for R in $\mathcal{H} - S$, and $c \notin V(\mathcal{R})$.

For any $F \in \mathcal{F}$, there is an F-edge avoiding $\{a, b, c\}$, because $a \in W$, and b and c do not cover every edge of $\mathcal{H}'|_F$ (a truncated multi-Fano plane).

Let $R' = \{r_1, r_2, r_3\} \in \mathcal{R}$ (where $r_i \in V_i$). We will find an R'-edge avoiding $\{a, b, c\}$. If b is not essential for R' in $\mathcal{H}'$, then R' has a neighbor $w \in W \cap V_1$ in B_2 with $w \neq b$. The R' edge $r_1 w r_3$ then avoids $\{a, b, c\}$ because $a \in W$, $b \neq w$, and $c \notin V(\mathcal{R})$. If b is essential for R' in $\mathcal{H}'$, then $b \in W$ and $R' \neq R$, so a is not essential for R' (because no vertex can be essential for two different members of $\mathcal{R}$ by matchability). Thus R' has a neighbor $w' \in W \cap V_1$ in B_1 with $w' \neq a$. The R'-edge $w' r_2 r_3$ then avoids $\{a, b, c\}$ because $w' \neq a$ and $b, c \notin V(\mathcal{R})$.

Therefore Lemma 8.5 applies, and we have $\nu(\mathcal{H}' - \{a, b, c\}) = \nu(\mathcal{H})$.

Case 4. $a \in W$ is not essential in $\mathcal{H} - S$, $b \notin V(\mathcal{R})$, and c is arbitrary.

For any $F \in \mathcal{F}$, there is an F-edge avoiding $\{a, b, c\}$, because $a \in W$, and b and c do not cover every edge of $\mathcal{H}'|_F$ (a truncated multi-Fano plane).

Let $R = \{r_1, r_2, r_3\} \in \mathcal{R}$ (where $r_i \in V_i$). We will find an R-edge avoiding $\{a, b, c\}$. If $c \in R$, then there is an R-edge avoiding $\{a, b, c\}$ because the matchability of B_3 ensures that there is an R-edge $r_1 r_2 w$ with $w \in W \cap V_3$, which clearly avoids $\{a, b, c\}$, since $a, b \notin V(\mathcal{R})$, and $c \in R$. Suppose $c \notin R$. Since a is not essential in $\mathcal{H}'$, R has a neighbor $w' \in W \cap V_1$ in B_1 with $w' \neq a$. The R-edge $w' r_2 r_3$ then avoids $\{a, b, c\}$ because $w' \neq a$, $b \notin V(\mathcal{R})$, and $c \notin R$.

Therefore Lemma 8.5 applies, and we have $\nu(\mathcal{H}' - \{a, b, c\}) = \nu(\mathcal{H})$. □

It is unfortunately necessary in Cases 3 and 4 to make sure that the non-essential W-vertex remains non-essential after removing the superfluous vertices. However, this condition is often very easy to check, since removing superfluous vertices from the hypergraph only affects the status of those W-vertices in their vertex class. This leads to the following observation:

Observation 8.7. *Let $\mathcal{H}$ be a 3-partite 3-graph with a matchable FR-partition $(\mathcal{F}, \mathcal{R}, W)$, and let $s \in W$ be a superfluous vertex. Then if $w \in W$ is in a different vertex class from s, it holds that w is non-essential in $\mathcal{H}$ if and only if it is non-essential in $\mathcal{H} - s$.*

8.3   Matchability and the Edge-Home Property

One nice consequence of the monster lemma is the following proposition, which will be key to our proof.

Definition 8.8. An FR-partition $(\mathcal{F}, \mathcal{R}, W)$ is *proper* if there is no $R \in \mathcal{R}$ and an edge of $\mathcal{H}$ consisting of three vertices of W which together induce a truncated Fano plane.

Being proper just means that we have not called anything an R if it could have been part of an F. Clearly home-base partitions are proper, because they do not contain any edges consisting of W-vertices. It turns out that a converse to this fact is also true.

Proposition 8.9. *A proper matchable FR-partition of a 3-partite 3-graph has the edge-home property.*

Proof. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, V_3, and let $(\mathcal{F}, \mathcal{R}, W)$ be a proper matchable FR-partition of $\mathcal{H}$. Let abc be an edge of $\mathcal{H}$. We aim to show that it is either an $\mathcal{F}$-edge or an $\mathcal{R}$-edge. Suppose it is not. We will aim for a contradiction by applying Lemma 8.5 to show $\mathcal{H} - \{a, b, c\}$ has a matching of size $\nu(\mathcal{H})$.

By assumption, abc is not in $\mathcal{H}|_F$ for any $F \in \mathcal{F}$, which means that every $F \in \mathcal{F}$ has an F-edge avoiding $\{a, b, c\}$, since the only way to cover a truncated Fano plane with vertices from different vertex classes is if they form one of its edges. We want to show that it also cannot cover every R-edge for any $R \in \mathcal{R}$.

Since the partition is matchable, each of the auxiliary bipartite graphs B_1, B_2, and B_3 have matchings saturating $\mathcal{R}$, say M_1, M_2, and M_3, respectively. Then each $R = \{r_1, r_2, r_3\} \in \mathcal{R}$ has three W-vertices, $w_i^R \in V_i$ assigned to it, so that $Rw_i^R \in M_i$, which means that $w_i^R r_j r_k$ are edges for each choice of $\{i, j, k\} = \{1, 2, 3\}$. By assumption, abc intersects R in at most one vertex (otherwise, it is an R-edge). If abc intersects R in one vertex, without loss of generality in V_1, then $w_1^R r_2 r_3$ is an R-edge disjoint from abc. If abc does not intersect R in any vertex, then it intersects all the R-edges $w_i^R r_j r_k$ for $\{i, j, k\} = \{1, 2, 3\}$ only if $abc = w_1^R w_2^R w_3^R$, which would mean that abc, $w_1^R r_2 r_3$, $r_1 w_2^R r_3$, and $r_1 r_2 w_3^R$ form a truncated Fano plane. If this is the case, then we claim that these are in fact the only edges on $\{a, b, c, r_1, r_2, r_3\}$, which would contradict the assumption that $(\mathcal{F}, \mathcal{R}, W)$ is proper.

Suppose these are not the only edges on $\{a, b, c, r_1, r_2, r_3\}$. Then there are two disjoint edges on $\{a, b, c, r_1, r_2, r_3\}$. Now pick one F-edge for each $F \in \mathcal{F}$, and take (for example) the edges $w_1^{R'} r_2' r_3'$ for each $R' \in \mathcal{R} \setminus \{R\}$. These edges form a matching of size $|\mathcal{F}| + |\mathcal{R}| - 1$, and they do not intersect $\{a, b, c, r_1, r_2, r_3\}$. Together with the two disjoint edges on $\{a, b, c, r_1, r_2, r_3\}$, we find a matching of size $|\mathcal{F}| + |\mathcal{R}| + 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Hence a, b, and c fulfill the conditions of Lemma 8.5, and $\mathcal{H} \setminus \{a, b, c\}$ would have a matching of size $\nu(\mathcal{H})$, which together with abc would be a matching of size $\nu(\mathcal{H}) + 1$ in $\mathcal{H}$, a contradiction. Therefore $\mathcal{H}$ has the edge-home property. $\square$

9 Cromulent Triples

The aim of this section is to define the appropriate substructure which will facilitate the inductive proof of our main theorem (Theorem 1.1). The key definition is that of a cromulent triple.

Definition 9.1. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3. A triple of nonempty sets (Y_1, Y_2, X) with $Y_1 \subseteq V_i$, $Y_2 \subseteq V_j$ and $X \subseteq V_k$,

where $\{i, j, k\} = \{1, 2, 3\}$ is called a *cromulent* triple if it fulfills the following conditions:

(1) $|Y_1| = |Y_2| \leq |X|$,

(2) $N_{\mathrm{lk}_{\mathcal{H}}(V_i)}(X) = Y_2$,

(3) There is a hypergraph matching in $\mathcal{H}|_{Y_1 \cup Y_2 \cup X}$ of size $|Y_1|$,

(4) The hypergraph $\mathcal{H}_0 = \mathcal{H} - (Y_1 \cup Y_2 \cup X)$ is a home-base hypergraph with $\nu(\mathcal{H}_0) = \nu(\mathcal{H}) - |Y_1|$,

(5) Given any home-base partition $(\mathcal{F}, \mathcal{R}, W)$ of $\mathcal{H}_0$, we have $N_{\mathrm{lk}_{\mathcal{H}}(V_j)}(X) \subseteq Y_1 \cup V(\mathcal{R}) \cup V(\mathcal{F})$.

Such a triple is called *perfectly cromulent* if it fulfills the following stronger version of condition (5):

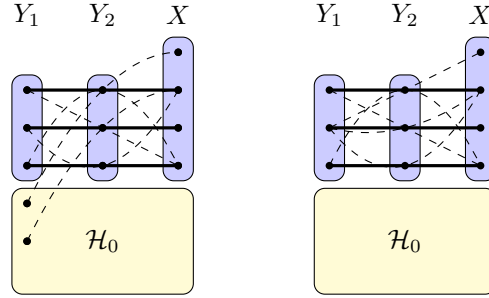(5*) $N_{\mathrm{lk}_{\mathcal{H}}(V_j)}(X) = Y_1$.

Figure 7: A cromulent triple (left) and a perfectly cromulent triple (right). In both cases, $\mathcal{H}_0$ is a home-base hypergraph with $\nu(\mathcal{H}_0) = \nu(\mathcal{H}) - |Y_1|$. The edges incident to X that reach into $\mathcal{H}_0$ only hit vertices in $V(\mathcal{F}) \cup V(\mathcal{R})$ in V_1, and such edges don't exist in perfectly cromulent triples. All edges incident to X are shown; there might be further edges incident to Y_i.

The first lemma of this section states that perfectly cromulent triples are the kind of substructure we should look for in order to prove our main theorem.

Lemma 9.2. *Let $\mathcal{H}$ be a 3-partite 3-graph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. If $\mathcal{H}$ has a perfectly cromulent triple, then $\mathcal{H}$ is a home-base hypergraph.*

Unfortunately, it is sometimes hard to ensure property (5*), and it will be easier to find just cromulent triples instead. Fortunately, we will be able to prove that this suffices.

Lemma 9.3. *If $\mathcal{H}$ is a 3-partite 3-graph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, then every cromulent triple of $\mathcal{H}$ is perfectly cromulent.*

These two lemmas combine to give the main result of this section as an immediate corollary:

Corollary 9.4. *Let $\mathcal{H}$ be a 3-partite 3-graph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. If $\mathcal{H}$ has a cromulent triple, then $\mathcal{H}$ is a home-base hypergraph.*

The proofs of the two lemmas follow similar lines, and so they will be handled in parallel. The basic idea is outlined below. We start with Lemma 9.2.

Let (Y_1, Y_2, X) be a perfectly cromulent triple, and let $\mathcal{H}_0 = \mathcal{H} - (Y_1 \cup Y_2 \cup X)$ be the hypergraph from the definition of cromulent triples. Let $(\mathcal{F}, \mathcal{R}, W)$ be a home-base partition of $\mathcal{H}_0$. Our goal will be to extend this partition into a home-base partition $(\mathcal{F}', \mathcal{R}', W')$ of $\mathcal{H}$. Fix a maximum hypergraph matching $\mathcal{M}$ in $\mathcal{H}|_{Y_1 \cup Y_2 \cup X}$. Each pair $y \in Y_1$, $y' \in Y_2$ that are together in an edge of $\mathcal{M}$ will participate in a new $R \in \mathcal{R}'$ together with a uniquely determined member of $W \cap V_3$. The vertices in X will be vertices of W', and by virtue of the matching saturating Y_1 and Y_2, they will ensure a matching saturating $\mathcal{R}'$ exists in the bipartite graph B_3'. The rest of the section will be devoted to finding the member of $W \cap V_3$ we can include in our new R's and proving that the resulting partition $(\mathcal{F}', \mathcal{R}', W')$ is indeed a home-base partition. Our fundamental tool in this proof will be Corollary 8.6, and we will finish by using Proposition 8.9.

If (Y_1, Y_2, X) was simply a cromulent triple, then much of the same proof as above still goes through in a more restricted form, and eventually we will be able to find a contradiction if (Y_1, Y_2, X) violated condition (5*), which will show Lemma 9.3.

We first introduce a notion which will be helpful for our upcoming proofs.

9.1 Heavy Vertex Covers

Recall the definition of essential subsets and superfluous vertices from Section 8.

The following is a particular type of vertex cover for home-base hypergraphs, which will be useful for the proofs in this and the next section.

Definition 9.5. Let $\mathcal{H}$ be a home-base hypergraph on vertex classes V_1, V_2, and V_3 with a home-base partition $(\mathcal{F}, \mathcal{R}, W)$, and let $i, j \in \{1, 2, 3\}$ with $i \neq j$. Let $C_i \subseteq W \cap V_i$ be the maximal essential set in B_i and let $\mathcal{U}_i \subseteq \mathcal{R}$ be the set with $|\mathcal{U}_i| = |C_i|$ and $N_{B_i}(\mathcal{U}_i) = C_i$. Then the union of the sets

- $C_i \cup ((V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_i)$

- $\left(\bigcup_{R \in \mathcal{R} \setminus \mathcal{U}_i} R \right) \cap V_j$

is called the *i-heavy (i, j)-cover* of $\mathcal{H}$.

Observation 9.6. *Every vertex in V_i which is not in the i-heavy (i, j)-cover is a superfluous vertex in $W \cap V_i$.*
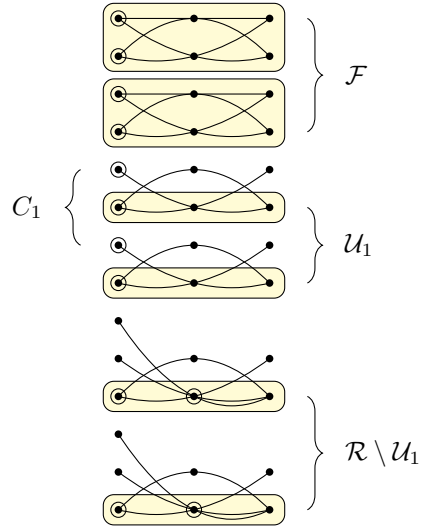
Figure 8: The 1-heavy $(1, 2)$-cover of a home-base hypergraph $\mathcal{H}$.

Proposition 9.7. *If $\mathcal{H}$ is a home-base hypergraph on vertex classes V_1, V_2, and V_3 with a home-base partition $(\mathcal{F}, \mathcal{R}, W)$, then for every pair $i, j \in \{1, 2, 3\}$ with $i \neq j$, the i-heavy (i, j)-cover is a minimal vertex cover of $\mathcal{H}_0$.*

Proof. Let T be the i-heavy (i, j)-cover of $\mathcal{H}$. Let $e \in E(\mathcal{H})$. Then by the edge-home property, e is at home in some $F \in \mathcal{F}$ or some $R \in \mathcal{R}$. If it is at home in F, then it contains some vertex in $F \cap V_i$, and so it intersects T. If it is at home in $R \in \mathcal{R} \setminus \mathcal{U}_i$, then it contains some vertex in $R \cap (V_i \cup V_j)$, and hence intersects T. The only remaining case is that e is at home in some $R' \in \mathcal{U}_i$. Let $V_i \cap e = \{v\}$. If $v \in V(\mathcal{F}) \cup V(\mathcal{R})$, then e intersects T. If $v \in W \cap V_i$, then vR' is an edge of B_i, and hence $v \in N_{B_i}(\mathcal{U}_i) = C_i$, which shows that e again intersects T. Thus T is a vertex cover of $\mathcal{H}$.

We now calculate the size of T. By the definition of the i-heavy (i, j)-cover, we get $|T| = 2\,|\mathcal{F}| + |\mathcal{R}| + |C_i| + |\mathcal{R}| - |\mathcal{U}_i|$. Since $|C_i| = |\mathcal{U}_i|$, we get $|T| = 2\,|\mathcal{F}| + 2\,|\mathcal{R}| = 2\,|\mathcal{F} \cup \mathcal{R}| = 2\nu(\mathcal{H})$, and because home-base hypergraphs are tight for Ryser's Conjecture by Proposition 7.5, we get $|T| = \tau(\mathcal{H})$ as desired. $\qquad\square$

9.2 Facts About Cromulent Triples

We start with some lemmas about cromulent and perfectly cromulent triples. Note that properties (2) and (5*) make the roles of Y_1 and Y_2 symmetric in perfectly cromulent triples. This gives us the following observation:

Observation 9.8. (Y_1, Y_2, X) *is a perfectly cromulent triple if and only if* (Y_1, Y_2, X) *and* (Y_2, Y_1, X) *are both cromulent triples.*

Most of the proofs in this section work for cromulent triples, and can be strengthened for perfectly cromulent triples by using Observation 9.8.

Assumptions. For the rest of this section, let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph with vertex classes V_1, V_2, and V_3 such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, and assume it has a cromulent triple (Y_1, Y_2, X). We will assume without loss of generality that $Y_1 \subseteq V_1$, $Y_2 \subseteq V_2$, and $X \subseteq V_3$. We also fix a hypergraph matching $M \subseteq E(\mathcal{H}|_{Y_1 \cup Y_2 \cup X})$ of size $|Y_1|$. Let $\mathcal{H}_0 = \mathcal{H} - (Y_1 \cup Y_2 \cup X)$ be the corresponding home-base hypergraph, and fix a home-base partition $(\mathcal{F}, \mathcal{R}, W)$ of $\mathcal{H}_0$.

Lemma 9.9. *For every pair $(i, j) \in \{(1, 2), (1, 3), (2, 1)\}$ we have that for every $y \in Y_i$ there is an edge ywu, where $w \in W \cap V_j$, and $u \in V(\mathcal{H}_0) \setminus V(\mathcal{R})$. If (Y_1, Y_2, X) is perfectly cromulent, then this holds also for $(i, j) = (2, 3)$.*

Proof. We will construct a vertex set T of size $\tau(\mathcal{H}) - 1$ which intersects all edges of $\mathcal{H}$ except for the edges of the form in question. Since T cannot be a vertex cover by virtue of its small size, some such edge must exist. Let T be the union of the sets $Y_1 \cup Y_2 \setminus \{y\}$, $(V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_j$, and $V(\mathcal{R}) \cap V_k$, where $k \in \{1, 2, 3\} \setminus \{i, j\}$. Since we have taken two vertices from each $F \in \mathcal{F}$ and two vertices from each $R \in \mathcal{R}$, and $2|Y_1| - 1$ additional vertices, we get $|T| = 2|\mathcal{F} \cup \mathcal{R}| + 2|Y_1| - 1 = 2\nu(\mathcal{H}_0) + 2|Y_1| - 1 = 2\nu(\mathcal{H}) - 1 = \tau(\mathcal{H}) - 1$, hence T is not a vertex cover of $\mathcal{H}$.

It is clear that T includes a cover of all edges of $\mathcal{H}_0$, so any uncovered edge must contain y or intersect X. It turns out that any edge e intersecting X is also covered by T. If $i = 1$, then e is covered by $N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X) = Y_2 \subseteq T$. If $i = 2$, then $j = 1$ and e is covered by $N_{\mathrm{lk}_{\mathcal{H}}(V_2)}(X) \subseteq Y_1 \cup (V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_1 \subseteq T$. Therefore, any edge not covered by T must contain y and two vertices of $\mathcal{H}_0$. The V_j-vertex must be a W-vertex because $(V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_j \subseteq T$, and the V_k-vertex cannot be in $V(\mathcal{R})$ because $V(\mathcal{R}) \cap V_k \subseteq T$. $\qquad\square$

Lemma 9.10. *For every pair $(i, j) \in \{(1, 2), (1, 3), (2, 1)\}$ we have that for every $y \in Y_i$ there is an edge ysu, where $s \in W \cap V_j$ is superfluous, and $u \in V(\mathcal{H}_0)$. If (Y_1, Y_2, X) is perfectly cromulent, then this holds also for $(i, j) = (2, 3)$.*

Proof. We will construct a vertex set T of size $\tau(\mathcal{H}) - 1$ which intersects all edges of $\mathcal{H}$ except for the edges of the form in question. Since T cannot be a vertex cover by virtue of its small size, some such edge must exist. Let T be the union of $Y_1 \cup Y_2 \setminus \{y\}$ and the j-heavy (j, i)-cover of $\mathcal{H}_0$. Since we have taken $\tau(\mathcal{H}_0)$ vertices from $\mathcal{H}_0$ and $2|Y_1| - 1$ additional vertices, we get $|T| = 2|\mathcal{F} \cup \mathcal{R}| + 2|Y_1| - 1 = \tau(\mathcal{H}) - 1$ (as calculated before). As in the proof of Lemma 9.9, the V_i-vertex of any uncovered edge must be y, and the other vertices are in $V(\mathcal{H}_0)$. The V_j-vertex of an uncovered edge must be a superfluous vertex because besides $(V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_j$, the maximal essential subset $C_j \subseteq W \cap V_j$ of B_j is also included in T (and every W-vertex outside of the maximal essential subset is by definition superfluous). $\qquad\square$

Lemma 9.11. *For $i = 1$ and $j = 3$ we have that for every $y \in Y_i$, if yvs is an edge of $\mathcal{H}$ with $v \in V(\mathcal{H}_0)$ and $s \in V_j$ a superfluous vertex, then there is an edge $yv's$ with $v' \in V(\mathcal{H}_0) \setminus V(\mathcal{R})$. If (Y_1, Y_2, X) is perfectly cromulent, then this holds also for $(i, j) = (2, 3)$.*

Proof. We may assume $v \in V(\mathcal{R})$, otherwise we are done. Let $y' \in Y_2$ be the V_2-vertex of the edge of $\mathcal{M}$ containing y.

By Lemma 9.9 (with $(i,j) = (2,1)$ for $y' \in Y_2$), there is an edge $wy'u$ with $w \in W \cap V_1$ and $u \in V(\mathcal{H}_0) \setminus V(\mathcal{R})$. We claim $s = u$.

Suppose not. Then yvs and $wy'u$ are disjoint edges. We can apply Case (2) of Corollary 8.6 with $a = v$, $b = w$, $c = u$, and $S = \{s\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{s, u, v, w\}$. This matching together with the edges yvs, $wy'u$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction. Hence $s = u$.

By Lemma 9.10 (with $(i,j) = (1,2)$ for $y \in Y_1$), there is an edge $yv'u'$ with v' a superfluous vertex in $W \cap V_2$. If $u' \neq s$, then $yv'u'$ and $wy's$ are disjoint edges. We can apply Case (4) of Corollary 8.6 with $a = v'$, $b = w$, $c = u'$, and $S = \{s\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{s, v', w, u'\}$. This matching together with the edges $yv'u'$, $wy's$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore $u' = s$, and thus $yv's$ is the edge we are looking for. $\square$

The next lemma is a strengthening of Lemma 9.11 in two ways: we can require more of our third vertex, and we can apply it to more combinations of i and j.

Lemma 9.12. *For $i = 1$ and for every $j \in \{2, 3\}$ we have that for every $y \in Y_i$, if yvs is an edge of $\mathcal{H}$ with $v \in V(\mathcal{H}_0)$ and $s \in V_j$ a superfluous vertex, then there is an edge $ys's$ with s' also superfluous. If (Y_1, Y_2, X) is perfectly cromulent, then this holds also for $i = 2$ and $j \in \{1, 3\}$.*

Proof. Let yvs be an edge with $v \in V(\mathcal{H}_0)$ and $s \in V_j$ superfluous. Let $y' \in Y_2$ be the V_2-vertex of the edge of $\mathcal{M}$ containing y. There are two cases.

Case 1. $i = 1$, $j = 3$.

By Lemma 9.11 (with $(i,j) = (1,3)$), we may assume $v \in V(\mathcal{H}_0) \setminus V(\mathcal{R})$. By Lemma 9.10 (with $(i,j) = (2,1)$ for $y' \in Y_2$), there is an edge $s''y'u$ with $s'' \in V_i$ a superfluous vertex. If $s \neq u$, then yvs and $s''y'u$ are disjoint edges, and we will reach a contradiction as in the previous lemma. We can apply Case (4) of Corollary 8.6 with $a = s''$, $b = v$, $c = u$, and $S = \{s\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{s, s'', u, v\}$. This matching together with the edges yvs, $s''y'u$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

It follows that $s = u$. Lemma 9.10 (with $(i,j) = (1,2)$ for $y \in Y_1$) tells us that there is an edge $ys'u'$ with $s' \in V_2$ superfluous. It must be the case that $s = u'$ because otherwise $ys'u'$ and $s''y's$ are disjoint edges, and we would reach a similar contradiction. We can apply Case (4) of Corollary 8.6 with $a = s''$, $b = s'$, $c = u'$, and $S = \{s\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{s, s', s'', u'\}$. This matching together with the edges $ys'u'$, $s''y's$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore there is an edge $ys's$, as required.

Case 2. $i = 1$, $j = 2$.

By Lemma 9.10) (with $(i,j) = (1,3)$ for $y \in Y_1$) there is an edge $yr's'$ with $s' \in V_3$ superfluous, and then by Case 1, above, there is an edge yrs' with $r \in V_2$ and $s' \in V_3$ both superfluous. By Lemma 9.10 (with $(i,j) = (2,1)$ for $y' \in Y_2$), there is an edge $qy'u$ with $q \in V_1$ a superfluous vertex and $u \in V(\mathcal{H}_0)$. If $u \neq s'$, then we will again reach a contradiction. Suppose yrs' and $qy'u$ are disjoint. We can apply Case (4) of Corollary 8.6 with $a = q$, $b = r$, $c = u$, and $S = \{s'\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{q, r, s', u\}$. This matching together with the edges yrs', $qy'u$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore $u = s'$. A similar contradiction is reached by ysv and $qy's'$ if $v \neq s'$, so that cannot be the case either. Suppose ysv and $qy's'$ are disjoint. We can apply Case (4) of Corollary 8.6 with $a = q$, $b = s$, $c = v$, and $S = \{s'\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{q, s, s', v\}$. This matching together with the edges ysv, $qy's'$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore we have found our edge yss'. $\qquad\square$

Lemma 9.13. *Let $y \in Y_1$ and $y' \in Y_2$ be in an edge of $\mathcal{M}$ together. Then there is a unique superfluous vertex $z_{y,y'} \in V_3$ such that*

(i) *There are edges $yvz_{y,y'}$ and $uy'z_{y,y'}$ for some vertices $u, v \in V(\mathcal{H}_0)$,*

(ii) *If $yv's'$ or $u'y's'$ is an edge with s' superfluous, then $s' = z_{y,y'}$.*

Proof. By Lemma 9.10 (with $(i,j) = (1,3)$ for $y \in Y_1$) there is an edge yvs with $v \in V(\mathcal{H}_0)$ and $s \in V_3$ superfluous. We claim that s satisfies (i) and (ii).

To see (i), we only need to find $uy's$, since we have yvs. By Lemma 9.12 (with $(i,j) = (1,2)$), we may assume v is superfluous as well. By Lemma 9.10 (with $(i,j) = (2,1)$ for $y' \in Y_2$), we have an edge $s'y'u'$ with $s' \in W \cap V_1$ superfluous. Suppose $u' \neq s$. Then yvs and $s'y'u'$ are disjoint edges. We can apply Case (4) of Corollary 8.6 with $a = v$, $b = s'$, $c = u'$, and $S = \{s\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{s, s', u', v\}$. This matching together with the edges yvs, $s'y'u'$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore $u' = s$, and we have the desired edge $s'y's$.

We now show (ii). Let $yv's'$ and $u'y's''$ be edges of $\mathcal{H}$ with $s', s'' \in V_3$ both superfluous vertices. By Lemma 9.12 (with $(i,j) = (1,2)$), we may assume v' is superfluous as well. If $s' \neq s''$, then $yv's'$ and $u'y's''$ are disjoint edges. This leads to a contradiction as before. We can apply Case (4) of Corollary 8.6 with $a = v'$, $b = s'$, $c = u'$, and $S = \{s''\}$ to find a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0 - \{s', s'', u', v'\}$. This matching together with the edges $yv's'$, $u'y's''$, and the rest of $\mathcal{M}$ (besides the edge containing y and y') forms a matching of size $\nu(\mathcal{H}_0) + 2 + |Y_i| - 1 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore it must be the case that $s' = s''$, which in particular means that $s' = s'' = s$, since we could have substituted yvs or $uy's$ for $yv's'$ or $u'y's''$, respectively. $\qquad\square$

Our aim is to make each set $\{y, y', z_{y,y'}\}$ into an R for our home-base partition. We will first show that the $z_{y,y'}$'s are all distinct, and then we will make use of Lemma 8.9 to show that combining the new R's with the home-base partition of $\mathcal{H}_0$ forms a home-base partition of $\mathcal{H}$.

Lemma 9.14. *For each (y, y')-pair, the associated $z_{y,y'}$ is distinct, and there is a matching saturating $\mathcal{R}$ in the subgraph of B_3 induced by $\mathcal{R} \cup (V_3 \cap W \setminus Z)$, where Z is the set of all $z_{y,y'}$'s.*

Proof. Define the bipartite graph K with parts $\mathcal{R} \cup Y_1$ and $W \cap V_3$, where there is an edge between $R \in \mathcal{R}$ and $w \in W \cap V_3$ precisely when there is an R-edge containing w, and there is an edge between $y \in Y_1$ and $w \in W \cap V_3$ precisely when $w = z_{y,y'}$, where y' is the partner of y in the pairing between Y_1 and Y_2. We claim that K has a matching saturating $\mathcal{R} \cup Y_1$.

We will apply Hall's theorem, so let $\mathcal{R}_0 \subseteq \mathcal{R}$ and $Y_0 \subseteq Y_1$. We construct a vertex cover T of $\mathcal{H}$. Let C_3 be the maximal essential set in the subgraph of K induced by $\mathcal{R}$ and $W \cap V_3$ (this is the graph B_3 associated with $\mathcal{H}_0$), and let $\mathcal{U}_3 \subseteq \mathcal{R}$ be such that $N_K(\mathcal{U}_3) = C_3$, which exists by the definition of essential. Let T be the union of the sets $(Y_1 \cup Y_2) \setminus Y_0$, $N_K(\mathcal{R}_0 \cup Y_0)$, $(V(\mathcal{R}) \cup V(\mathcal{F})) \cap V_3$, C_3, and $\bigcup_{R \in \mathcal{R} \setminus (\mathcal{U}_3 \cup \mathcal{R}_0)} (R \cap V_1)$. Note the similarities to the 3-heavy $(3,1)$-cover of $\mathcal{H}_0$.

We must show that T is indeed a vertex cover. Let $e \in E(\mathcal{H}_0)$. Then e is either an $\mathcal{F}$-edge or an $\mathcal{R}$-edge. If it is an $\mathcal{F}$-edge, it is covered by $V(\mathcal{F}) \cap V_3 \subseteq T$. If it is an $\mathcal{R}$-edge, then it is covered by $(V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_3 \subseteq T$, unless its V_3-vertex is in W, so assume that is the case. Let e be an R-edge. If $R \in \mathcal{R}_0$, then $e \cap V_3 \in N_K(R) \subseteq T$. If $R \in \mathcal{U}_3$, then $e \cap V_3 \in C_3 \subseteq T$. If $R \in \mathcal{R} \setminus (\mathcal{U}_3 \cup \mathcal{R}_0)$, then $e \cap V_1 = R \cap V_1 \subseteq T$. This shows that T covers every edge of $\mathcal{H}_0$. All edges incident to X intersect Y_2, so any uncovered edge must be incident to Y_0 and two vertices of $\mathcal{H}_0$. All such edges whose V_3-vertex is not superfluous intersect T, since $C_3 \cup (V(\mathcal{R}) \cup V(\mathcal{F})) \cap V_3 \subseteq T$. Thus, the only edges we have to worry about are those incident to some $y \in Y_0$ and a superfluous vertex in V_3. Then by Lemma 9.13, the V_3-vertices of those edges are the corresponding $z_{y,y'}$, and hence those edges intersect $N_K(Y_0) \subseteq T$. This shows that T is a vertex cover.

We now calculate the size of T. By the definition of T, we calculate $|T| = |Y_1| + |Y_2| - |Y_0| + |N_K(\mathcal{R}_0 \cup Y_0)| + 2|\mathcal{F}| + |\mathcal{R}| + |C_3| - |C_3 \cap N_K(\mathcal{R}_0)| + |\mathcal{R}| - |\mathcal{U}_3 \cup \mathcal{R}_0|$. Because it is a vertex cover, we must have $|T| \geq \tau(\mathcal{H})$. Since $\nu(\mathcal{H}) = \nu(\mathcal{H}_0) + |Y_1|$ by the definition of cromulent triple, and since $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, we have $\tau(\mathcal{H}) = 2\nu(\mathcal{H}_0) + 2|Y_1| = 2|\mathcal{F} \cup \mathcal{R}| + |Y_1| + |Y_2|$. Combining this with the fact that $\tau(\mathcal{H}) \leq |T|$ yields the inequality $|Y_0| + |\mathcal{U}_3 \cup \mathcal{R}_0| + |C_3 \cap N_K(\mathcal{R}_0)| \leq |N_K(\mathcal{R}_0 \cup Y_0)| + |C_3|$. By the inclusion-exclusion principle we can rewrite this as $|Y_0| + |\mathcal{U}_3| + |\mathcal{R}_0| - |\mathcal{U}_3 \cap \mathcal{R}_0| + |C_3 \cap N_K(\mathcal{R}_0)| \leq |N_K(\mathcal{R}_0 \cup Y_0)| + |C_3|$. Since $C_3 = N_K(\mathcal{U}_3)$, we clearly have $C_3 \cap N_K(\mathcal{R}_0) \supseteq N_K(\mathcal{U}_3 \cap \mathcal{R}_0)$. Since B_3 has a matching saturating $\mathcal{R}$, by Hall's Theorem, we must have $|\mathcal{U}_3 \cap \mathcal{R}_0| \leq |N_K(\mathcal{U}_3 \cap \mathcal{R}_0)|$. Combining this with our previous inequality, we then get $|Y_0| + |\mathcal{U}_3| + |\mathcal{R}_0| - |\mathcal{U}_3 \cap \mathcal{R}_0| + |\mathcal{U}_3 \cap \mathcal{R}_0| \leq |N_K(\mathcal{R}_0 \cup Y_0)| + |C_3|$, which simplifies to $|Y_0| + |\mathcal{R}_0| \leq |N_K(\mathcal{R}_0 \cup Y_0)|$, since $|\mathcal{U}_3| = |C_3|$. This last inequality shows that we can apply

Hall's Theorem to find a matching in K saturating $\mathcal{R} \cup Y_0$, which proves the lemma. □
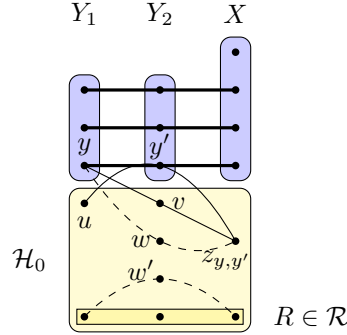

Figure 9: An illustration of Lemma 9.15. The hypergraph edges indicated by dashed lines correspond to the two types of edges in the bipartite graph K_1. Here $w, w' \in W \cap V_2$.

Lemma 9.15. *For $i = 2$, let K_i be the bipartite graph with parts $\mathcal{R} \cup Y_{3-i}$ and $W \cap V_i$, where there is an edge between $R \in \mathcal{R}$ and $w \in W \cap V_i$ precisely when there is an R-edge containing w, and there is an edge between $y \in Y_{3-i}$ and $w \in W \cap V_i$ precisely when there is an edge $y w z_{y,y'}$, where y' is the partner of y in the pairing between Y_1 and Y_2. Then K_i has a matching saturating $\mathcal{R} \cup Y_{3-i}$. If (Y_1, Y_2, X) is perfectly cromulent, then this holds also for $i = 1$.*

Proof. We will apply Hall's theorem, so let $\mathcal{R}_0 \subseteq \mathcal{R}$ and $Y_0 \subseteq Y_{3-i}$. We construct a vertex cover T of $\mathcal{H}$. Let C_i be the maximal essential set in the subgraph of K_i induced by $\mathcal{R}$ and $W \cap V_i$ (this is the graph B_i associated with $\mathcal{H}_0$), and let $\mathcal{U}_i \subseteq \mathcal{R}$ be such that $N_{K_i}(\mathcal{U}_i) = C_i$, which exists by the definition of essential. Let T be the union of the sets $(Y_1 \cup Y_2) \setminus Y_0$, $N_{K_i}(\mathcal{R}_0 \cup Y_0)$, $(V(\mathcal{R}) \cup V(\mathcal{F})) \cap V_i$, C_i, and $\bigcup_{R \in \mathcal{R} \setminus (\mathcal{U}_i \cup \mathcal{R}_0)}(R \cap V_3)$. Note the similarities to the i-heavy $(i, 3)$-cover of $\mathcal{H}_0$.

We must show that T is indeed a vertex cover. Let $e \in E(\mathcal{H}_0)$. Then e is either an $\mathcal{F}$-edge or an $\mathcal{R}$-edge. If it is an $\mathcal{F}$-edge, it is covered by $V(\mathcal{F}) \cap V_i \subseteq T$. If it is an $\mathcal{R}$-edge, then it is covered by $(V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_i \subseteq T$, unless its V_i-vertex is in W, so assume that is the case. Let e be an $\mathcal{R}$-edge. If $R \in \mathcal{R}_0$, then $e \cap V_i \in N_K(R) \subseteq T$. If $R \in \mathcal{U}_i$, then $e \cap V_i \in C_i \subseteq T$. If $R \in \mathcal{R} \setminus (\mathcal{U}_i \cup \mathcal{R}_0)$, then $e \cap V_3 = R \cap V_3 \subseteq T$. This shows that T covers every edge of $\mathcal{H}_0$. All edges incident to X intersect Y_2, which if $i = 2$ is part of T, and if $i = 1$, then (Y_1, Y_2, X) is assumed to be perfectly cromulent, in which case all edges incident to X are incident to $Y_1 \subseteq T$. Therefore, any uncovered edge must be incident to Y_0 and two vertices of $\mathcal{H}_0$. All such edges whose V_3-vertex is not superfluous intersect T, since $C_i \cup (V(\mathcal{R}) \cup V(\mathcal{F})) \cap V_i \subseteq T$. Thus, the only edges we have to worry about are those incident to some $y \in Y_0$ and a superfluous vertex $s \in V_i$. By Lemma 9.12 (with $(i, j) = (3 - i, i)$), there is an edge containing y and s, whose V_3-vertex is also superfluous. By Lemma 9.13, the V_3-vertices

of those edges are the corresponding $z_{y,y'}$, and hence their V_2-vertices are in $N_{K_i}(Y_0) \subseteq T$ by the definition of K_i. This shows that T is a vertex cover.

We now calculate the size of T. By the definition of T, we calculate $|T| = |Y_1| + |Y_2| - |Y_0| + |N_{K_i}(\mathcal{R}_0 \cup Y_0)| + 2|\mathcal{F}| + |\mathcal{R}| + |C_i| - |C_i \cap N_{K_i}(\mathcal{R}_0)| + |\mathcal{R}| - |\mathcal{U}_i \cup \mathcal{R}_0|$. Because it is a vertex cover, we must have $|T| \geq \tau(\mathcal{H})$. Since $\nu(\mathcal{H}) = \nu(\mathcal{H}_0) + |Y_1|$ by the definition of cromulent triple, and since $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, we have $\tau(\mathcal{H}) = 2\nu(\mathcal{H}_0) + 2|Y_1| = 2|\mathcal{F} \cup \mathcal{R}| + |Y_1| + |Y_2|$. Combining this with the fact that $\tau(\mathcal{H}) \leq |T|$ yields the inequality $|Y_0| + |\mathcal{U}_i \cup \mathcal{R}_0| + |C_i \cap N_{K_i}(\mathcal{R}_0)| \leq |N_{K_i}(\mathcal{R}_0 \cup Y_0)| + |C_i|$. By the inclusion-exclusion principle we can rewrite this as $|Y_0| + |\mathcal{U}_i| + |\mathcal{R}_0| - |\mathcal{U}_i \cap \mathcal{R}_0| + |C_i \cap N_{K_i}(\mathcal{R}_0)| \leq |N_{K_i}(\mathcal{R}_0 \cup Y_0)| + |C_i|$. Since $C_i = N_{K_i}(\mathcal{U}_i)$, we clearly have $C_i \cap N_{K_i}(\mathcal{R}_0) \supseteq N_{K_i}(\mathcal{U}_i \cap \mathcal{R}_0)$. Since B_i has a matching saturating $\mathcal{R}$, by Hall's Theorem, we must have $|\mathcal{U}_i \cap \mathcal{R}_0| \leq |N_{K_i}(\mathcal{U}_i \cap \mathcal{R}_0)|$. Combining this with our previous inequality, we then get $|Y_0| + |\mathcal{U}_i| + |\mathcal{R}_0| - |\mathcal{U}_i \cap \mathcal{R}_0| + |\mathcal{U}_i \cap \mathcal{R}_0| \leq |N_{K_i}(\mathcal{R}_0 \cup Y_0)| + |C_i|$, which simplifies to $|Y_0| + |\mathcal{R}_0| \leq |N_{K_i}(\mathcal{R}_0 \cup Y_0)|$, since $|\mathcal{U}_i| = |C_i|$. This last inequality shows that we can apply Hall's Theorem to find a matching in K_i saturating $\mathcal{R} \cup Y_0$, which proves the lemma. $\square$

9.3   The Proof of Corollary 9.4

It suffices to prove Lemmas 9.2 and 9.3.

Proof of Lemma 9.2. Let (Y_1, Y_2, X) be a perfectly cromulent triple. We set $\mathcal{R}' = \mathcal{R} \cup \{\{y, y', z_{y,y'}\} : y \in Y_1, y' \in Y_2 \text{ in an edge of } \mathcal{M} \text{ together with } y\}$, and $W' = W \cup X \setminus \{z_{y,y'} : y \in Y_1, y' \in Y_2 \text{ in an edge of } \mathcal{M} \text{ together with } y\}$, where $z_{y,y'}$ is the superfluous vertex in V_3 from Lemma 9.13. By applying Lemma 9.14, we find that $(\mathcal{F}, \mathcal{R}', W')$ is an FR-partition, since $\nu(\mathcal{H}) = \nu(\mathcal{H}_0) + |Y_1| = |\mathcal{F} \cup \mathcal{R}| + |Y_1| = |\mathcal{F} \cup \mathcal{R}'|$. Applying 9.15 for $i = 1, 2$ we get that $(\mathcal{F}, \mathcal{R}', W')$ has a matching in B_1' and B_2'. We can combine the partial matching in B_3' that we get from Lemma 9.14 with the edges of $\mathcal{M}$ going to X to complete it. Thus $(\mathcal{F}, \mathcal{R}', W')$ is a matchable FR-partition. It is clearly also proper, because there are no edges with three vertices in W' by virtue of the fact that no such edge is in $\mathcal{H}_0$ and all edges going to X have their other vertices in Y_1 and Y_2. Thus, by Proposition 8.9, we in fact have a home-base partition. $\square$

Proof of Lemma 9.3. Let (Y_1, Y_2, X) be a cromulent triple. We now mean to rule out the possibility that any edge incident to X is also incident to an $\mathcal{F}$- or $\mathcal{R}$-vertex of $\mathcal{H}_0$. Lemma 9.15 means that we can find a hypergraph matching $\mathcal{M}'$ of size $|Y_1|$ in $\mathcal{H}$ consisting of edges of the form yss' with $y \in Y_1$, and s, s' superfluous vertices in $\mathcal{H}_0$. Suppose there were an edge $uy'x$ for some $u \in (V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_1$, $y' \in Y_2$, and $x \in X$. By the matchability of B_1, we can choose a matching of WRR-edges for each $R \in \mathcal{R}$, which avoids u, since $u \notin W$. We can also clearly find a matching of $\mathcal{F}$-edges avoiding u. Combining these matchings with $\mathcal{M}'$ yields a hypergraph matching of size $\nu(\mathcal{H})$ which is disjoint from $uy'x$. This is impossible, so such an edge cannot exist. Therefore (Y_1, Y_2, X) is a perfectly cromulent triple. $\square$

Therefore, we have shown that if we have a cromulent triple, we have a home-base hypergraph. The next section is devoted to finding cromulent triples under various assumptions.

10 Searching for Cromulent Triples

Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3, and with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. We want to find a home-base partition of $\mathcal{H}$. By Corollary 9.4, we are done if $\mathcal{H}$ has a cromulent triple. Therefore, our goal will be to find a cromulent triple inside our hypergraph. We will do this under a few assumptions, and we will later show that if all of these assumptions fail to hold, then we can prove $\mathcal{H}$ is a home-base hypergraph even without cromulent triples.

Finding cromulent triples will entail finding a subgraph which is a home-base hypergraph. We do this by finding a subgraph which is tight for Ryser's Conjecture and has a smaller matching number than $\mathcal{H}$, and then applying induction on Theorem 1.1. We would like to pinpoint exactly where in the proof we need to rely on induction. Therefore, we lay out the induction hypothesis here precisely.

Induction Hypothesis (IH(k))**.** If $\mathcal{H}$ is a 3-partite 3-graph with $\nu(\mathcal{H}) \leq k$ and $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, then $\mathcal{H}$ is a home-base hypergraph.

The first assumption under which we will find a cromulent triple is if we have a good set (see Definition 5.6).

10.1 Good Subsets Lead to Cromulent Triples

Lemma 10.1. *Suppose IH($k - 1$) holds. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3 such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H}) = 2k$. If $X \subseteq V_3$ is a good set for $\mathrm{lk}_{\mathcal{H}}(V_1)$, then the triple (Y_1, Y_2, X) is perfectly cromulent, where $Y_1 = N_{\mathrm{lk}_{\mathcal{H}}(V_2)}(X)$ and $Y_2 = N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$.*

Proof. Let $X \subseteq V_3$ be a good set, and let $Y_2 = N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$. Let $y \in Y_2$, and let $\mathcal{H}_y = \mathcal{H} - \{vyz \in E(\mathcal{H}) : v \in V_1, z \in V_3 \setminus X\}$. Since the deleted edges can be covered by one vertex (y), we clearly have $\tau(\mathcal{H}_y) \geq \tau(\mathcal{H}) - 1$, and of course $\nu(\mathcal{H}_y) \leq \nu(\mathcal{H})$ as $\mathcal{H}_y \subseteq \mathcal{H}$. It is easy to see that $\mathrm{lk}_{\mathcal{H}_y}(V_1) = \mathrm{lk}_{\mathcal{H}}(V_1) - \{yz \in E(\mathrm{lk}_{\mathcal{H}}(V_1)) : z \in V_3 \setminus X\}$. Therefore, because X is good, we have $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(V_1)\right)\right) \geq \eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_1)\right)\right) + 1$. Recall that by Theorem 2.4, we have $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_1)\right)\right) = \nu(\mathcal{H})$. Thus, we in fact have $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(V_1)\right)\right) \geq \nu(\mathcal{H}) + 1$. By Proposition 3.2, there is a subset $S \subseteq V_1$ for which we have $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(S)\right)\right) \leq \nu(\mathcal{H}_y) - (|V_1| - |S|)$ and $|S| \geq |V_1| - (2\nu(\mathcal{H}_y) - \tau(\mathcal{H}_y))$. (Note that $|V_1| \geq 2$, so Proposition 3.2 part (ii) applies.) Plugging in the inequalities for τ and ν, we get

$$\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(S)\right)\right) \leq \nu(\mathcal{H}) - (|V_1| - |S|)$$

and

$$|S| \geq |V_1| - (2\nu(\mathcal{H}) - \tau(\mathcal{H}) + 1) = |V_1| - 1$$

since $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$.

We have seen that V_1 itself does not fulfil the first of these inequalities, so S must be a proper subset of V_1, and thus by the second inequality, $S = V_1 \setminus \{a\}$ for some $a \in V_1$. A priori, we do not know if this a is unique for each $y \in Y_2$, so denote by A_y the set of all V_1-vertices a for which $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(V_1 \setminus \{a\})\right)\right) \leq \nu(\mathcal{H}) - 1$.

Let $a \in A_y$ and let $S = V_1 \setminus \{a\}$. By Theorem 2.2, we have $\nu(\mathrm{lk}_{\mathcal{H}_y}(S)) \leq 2\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(S)\right)\right) \leq 2\nu(\mathcal{H}) - 2 = \tau(\mathcal{H}) - 2$, which implies that $\nu(\mathrm{lk}_{\mathcal{H}}(S)) \leq \tau(\mathcal{H}) - 1$ because at most one edge of each maximum matching has been erased when passing from $\mathcal{H}$ to $\mathcal{H}_y$ in the link of S. We must have $\tau(\mathcal{H}_y) = \tau(\mathcal{H}) - 1$ because if $\tau(\mathcal{H}_y) = \tau(\mathcal{H})$, then by inequality (i) of Proposition 3.2, we would have $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(S)\right)\right) \geq \tau(\mathcal{H}_y)/2$ (since $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_y}(S)\right)\right)$ is an integer and $\tau(\mathcal{H}_y) = \tau(\mathcal{H})$ is even), which is a contradiction. We can in fact show $\nu(\mathrm{lk}_{\mathcal{H}}(S)) = \tau(\mathcal{H}) - 1$, from which $\nu(\mathrm{lk}_{\mathcal{H}_y}(S)) = \tau(\mathcal{H}) - 2$ then follows, by considering the vertex cover T_S of $\mathcal{H}$ consisting of a and a minimum vertex cover of $\mathrm{lk}_{\mathcal{H}}(S)$ (which, by König's Theorem, has size $\nu(\mathrm{lk}_{\mathcal{H}}(S))$).

This means that every maximum matching in $\mathrm{lk}_{\mathcal{H}}(S)$ must contain an edge which is not in $\mathrm{lk}_{\mathcal{H}_y}(S)$. Set $Z = V_3 \setminus X$ and $W = V_2 \setminus Y_2$. We get the following structure for the maximum matchings:

Claim. *For every $y \in Y_2$ and for every $a \in A_y$ every maximum matching in $\mathrm{lk}_{\mathcal{H}}(V_1 \setminus \{a\})$ contains an edge yz for some $z \in Z$, and then saturates $Y_2 \setminus \{y\}$ using (X, Y_2)-edges and saturates $Z \setminus \{z\}$ using (Z, W)-edges.*

Proof. Let $S = V_1 \setminus \{a\}$. As observed, every maximum matching in $\mathrm{lk}_{\mathcal{H}}(S)$ contains an edge from y to Z. Since X is good (hence decent), it satisfies property (1) of Definition 5.1, so $\nu(\mathrm{lk}_{\mathcal{H}}(V_1)) = |Y_2| + |Z|$. Then because there are no edges between X and W, it follows that every maximum matching in $\mathrm{lk}_{\mathcal{H}}(V_1)$ saturates Y_2 with edges incident to X and saturates Z with edges incident to W. Since $\nu(\mathrm{lk}_{\mathcal{H}}(S)) = \tau(\mathcal{H}) - 1 = \nu(\mathrm{lk}_{\mathcal{H}}(V_1)) - 1$, we cannot have more than one matching edge between Y_2 and Z. Thus the claim follows. $\square$

This structure immediately implies that the sets A_y are pairwise disjoint.

Claim. *If $y, y' \in Y_2$ with $y \neq y'$, then $A_y \cap A_{y'} = \emptyset$.*

Proof. Let $a \in A_y$, and let $S = V_1 \setminus \{a\}$. Then we know that a maximum matching in $\mathrm{lk}_{\mathcal{H}}(S)$ contains a (y, Z)-edge and the rest of its edges are between X and Y_2 and between Z and W. Thus the only edge between Y_2 and Z in the matching is incident to y. For $a' \in A_{y'}$, the structure of the maximum matchings in $\mathrm{lk}_{\mathcal{H}}(V_1 \setminus \{a'\})$ is different, and thus $a \neq a'$, hence the sets A_y and A'_y must be disjoint. $\square$

Since every A_y is non-empty, we thus clearly have $\left|\bigcup_{y \in Y_2} A_y\right| \geq |Y_2|$.

Claim. *For every $a \in \bigcup_{y \in Y_2} A_y$, every maximum (X, Y_2)-matching in $\mathrm{lk}_{\mathcal{H}}(V_1)$ must have one edge which extends only to a.*

Proof. Suppose there were a maximum (X, Y_2)-matching M' in $\mathrm{lk}_{\mathcal{H}}(V_1)$ in which every edge extended to an element of $S = V_1 \setminus \{a\}$. Then we could take a maximum (V_2, V_3)-matching in $\mathrm{lk}_{\mathcal{H}}(S)$ (which must contain a (y, Z)-edge) and replace the part of the matching which hits Y_2 with M'. Because X has no neighbors outside of Y_2, this modified matching is a matching and is at least as big as the original one and therefore also maximum. This does not use a (y, Z)-edge, so we have a contradiction. Thus M' must contain an edge which does not extend to S, and hence extends only to a. $\qquad\square$

From this claim, we see that $\left| \bigcup_{y \in Y_2} A_y \right| = |Y_2|$, since there can be at most as many vertices in $\bigcup_{y \in Y_2} A_y$ as edges in a maximum (X, Y_2)-matching in $\mathrm{lk}_{\mathcal{H}}(V_1)$, of which there are precisely $|Y_2|$.

Claim. $Y_1 = \bigcup_{y \in Y_2} A_y$ and there is a hypergraph matching in $\mathcal{H}_{Y_1 \cup Y_2 \cup X}$ saturating Y_1 and Y_2.

Proof. We clearly have $Y_1 \supseteq \bigcup_{y \in Y_2} A_y$ by the previous claim. We will show the other inclusion as well. Consider any vertex $x \in Y_1$. It follows from the definitions of Y_1 and Y_2 that there is an (X, Y_2)-edge e in $\mathrm{lk}_{\mathcal{H}}(V_1)$ such that $e \cup \{x\} \in E(\mathcal{H})$. Since X is good, e appears in a maximum matching M. For every $y \in Y_2$ and every $a \in A_y$, one edge of the matching between X and Y_2 must extend to a (recall that to be maximum, M must saturate Y_2 using (Y_2, X)-edges and must saturate Z using (Z, W)-edges). Since the A_y's are all disjoint, the matching extends to a hypergraph matching saturating Y_2 and $\bigcup_{y \in Y_2} A_y$. Since e extends to $\bigcup_{y \in Y_2} A_y$, it follows that $x \in \bigcup_{y \in Y_2} A_y$ and hence $Y_1 = \bigcup_{y \in Y_2} A_y$. This proves the claim. $\qquad\square$

Now we almost have that (Y_1, Y_2, X) is perfectly cromulent. We just need to show that $\mathcal{H}_0 = \mathcal{H} \setminus (Y_1 \cup Y_2 \cup X)$ is a home-base hypergraph with $\nu(\mathcal{H}_0) = \nu(\mathcal{H}) - |Y_1|$.

Consider the graph $\mathcal{H}_1 = \mathcal{H} \setminus (Y_1 \cup Y_2)$. Since we have removed only $2|Y_1|$ vertices from $\mathcal{H}$, it follows that $\tau(\mathcal{H}_1) \geq \tau(\mathcal{H}) - 2|Y_1|$. We must have $\nu(\mathcal{H}_1) \leq \nu(\mathcal{H}) - |Y_1|$ because to any matching in $\mathcal{H}_1$, we may add the matching of size $|Y_1|$ we just showed exists to it to produce a matching in $\mathcal{H}$ (because no matching edge in the original matching is incident to $Y_1 \cup Y_2 \cup X$). Because $\tau(\mathcal{H}_1) \leq 2\nu(\mathcal{H}_1)$, we must have equality in both cases, whence $\tau(\mathcal{H}_1) = 2\nu(\mathcal{H}_1) = 2\nu(\mathcal{H}) - 2|Y_1|$. Note however that X is a set of isolated vertices in $\mathcal{H}_1$, and so removing them changes neither the matching size nor the covering number. Hence $\mathcal{H}_0 = \mathcal{H}_1 \setminus X$ also has $\tau(\mathcal{H}_0) = 2\nu(\mathcal{H}_0) = 2\nu(\mathcal{H}) - 2|Y_1|$. By induction on the matching number of the Ryser-tight hypergraph, $\mathcal{H}_0$ is a home-base hypergraph. This proves that (Y_1, Y_2, X) is a perfectly cromulent triple. $\qquad\square$

This lemma shows that if $\mathrm{lk}_{\mathcal{H}}(V_i)$ has a good set for any i, then we find a perfectly cromulent triple.

10.2   No Good Sets

From now on we assume that $\mathrm{lk}_{\mathcal{H}}(V_1)$ has no good set. Recall that by Theorem 2.4, we know that $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_1)\right)\right) = \nu(\mathcal{H})$, and so by Lemma 5.7 $\mathrm{lk}_{\mathcal{H}}(V_1)$ has a perfect matching. Moreover for every minimal equineighbored set $X \subseteq V_3$ both it and its neighborhood $N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$ have size 2 and together induce a C_4 (possibly with parallel edges). Our next assumption will be that there are two disjoint hyperedges incident to some minimal equineighbored set.

Lemma 10.2. *Suppose IH(k−1) holds. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3 such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H}) = 2k$, and let $\mathrm{lk}_{\mathcal{H}}(V_1)$ have no good sets. Suppose there is a minimal equineighbored set $X \subseteq V_3$ in $\mathrm{lk}_{\mathcal{H}}(V_1)$ such that there are two disjoint hyperedges zyx and $z'y'x'$ of $\mathcal{H}$ with $x, x' \in X$. Let $Y_1 = \{z, z'\} \subseteq V_1$ and $Y_2 = \{y, y'\} \subseteq V_2$. Then (Y_1, Y_2, X) is a cromulent triple.*

Proof. We check that conditions (1)-(5) of (Y_1, Y_2, X) being a cromulent triple hold.

For Condition (1) note that $|Y_1| = |Y_2| = |X| = 2$, since by Lemma 5.7 X has size 2.

Then $X = \{x, x'\}$ and because X is equineighbored, the neighborhood of X is also of size 2, that is, $N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X) = \{y, y'\}$. So Condition (2) is satisfied.

For Condition (3) note that by assumption there are two disjoint hyperedges zyx and $z'y'x'$ in $\mathcal{H}|_{Y_1 \cup Y_2 \cup X}$ and that $|Y_1| = 2$.

For Condition (4) we first prove that $\tau(\mathcal{H}_0) = 2\nu(\mathcal{H}_0) = 2(\nu(\mathcal{H}) - |Y_1|)$. Then we can use IH$(k-1)$ to derive the existence of a home-base partition of $\mathcal{H}_0$. First, consider the graph $\mathcal{H}_1 = \mathcal{H} \setminus (Y_1 \cup Y_2)$. Since we have removed only $2|Y_1|$ vertices from $\mathcal{H}$, it follows that $\tau(\mathcal{H}_1) \geq \tau(\mathcal{H}) - 2|Y_1|$. We must have $\nu(\mathcal{H}_1) \leq \nu(\mathcal{H}) - |Y_1|$ because X consists of isolated vertices in $\mathcal{H}_1$, so we may add zyx and $z'y'x'$ to any matching in $\mathcal{H}_1$ to obtain a matching 2 larger in $\mathcal{H}$. Because $\tau(\mathcal{H}_1) \leq 2\nu(\mathcal{H}_1)$, we must have equality in both cases, whence $\tau(\mathcal{H}_1) = 2\nu(\mathcal{H}_1) = 2\nu(\mathcal{H}) - 2|Y_1|$. Note however that because X is a set of isolated vertices in $\mathcal{H}_1$, removing them changes neither the matching size nor the covering number. Hence $\mathcal{H}_0 = \mathcal{H}_1 \setminus X$ also has $\tau(\mathcal{H}_0) = 2\nu(\mathcal{H}_0) = 2\nu(\mathcal{H}) - 2|Y_1|$. Thus $\mathcal{H}_0$ has a home-base partition $(\mathcal{F}, \mathcal{R}, W)$.

The proof of Condition (5) is far more involved and will use a number of internal lemmas, so we give a brief overview. Our goal will be to find a contradiction by providing a larger matching than $\nu(\mathcal{H})$ if there is an edge of $\mathcal{H}$ incident to X and a W-vertex of $\mathcal{H}_0$. This matching will consist of a maximum matching in $\mathcal{H}_0$ and a few extra edges whose existence will be guaranteed by the high vertex cover number of $\mathcal{H}$. We utilize the fact that we are quite flexible in choosing a matching for $\mathcal{H}_0$, so that we can usually avoid the vertices of the extra edges when we choose our matching. Recall the definition of superfluous vertices and i-heavy (i, j)-covers from Section 9.

Lemma 10.3. *There is no edge wyx with $w \in W$. Similarly there is no $wy'x'$.*

Proof. Suppose wyx is an edge. Take the following partial cover of $\mathcal{H}$: y, y', and z' plus the 2-heavy $(2,3)$-cover of $\mathcal{H}_0$. Since this set of vertices is one too small to be a cover, this implies the existence of an edge zsp avoiding it, where s is superfluous in $\mathcal{H}_0$, and $p \in V(\mathcal{H}_0)$. Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_2-vertex is superfluous. By Case (4) of Corollary 8.6 applied to $\mathcal{H}_0$ with $a = s$, $b = w$, $c = p$, and $S = \emptyset$, we can find a matching of size $\nu(\mathcal{H}_0)$ inside $\mathcal{H}_0$ avoiding $\{s, w, p\}$. This matching together with the edges $z'y'x'$, wyx, and zsp gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction. $\square$

Lemma 10.4. *If there is an edge of $\mathcal{H}$ incident to X and a vertex of $W \cap V_1$, then there are two disjoint edges of $\mathcal{H}$ whose V_1-vertices are in W, at least one being superfluous, whose V_2-vertices are y and y', and exactly one of whose V_3-vertices are in $V(\mathcal{H}_0)$.*

Proof. Suppose there is an edge incident to $w \in W \cap V_1$ and X. Without loss of generality suppose it is incident to x. Then by Lemma 10.3, it is not incident to y, so it must be the edge $wy'x$.

Suppose that w is superfluous in $\mathcal{H}_0$. Then we will show that wyx' is also an edge of $\mathcal{H}$ and that $wy'x$ and wyx' are the only edges extending $y'x$ or yx'.

Since X is a minimal equineighbored of size 2, we have $yx' \in E(\mathrm{lk}_{\mathcal{H}}(V_1))$, and hence there is some edge $vyx' \in E(\mathcal{H})$. Suppose $v \neq w$. Take the partial cover consisting of $\{y, y'\}$ plus the 2-heavy $(2,3)$-cover of $\mathcal{H}_0$. If $v \in \{z, z'\}$, then add v to the partial cover. If $v \in R_1 \in \mathcal{R}$, then add instead the vertex in $R_1 \cap V_3$ to the partial cover. This leaves an edge of the form $(z$ or $z')sp$ where $s \in V_2$ is superfluous in $\mathcal{H}_0$ and $p \notin R_1$ (in case $v \in V(\mathcal{R})$, hence R_1 exists) which is disjoint from vyx'. Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_2-vertex is superfluous. If $v \in \{z, z'\}$, then we can apply Case (4) of Corollary 8.6 to $\mathcal{H}_0$ with $a = w$, $b = s$, $c = p$, and $S = \emptyset$. If $v \in V(\mathcal{R})$, then we can apply Case (2) of Corollary 8.6 to $\mathcal{H}_0$ with $a = v$, $b = p$, $c = s$, and $S = \{w\}$. And if $v \in V(\mathcal{H}_0) \setminus V(\mathcal{R})$, then we can apply Case (4) of Corollary 8.6 to $\mathcal{H}_0$ with $a = s$, $b = v$, $c = p$, and $S = \{w\}$. In any case, we find a matching in $\mathcal{H}_0$ of size $\nu(\mathcal{H}_0)$ avoiding $\{w, v, s, p\}$. Then this matching together with $wy'x$, vyx', and $(z$ or $z')sp$ gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction.

Therefore the only edge extending yx' is wyx', and because wyx' is an edge, a similar argument shows that $wy'x$ is the only edge extending $y'x$.

Take a partial cover $\{z, z', w\}$ plus the 1-heavy $(1,2)$-cover of $\mathcal{H}_0$. This leaves an edge $w'(y$ or $y')p$ where w' is superfluous and $w' \neq w$. Indeed, an edge not intersecting the partial cover is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_1-vertex is superfluous. Also $p \notin \{x, x'\}$, since $w' \neq w$. It is disjoint from one of wyx' and $wy'x$, so $w'(y$ or $y')p$ together with whichever of wyx' and $wy'x$ it is disjoint from are the two disjoint edges we are after.

Suppose on the other hand, that there is no edge incident to $\{x, x'\}$ which extends to a superfluous vertex in V_1. Then in particular w is not superfluous in $\mathcal{H}_0$. Take the partial cover $\{z, z', y'\}$ plus the 1-heavy $(1,3)$-cover of $\mathcal{H}_0$. This

leaves an edge syp where s is superfluous in $\mathcal{H}_0$, and hence $s \neq w$. Indeed, an edge not intersecting the partial cover is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_1-vertex is superfluous. Also $p \notin \{x, x'\}$, since s is superfluous. Thus $wy'x$ and syp are the two disjoint edges we are after. $\qquad \square$

Thus, suppose that there is an edge incident to $W \cap V_1$ and X. Then by Lemma 10.4, there are two disjoint edges e and f whose vertices intersect $V(\mathcal{H}_0)$ in $s, w \in W \cap V_1$ and $p \in V_3$. At least one of s and w is superfluous in $\mathcal{H}_0$, so suppose without loss of generality that s is the superfluous one. We consider several cases, depending on the location of p. In each case we will reach a contradiction.

Case 1. $p \in V(\mathcal{F})$.

Take the partial cover $\{y, y', z\}$, plus the 3-heavy $(3, 2)$-cover of $\mathcal{H}_0$. This gives an edge $z'p's'$ where s' is superfluous (hence $s' \neq p$). Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_3-vertex is superfluous. We can apply Case (1) of Corollary 8.6 with $a = p$, $b = w$, $c = p'$, and $S = \{s, s'\}$ to obtain a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0$ avoiding $\{s, s', w, p', p\}$. This matching together with the edges e, f, and $z'p's'$ gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction.

Case 2. $p \in R_1 \in \mathcal{R}$.

Take the partial cover $\{y, y'\}$ together with the vertex in $R_1 \cap V_2$ and the 3-heavy $(3, 2)$-cover of $\mathcal{H}_0$. This gives an edge $(z$ or $z')p's'$ where s' is superfluous (note $s' \neq p$) and p' is not in R_1. Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_3-vertex is superfluous. We can apply Case (2) of Corollary 8.6 with $a = p$, $b = p'$, $c = w$, and $S = \{s, s'\}$ to obtain a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0$ avoiding $\{s, s', w, p', p\}$. This matching together with the edges e, f, and $(z$ or $z')p's'$ gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction.

Case 3. $p \in W$ is essential for $R_1 \in \mathcal{R}$.

Take the partial cover $\{y, y'\}$, the V_2-vertex essential for R_1 if it exists, plus the 3-heavy $(3, 2)$-cover of $\mathcal{H}_0$. This gives an edge $(z$ or $z')p's'$ where s' is superfluous (hence $s' \neq p$) and p' is not essential for R_1. Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_3-vertex is superfluous. We can apply Case (3) of Corollary 8.6 with $a = p$, $b = p'$, $c = w$, and $S = \{s, s'\}$ to obtain a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0$ avoiding $\{s, s', w, p', p\}$. This matching together with the edges e, f, and $(z$ or $z')p's'$ gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction.

Case 4. $p \in W$ is not essential but not superfluous.

Take the partial cover $\{y, y'\}$ plus the 3-heavy $(3, 2)$-cover of $\mathcal{H}_0$. This gives an edge $(z$ or $z')p's'$ where s' is superfluous, hence $s' \neq p$. Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_3-vertex is superfluous. By Lemma 8.4, p does not become essential after removing a superfluous vertex from V_3. Then we can apply Case (4) of Corollary 8.6 with $a = p$, $b = w$, $c = p'$, and $S = \{s, s'\}$ to obtain a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0$ avoiding $\{s, s', w, p', p\}$. This

matching together with the edges e, f, and $(z$ or $z')p's'$ gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction.

Case 5. $p \in W$ is superfluous.

Take the partial cover $\{y, y', p\}$ plus the 2-heavy $(2,3)$-cover of $\mathcal{H}_0$. This gives an edge $(z$ or $z')s'p'$ where s' is superfluous and $p' \neq p$. Indeed, an edge not intersecting the partial cover must avoid Y_2, hence also X, is not in $E(\mathcal{H}_0)$, and by Observation 9.6, its V_2-vertex is superfluous. We can apply Case (4) of Corollary 8.6 with $a = s'$, $b = w$, $c = p'$, and $S = \{s, p\}$ to obtain a matching of size $\nu(\mathcal{H}_0)$ in $\mathcal{H}_0$ avoiding $\{s, s', w, p', p\}$. This matching together with the edges e, f, and $(z$ or $z')s'p'$ gives a matching of size $\nu(\mathcal{H}_0) + 3 = \nu(\mathcal{H}) + 1$, a contradiction.

Thus we conclude that there can be no edge incident to $W \cap V_1$ and X, so Condition (5) must hold, and hence (Y_1, Y_2, X) is a cromulent triple. $\square$

Thus, if we either have a good set, or if we have no good set and there are two disjoint hyperedges incident to a minimal equineighbored subset of some link graph, then we find a cromulent triple, and hence have found a home-base partition by Corollary 9.4. Therefore, the only hypergraphs left to check are those which have no good set and where the hyperedges incident to any equineighbored subset of any link graph form intersecting hypergraphs. This case is handled in the next section.

11 The End Game

We start with the following easy proposition which will be useful in what is to come:

Proposition 11.1. *Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3 such that each link $\mathrm{lk}_{\mathcal{H}}(V_i)$ has a perfect matching. Suppose $X \subseteq V_j$ is a minimal equineighbored set of $\mathrm{lk}_{\mathcal{H}}(V_i)$ with $|X| = 2$, and suppose X is not incident to two disjoint edges of $\mathcal{H}$. Then the edges incident to X form a truncated multi-Fano plane.*

Proof. Since X is a minimal equineighbored set of size 2 and $\mathrm{lk}_{\mathcal{H}}(V_i)$ has no isolated vertices, it follows easily that the edges of $\mathrm{lk}_{\mathcal{H}}(V_i)$ incident to X form a C_4 (possibly with parallel edges). By assumption, the edges incident to X form an intersecting hypergraph. Since the hyperedges incident to X all intersect, each pair of opposite edges in the C_4 must extend to one vertex in V_i. If this is the same vertex v for all pairs, then $N_{\mathrm{lk}_{\mathcal{H}}(V_k)}(X) = \{v\}$, where V_k is the third vertex class besides V_i and V_j. This contradicts the fact that $\mathrm{lk}_{\mathcal{H}}(V_k)$ has a perfect matching, so each pair extends to a different vertex, which gives the truncated Fano plane. If there are parallel edges in the C_4, this analysis shows that they also have to extend to the same vertex as the edges to which they are parallel, hence we have a truncated multi-Fano plane. $\square$

We aim to prove the following lemma, which is the missing ingredient in our proof of Theorem 1.1.

Lemma 11.2. *Suppose IH(k − 1) holds. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3 such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H}) = 2k$. Suppose that $\mathcal{H}$ does not have a cromulent triple. Then there is an $X \subseteq V_3$, which is a minimal equineighbored set for $\mathrm{lk}_{\mathcal{H}}(V_1)$ such that for its neighborhood $Y = N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$ we also have $N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(Y) = X$.*

Proof. We have shown in Lemma 10.1 that we have a cromulent triple if there is at least one good set, which means we are working under the assumption that $\mathrm{lk}_{\mathcal{H}}(V_1)$ has no good set. By Lemma 5.7, we then know that $\mathrm{lk}_{\mathcal{H}}(V_1)$ has a perfect matching and that every minimal equineighbored set is of size 2 and hence is incident to a C_4. Therefore, it is clear that every edge incident to a minimal equineighbored set participates in a perfect matching, so we have shown that every minimal equineighbored set is still decent.

If $X \subseteq V_3$ is a minimal equineighbored set, for $y \in N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$ define the bipartite graph $G_y = \mathrm{lk}_{\mathcal{H}}(V_1) - \{yz \in E(\mathrm{lk}_{\mathcal{H}}(V_1)) : z \in V_3 \setminus X\}$. Since X is decent but not good, it must be that for some $y \in N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$ we have

$$\eta\left(\mathcal{M}\left(G_y\right)\right) \leq \eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_1)\right)\right).$$

A similar statement holds if $X \subseteq V_2$.

Now suppose for the sake of contradiction to the statement of Lemma 11.2 that for every minimal equineighbored subset X in $\mathrm{lk}_{\mathcal{H}}(V_1)$, its neighborhood Y has neighbors outside of X. Again, Theorem 2.4 gives that $\mathrm{lk}_{\mathcal{H}}(V_1)$ is extremal for Theorem 2.2, and hence it has a CP-decomposition by Theorem 4.5. We know that any CP-decomposition of $\mathrm{lk}_{\mathcal{H}}(V_1)$ contains some P_4's, since otherwise the graph would consist entirely of disjoint C_4's, which is not the case if there are edges between Y and $V_3 \setminus X$.

Claim. *The graph $\mathrm{lk}_{\mathcal{H}}(V_1)$ contains a minimal equineighbored set $X \subseteq V_3$ for which both elements of $N(X)$ have neighbors outside X in $\mathrm{lk}_{\mathcal{H}}(V_1)$.*

Proof. Let Z be the set of endpoints of P_4's in V_3 for some CP-decomposition of $\mathrm{lk}_{\mathcal{H}}(V_1)$ with respect to some perfect matching M. Then Z is equineighbored because the edges incident to the endpoints in V_3 all must contain an interior vertex in V_2 either of the same P_4 or of some other one. The set of interior vertices of P_4's in V_2 is matched by M to the set of endpoints of P_4's in V_3, so these are the same size. Therefore $|Z| = |N(Z)|$. Since Z is equineighbored, it contains a minimal equineighbored subset X.

Since X consists of endpoints of P_4's and $N(X)$ consists of interior vertices of P_4's, the vertices in $N(X)$ all have neighbors outside X: the other interior vertices of their respective P_4's. $\square$

Fix a perfect matching M of the link graph $\mathrm{lk}_{\mathcal{H}}(V_1)$. Let $X_3 \subseteq V_3$ be a minimal equineighbored set for which both elements of $N(X_3)$ have neighbors outside X_3, and let $N(X_3) = \{y, y'\}$. Let $X_3 = \{x, x'\}$ so that $yx, y'x' \in M$. Without loss of generality, let y' be a vertex of $N(X_3)$ that witnesses the failure of X_3 to be good; that is, we have

$$\eta\left(\mathcal{M}\left(G_{y'}\right)\right) \leq \eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}}(V_1)\right)\right).$$

Then by Theorem 4.5, $G_{y'}$ has a CP-decomposition with respect to M (since no edges of M were erased, and hence $G_{y'}$ is still extremal for Theorem 2.2). We claim that in every CP-decomposition of $G_{y'}$, the two vertices of X_3 are together in one of the C_4's of the decomposition. The edge $x'y'$ is an edge of M, so it must be in some C_4 or P_4 of the CP-decomposition. Since $N_{G_{y'}}(y') = X$, and $N_{G_{y'}}(x') = N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X_3)$, this C_4 or P_4 must be contained in $G_{y'}[X_3 \cup N(X_3)]$. But we know the edges in $G_{y'}[X_3 \cup N(X_3)]$ form a C_4, so $x'y'$ can't be contained in a P_4 of the CP-decomposition (one of the edges xy' and $x'y$ would not be incident to an interior vertex of any P_4).

Let Z_2 be the set of vertices in V_2 reachable by M-alternating paths in $G_{y'}$ starting at y with an edge not in M (including y itself). Note that $Y \subseteq Z_2$. We have $\left| N_{G_{y'}}(Z_2) \right| = |Z_2|$ because every vertex of V_3 we reach is matched to a vertex of V_2 which is included in Z_2. Then Z_2 contains a minimal equineighbored set X_2. Note that X_2 is disjoint from Y, since $X_2 \backslash Y$ must also be equineighbored (because X_3 is taken out of the neighborhood), and $X_2 \backslash Y$ is not empty because $\left| N_{G_{y'}}(Y) \right| > 2$. This means also that X_2 has exactly the same neighborhood in $G_{y'}$ and in $\mathrm{lk}_{\mathcal{H}}(V_1)$, and so it is also a minimal equineighbored set for $\mathrm{lk}_{\mathcal{H}}(V_1)$. Therefore, $|X_2| = 2$ and the edges incident to X_2 form a C_4.

Lemma 11.3. *In any CP-decomposition of $G_{y'}$ all vertices of $Z_2 \setminus N(X_3)$ are endpoints of P_4's, and all vertices of $N(Z_2 \setminus N(X_3))$ are interior vertices of P_4's.*

Proof. Since the $(y', V_3 \setminus X_3)$-edges are erased, any CP-decomposition of $G_{y'}$ must have a C_4 on $X_3 \cup N(X_3)$. So any M-alternating path going out from y (not to X_3) must go first to an interior vertex of a P_4, which is matched to an endpoint of that P_4, and so on, alternating between interior vertices and endpoints. So the neighbors of $Z_2 \setminus N(X_3)$ are interior vertices and the vertices of $Z_2 \setminus N(X_3)$ are endpoints. $\square$

This shows in particular that both vertices of X_2 are endpoints of P_4's, and both vertices of $N(X_2)$ are interior vertices of P_4's, and hence both have neighbors outside of X_2.

Lemma 11.4. *If $X \subseteq V_3$ and $X' \subseteq V_2$ are minimal equineighbored subsets of $\mathrm{lk}_{\mathcal{H}}(V_1)$ with $X' \cap N(X) = \emptyset$, and there is an M-alternating path from $N(X)$ to $N(X')$ starting with a non-matching edge, then the edges incident to X and the edges incident to X' extend to the same two vertices $\{z, z'\} \subseteq V_1$.*

Proof. We have seen that each link graph $\mathrm{lk}_{\mathcal{H}}(V_i)$ has a perfect matching, and we know $|X| = 2$ and is not incident to two disjoint hyperedges, so by Proposition 11.1, the edges incident to X form a truncated Fano plane.

Let $N(X) = \{y, y'\}$, and let $N(X') = \{w, w'\}$, where without loss of generality y is the last vertex of $N(X)$ visited on the M-alternating path, and w is the first vertex of $N(X')$ visited. Let $G_{y',w'}$ be the graph formed by erasing *both* the $(y', V_3 \setminus X)$-edges and the $(w', V_2 \setminus X_2)$-edges from $\mathrm{lk}_{\mathcal{H}}(V_1)$. We will show

that $G_{y',w'}$ does *not* have a CP-decomposition. Suppose it did. Then fix a CP-decomposition of $G_{y',w'}$. Both X and X' would need to consist of vertices of a C_4 in the CP-decomposition of $G_{y',w'}$, as previously observed for $G_{y'}$. However since there is an M-alternating path from y to w starting with a non-matching edge, we will see that this leads to a contradiction. Consider the first edge yv of this path. It is not an edge of a C_4 or P_4 of the CP-decomposition, so it must be incident to an interior vertex of some P_4, and since y is not an interior vertex of a P_4 of the CP-decomposition, it follows that v is. The next edge is an edge of M which pairs the interior vertex v with an endpoint. The next edge must be incident to an interior vertex of some P_4, hence its other vertex is again an interior vertex of that P_4. Continuing in this manner, one sees that the even vertices of the path (y being the first vertex) are interior vertices of P_4's of the CP-decomposition. However, since w is one of the even vertices, this contradicts the fact that w is a vertex of a C_4 of the CP-decomposition. Therefore no CP-decomposition is possible, and hence by the contrapositive of Theorem 4.5, we must have

$$\eta\left(\mathcal{M}\left(G_{y',w'}\right)\right) \geq \frac{\nu(G_{y',w'})}{2} + 1 = \frac{\nu(\mathrm{lk}_{\mathcal{H}}(V_1))}{2} + 1 = \nu(\mathcal{H}) + 1, \qquad (11.1)$$

where the last equality is by Theorem 2.4.

Consider the hypergraph $\mathcal{H}_{y',w'}$ that results by removing from $\mathcal{H}$ the edges inducing the $(y', V_3 \setminus X)$-edges and the $(w', V_2 \setminus X_2)$-edges in $\mathrm{lk}_{\mathcal{H}}(V_1)$. Then clearly $\mathrm{lk}_{\mathcal{H}_{y',w'}}(V_1) = G_{y',w'}$. We have $\tau(\mathcal{H}_{y',w'}) \geq \tau(\mathcal{H}) - 2$, since we can cover all of the deleted edges with two vertices, and we clearly have $\nu(\mathcal{H}_{y',w'}) \leq \nu(\mathcal{H})$. Therefore by parts (ii) and (iii) of Proposition 3.2, there is some $S \subseteq V_1$ such that $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_{y',w'}}(S)\right)\right) \leq \nu(\mathcal{H}) - (|V_1| - |S|)$ and $|S| \geq |V_1| - 2$. (Note that if $|V_1| > 2$ then Proposition 3.2 is applicable, and otherwise the conclusion of the lemma is immediate.) We know $S \neq V_1$ because the first inequality fails for V_1, as we have just concluded in the preceding paragraph.

Combining the inequality for $\eta\left(\mathcal{M}\left(\mathrm{lk}_{\mathcal{H}_{y',w'}}(S)\right)\right)$ with the inequality in Theorem 2.2 gives that $\nu(\mathrm{lk}_{\mathcal{H}_{y',w'}}(S)) \leq 2\nu(\mathcal{H}) - 2(|V_1| - |S|)$. Recalling the vertex cover T_S of $\mathcal{H}$ consisting of $V_1 \setminus S$ and a minimal vertex cover of $\mathrm{lk}_{\mathcal{H}}(S)$ gives that $\nu(\mathrm{lk}_{\mathcal{H}}(S)) \geq \tau(\mathcal{H}) - (|V_1| - |S|)$ (by König's Theorem). Thus we have

$$\nu(\mathrm{lk}_{\mathcal{H}_{y',w'}}(S)) \leq \nu(\mathrm{lk}_{\mathcal{H}}(S)) - (|V_1| - |S|). \qquad (11.2)$$

Therefore, every maximum matching of $\mathrm{lk}_{\mathcal{H}}(S)$ has to contain an edge that gets erased in $\mathcal{H}_{y',w'}$. If xy and $x'y'$ are in $\mathrm{lk}_{\mathcal{H}}(S)$, then we can change any maximum matching to avoid a $(y', V_3 \setminus X)$-edge without changing the cardinality of the matching, and similarly for xy' and $x'y$. Analogously, we can avoid a $(w', V_2 \setminus X')$-edge if either pair of opposite edges of the C_4 incident to X' is contained in $\mathrm{lk}_{\mathcal{H}}(S)$. Therefore for one of the C_4's, no pair of opposite edges is contained in $\mathrm{lk}_{\mathcal{H}}(S)$. This implies that the two vertices of V_1 to which the edges of the C_4 extend are not in S, and hence in fact $|S| = |V_1| - 2$.

This of course means that every maximum matching of $\mathrm{lk}_{\mathcal{H}}(S)$ has to contain *two* edges that get erased in $\mathcal{H}_{y',w'}$, so no pair of opposite edges of either C_4

is contained in $\mathrm{lk}_{\mathcal{H}}(S)$, and hence the vertices of V_1 to which the edges extend are not in S. But each C_4 extends to exactly two vertices, as observed in Lemma 11.1, and since $|S| = |V_1| - 2$, they must be the same two vertices for X and X', as claimed. $\qquad\square$

Lemma 11.4 applied to X_2 and X_3 shows that $\mathcal{H}$ has two truncated Fano planes intersecting in two vertices $\{z, z'\} \subseteq V_1$. We will see that this leads to a contradiction.

Let $X_2 = \{v, v'\}$, and let $N(X_2) = \{w, w'\}$. Assume without loss of generality that the truncated Fano planes consist of the edges $\{zyx, zy'x', z'yx', z'y'x\}$ and $\{zvw, zv'w', z'vw', z'v'w\}$. Consider the hypergraph $\mathcal{H}' = \mathcal{H} - \{y, w, z, z'\}$, and note that X_3 and X_2 consist of isolated vertices in $\mathcal{H}'$, since all edges incident to them are incident to $\{z, z'\}$. Because we have deleted only four vertices, we clearly have $\tau(\mathcal{H}') \geq \tau(\mathcal{H}) - 4$. To any matching in $\mathcal{H}'$ we may add zyx and $z'vw$ to get a matching two larger in $\mathcal{H}$, so we must have $\nu(\mathcal{H}') \leq \nu(\mathcal{H}) - 2$. Combining this with the assumption that $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$ and the fact that Ryser's Conjecture is true for 3-partite hypergraphs we get the following sequence of inequalities:

$$\tau(\mathcal{H}') \leq 2\nu(\mathcal{H}') \leq 2\nu(\mathcal{H}) - 4 = \tau(\mathcal{H}) - 4 \leq \tau(\mathcal{H}').$$

Since the first and last expressions are the same, all inequalities are actually equalities, and hence $\mathcal{H}'$ is also extremal for Ryser's Conjecture, with $\nu(\mathcal{H}') = k - 2$. Therefore, by the inductive hypothesis IH$(k-1)$, $\mathcal{H}'$ has a home-base partition $(\mathcal{F}, \mathcal{R}, W)$.

We will find either a vertex cover of size $\tau(\mathcal{H}) - 1$, or a matching of size $\nu(\mathcal{H}) + 1$ in $\mathcal{H}$, either of which gives our desired contradiction.

Claim. *There exists an edge $e = ayb$ for some vertices $a \in W \cap V_1$ and $b \in (V(\mathcal{F}) \cup W) \cap V_3$.*

Proof. Consider the minimal vertex cover of $\mathcal{H}'$ consisting of $V(\mathcal{F}) \cap V_1$ and $V(\mathcal{R}) \cap (V_1 \cup V_3)$. If adding the three vertices z, z', and w to this set would form a vertex cover T of $\mathcal{H}$, we would have a contradiction and be done, so we may assume that there is some edge $e \in E(\mathcal{H})$ which avoids T. Its V_1-vertex must be in W, since $(V(\mathcal{F}) \cup V(\mathcal{R})) \cap V_1 \cup \{z, z'\} \subseteq T$. Its V_3-vertex must be in $V(\mathcal{F}) \cup W$, since $V(\mathcal{R}) \cap V_3 \cup \{w\} \subseteq T$ and any edge incident to X_3 intersects T in $\{z, z'\}$. Its V_2-vertex cannot be in $V(\mathcal{H}')$, since otherwise e would be an edge of $\mathcal{H}'$ and hence intersect T, and its V_2-vertex also cannot be in X_2, since all edges incident to X_2 intersect T in $\{z, z'\}$. Therefore e must go through y, so it is of the form ayb for some $a \in W \cap V_1$ and $b \in (V(\mathcal{F}) \cup W) \cap V_3$. $\qquad\square$

Claim. *We have $y' \in W$.*

Proof. Suppose we can find a maximum matching in $\mathcal{H}'$ avoiding a, y', and b. Then this matching plus the three disjoint edges $zy'x'$, $z'v'w$, and ayb would form a matching of size $\nu(\mathcal{H}) + 1$ in $\mathcal{H}$, a contradiction.

By the monster lemma (Lemma 8.5), we can find such a matching of size $\nu(\mathcal{H}')$ in $\mathcal{H}' - \{a, y', b\}$ if there is an F-edge avoiding $\{a, y', b\}$ for each $F \in \mathcal{F}$, and an R-edge avoiding $\{a, y', b\}$ for each $R \in \mathcal{R}$. Since $a \in W$, and y' and b are in different vertex classes, we do not cover all F-edges for any $F \in \mathcal{F}$. Since $a, b \notin V(\mathcal{R})$, we could pick an RWR-edge for any $R \in \mathcal{R}$ avoiding $\{a, y', b\}$ *unless* y' is a W-vertex essential for some $R \in \mathcal{R}$. This means that if $y' \notin W$, we have the desired contradictory matching, and hence we may assume $y' \in W$. $\qquad\square$

Claim. *There exists an edge $e' = a'yb'$ for some superfluous vertex $a' \in W \cap V_1$ and some vertex $b' \in V(\mathcal{H}') \cap V_3$.*

Proof. Consider the 1-heavy $(1, 3)$-cover of $\mathcal{H}'$ (see Section 9 for the definition), which is a minimal vertex cover of $\mathcal{H}'$. If adding the three vertices z, z', and w to this set would form a vertex cover T' of $\mathcal{H}$, we would again have a contradiction, so we may assume that some edge $e' \in E(\mathcal{H})$ avoids T'. Its V_1-vertex must be a superfluous W-vertex, since all other V_1-vertices are in T'. Its V_3-vertex must be in $V(\mathcal{H}')$, since $w \in T'$ and any edge incident to X_3 intersects T' in $\{z, z'\}$. Its V_2-vertex cannot be in $V(\mathcal{H}')$, since otherwise e' would be an edge of $\mathcal{H}'$ and hence intersect T', and its V_2-vertex also cannot be in X_2, since all edges incident to X_2 intersect T' in $\{z, z'\}$. Therefore e' must go through y, so it is of the form $a'yb'$ for some superfluous vertex $a' \in W \cap V_1$ and some $b' \in V(\mathcal{H}') \cap V_3$. $\qquad\square$

By the last two claims, the conditions of part (4) of Corollary 8.6 of the monster lemma hold for $\mathcal{H}'$, with vertices a' (for a), y' (for b), and b' (for c), and set $\emptyset$ (for S). Hence there is a matching of size $\nu(\mathcal{H}')$ in $\mathcal{H}'$ avoiding a', y', and b'. Combining this matching with the three disjoint edges $zy'x'$, $z'v'w$, and $a'yb'$ yields a matching of size $\nu(\mathcal{H}) + 1$, a contradiction.

Therefore, in all cases we have found a contradiction, and since we have assumed the negation of the statement of Lemma 11.2, we have proven the lemma. $\qquad\square$

12   The Proof of Theorem 1.1

Proof of Theorem 1.1. The proof is by induction. IH(0) holds trivially: Let $\mathcal{H}$ be a 3-partite 3-graph with $\nu(\mathcal{H}) = 0$. Then $\mathcal{H}$ has no edges, so $(\emptyset, \emptyset, V(\mathcal{H}))$ is a home-base partition of $\mathcal{H}$ as can easily be seen. Now let $k \geq 1$ and assume IH($k-1$) holds. We will show IH(k).

Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes V_1, V_2, and V_3 such that $\tau(\mathcal{H}) = 2\nu(\mathcal{H}) = 2k$. If it has a cromulent triple, then by Corollary 9.4, it is a home-base hypergraph, and we are done.

Therefore, assume there is no cromulent triple. Then by Lemma 11.2 there is a minimal equineighbored $X \subseteq V_3$ such that for $Y = N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(X)$ we also have $N_{\mathrm{lk}_{\mathcal{H}}(V_1)}(Y) = X$. By Proposition 11.1, the edges incident to X form a truncated Fano plane F. Let A be the set of V_1-vertices of the hyperedges of F. Set $\mathcal{H}_1 = \mathcal{H} \setminus A$. Since we have removed two vertices, we have $\tau(\mathcal{H}_1) \geq \tau(\mathcal{H}) - 2$, and since any matching in $\mathcal{H}_1$ can be enlarged by adding an edge of F (as no

edge of $\mathcal{H}_1$ is incident to X or Y), we have $\nu(\mathcal{H}_1) \leq \nu(\mathcal{H}) - 1$. Combining these inequalities with the fact that $\tau(\mathcal{H}_1) \leq 2\nu(\mathcal{H}_1)$ yields that all three inequalities are actually equalities. Since X and Y consist of isolated vertices, the same holds true for $\mathcal{H}_0 = \mathcal{H}_1 \setminus (Y \cup X)$. Thus, we can apply induction to get a home-base partition of $\mathcal{H}_0$ and add the F to it to get a proper matchable FR-partition of $\mathcal{H}$, which by Lemma 8.9 is a home-base partition.

Thus in all cases, $\mathcal{H}$ is a home-base hypergraph, so IH(k) holds.

Therefore Theorem 1.1 holds by induction. $\qquad\square$

For interest, we can directly show also that IH(1) holds.

Proposition 12.1. *Let $\mathcal{H}$ be a 3-partite 3-graph with $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = 2$. Then $\mathcal{H}$ is a home-base hypergraph.*

Proof. Suppose $\mathcal{H}$ is an intersecting 3-partite 3-graph with $\tau(\mathcal{H}) = 2$. If every pair of edges intersect in two vertices, then it is easy to see that there must then be two vertices which are in every edge, and thus $\mathcal{H}$ would in fact have a vertex cover of size 1 (pick any one of the two vertices). Therefore there must be two edges which intersect in one vertex. Label these edges abc and ade. Since a alone does not form a vertex cover, there must be an edge which misses a, but it must intersect both of these edges, each in a different vertex class of $\mathcal{H}$. Thus WLOG, we have the edge fbe. If fdc is also an edge of $\mathcal{H}$, then we have an F. In this case, no further edge can be present unless it is parallel to one of the existing edges, since no other edge can intersect all four of these edges. Therefore in this case, $\mathcal{H}$ is indeed a home-base hypergraph which consists of a single F.

If fdc is not an edge of $\mathcal{H}$, then we let $R = \{a, b, e\}$, and we claim that every edge of $\mathcal{H}$ contains at least two of the vertices a, b, or e. If an edge misses any two of these vertices, then its third vertex must be the vertex outside of R of the edge among abc, ade, and fbe that contains those two vertices (since $\mathcal{H}$ is intersecting). Since this vertex is not in R either, by symmetry the same is true of each of the other edges we have given. Thus the edge must in fact be fdc, which is not the case by assumption. Thus $(\emptyset, \{R\}, V(\mathcal{H}) \setminus R)$ forms an FR-partition of $\mathcal{H}$ with the edge-home property. It is matchable because the graphs B_1, B_2, and B_3 contain edges Rf, Rd, and Rc, respectively, which obviously form matchings saturating $\{R\}$. Thus in this case, $\mathcal{H}$ is a home-base hypergraph consisting of a single R and at least three W-vertices. This proves the case $\nu(\mathcal{H}) = 1$. $\qquad\square$

13 Concluding Remarks and Open Questions

13.1 Equivalences

As promised, we prove here the "if" direction of Theorem 2.5 and the converse of Theorem 2.4, using the following result.

Theorem 13.1. *If bipartite graph G has a CP-decomposition then it is the link of a Ryser-extremal 3-graph.*

For the converse of Theorem 2.4, let G be a bipartite graph with $\eta\left(\mathcal{M}\left(G\right)\right) = \frac{\nu(G)}{2}$. By Theorem 2.5 G has a CP-decomposition. Therefore by Theorem 13.1 G is the link of a Ryser-extremal 3-graph $\mathcal{H}$.

For the converse of Theorem 2.5 let G be a bipartite graph with a CP-decomposition. By Theorem 13.1 G is the link of a Ryser-extremal 3-graph $\mathcal{H}$. Equation (2.1) of Theorem 2.4 then implies that $\eta\left(\mathcal{M}\left(G\right)\right) = \frac{\nu(G)}{2}$.

We remark that one can also prove the latter statement directly, without going through hypergraphs, by giving a suitable non-fillable triangulation of the boundary of the crosspolytope.

Proof of Theorem 13.1. Let G be a bipartite graph with a collection of $\nu(G)/2$ pairwise vertex-disjoint subgraphs, each of them a C_4 or a P_4, such that every edge of G is either an edge of one of the C_4's or is incident to an interior vertex of one of the P_4's. We will construct a home-base hypergraph $\mathcal{H}$ with G as one of its links.

Let V_1 and V_2 be the vertex classes of G. Let V_3 be a set of sufficiently many new vertices ($\nu(G)$ suffice). Let $\mathcal{H}$ be the empty 3-graph. Then $(\mathcal{F}, \mathcal{R}, W) = (\emptyset, \emptyset, \emptyset)$ is a home-base partition of $\mathcal{H}$. We will add edges to $\mathcal{H}$, maintaining a home-base partition $(\mathcal{F}, \mathcal{R}, W)$.

For each C_4 in the collection we do the following. Let $\{a, b, c, d\}$ be the vertices of the C_4, so that $a, c \in V_1$, $b, d \in V_2$, and $ab, bc, cd, da \in E(G)$. Take two unused vertices $e, f \in V_3 \setminus V(\mathcal{H})$, and add the edges abe, adf, cbf, and cde to $\mathcal{H}$. These edges form a truncated Fano plane. For each edge parallel to an edge of the C_4, add an edge parallel to the corresponding one of these edges to $\mathcal{H}$, forming a truncated multi-Fano plane. We can then add the set $F = \{a, b, c, d, e, f\}$ to $\mathcal{F}$, maintaining that $(\mathcal{F}, \mathcal{R}, W)$ is a home-base partition of $\mathcal{H}$. Clearly, the C_4 is now present in the link $\mathrm{lk}_{\mathcal{H}}(V_3)$ together with all its parallel edges.

Then, for each P_4 in the collection we do the following. Let $\{a, b, c, d\}$ be the vertices of the P_4, so that $a, c \in V_1$, $b, d \in V_2$, and $ab, bc, cd \in E(G)$. Take two unused vertices $e, f \in V_3 \setminus V(\mathcal{H})$, and add the edges abe, cbf, and cde to $\mathcal{H}$. For each edge parallel to an edge of the P_4, add an edge parallel to the corresponding one of these edges to $\mathcal{H}$. Add the set $R = \{b, c, e\}$ to $\mathcal{R}$, and add the vertices a, d, and f to W. The edges abe, cbf, and cde are R-edges with a W-vertex in V_1, V_3, and V_2, respectively. Thus a, d, and f can be matched to R in B_1, B_3, and B_2, respectively, without disturbing matchability, since the W-vertices are new. Clearly the P_4 is now present in the link $\mathrm{lk}_{\mathcal{H}}(V_3)$ along with all parallel edges, and note especially that its interior vertices are members of R.

Once we've processed all the C_4's and P_4's, any edges of G not yet present in the link $\mathrm{lk}_{\mathcal{H}}(V_3)$ are incident to an interior vertex of one of the P_4's. Let $xy \in E(G)$ be such an edge, and suppose y is an interior vertex of one of the P_4's. Then $y \in R$ for some $R \in \mathcal{R}$. Let $z \in R \cap V_3$. Then, we add the edge

xyz to $\mathcal{H}$. If x was not previously a vertex of $\mathcal{H}$, we add it to W, otherwise, we leave it where it is. Since xyz is an R-edge, $\mathcal{H}$ is still a home-base hypergraph with home-base partition $(\mathcal{F}, \mathcal{R}, W)$. After this addition, xy is present in the link $\mathrm{lk}_{\mathcal{H}}(V_3)$. We process every remaining edge this way.

If G has any isolated vertices, we add them to $\mathcal{H}$, putting them in W (these clearly do not disturb the home-base partition of $\mathcal{H}$). Now $\mathcal{H}$ is a home-base hypergraph with $\mathrm{lk}_{\mathcal{H}}(V_3) = G$, and we know that $\mathcal{H}$ is Ryser-extremal by Proposition 7.5. □

13.2   Lack of good sets does not exclude P_4's

A substantial part of our proof (Sections 10.2 and 11) is devoted to the case in which the link graphs of our Ryser-extremal hypergraph contain no good sets. We know that each of these link graphs is a bipartite graph with a CP-decomposition, and one way that such a graph can fail to have a good set is if its CP-decomposition consists entirely of C_4's. If it were true that every bipartite graph with a CP-decomposition containing a P_4 also had a good set, then our proof could be significantly shortened (since this would imply the existence of a cromulent triple, see the proof of Theorem 1.1 in Section 12).

Unfortunately however, this statement is not true in general, as the following bipartite graph shows. The graph G consists of six pairwise vertex-disjoint four-cycles $Q_1, \ldots, Q_6$, and two six-cycles T_1, T_2, that are vertex disjoint from each other but contain one vertex from each Q_i. The six-cycles T_1 and T_2 are placed so that, in each Q_i, two non-adjacent vertices have degree two (i.e., have edges only within Q_i), while the other two vertices have degree four. The edges of T_1 and T_2 go between the Q_i cyclically, that is, there are two edges going from Q_i to Q_{i+1} and from Q_6 to Q_1.

We claim that this graph G has a CP-decomposition with four P_4's and two C_4's, but has no good sets. To start building a CP-decomposition, one can take as C_4's an arbitrary pair of four-cycles that are opposite in the six-cycles, say Q_1 and Q_4. Then the four edges of T_1 and T_2 that are not incident to these can be used as the middle edges of four P_4's, giving the promised CP-decomposition (as all other edges are incident to one of these middle edges).

To check that there is no good subset S of one of the sides of this bipartite G, consider the following three cases. If S contains a vertex of degree four, then S is not even decent, since edges of T_i cannot participate in a maximum matching, contradicting property (2) of decency. If S consists of degree two vertices only, and $|S \cap V(Q_i)| = 1$ for some Q_i, then S is again not decent, because $|N(S)| > |S|$. Finally, if S consists of degree two vertices and $|S \cap V(Q_i)| = 0$ or 2 for every $i = 1, \ldots, 6$, then we obtain a contradiction to the connectedness condition of good sets as follows. Suppose without loss of generality that $|S \cap V(Q_1)| = 2$, and fix $y \in V(Q_1) \setminus S$. Now note that the graph G_y still has a CP-decomposition using the four-cycles Q_1 and Q_4 as C_4's (as described above). Consequently, the reverse direction of the CP-decomposition theorem implies that $\eta(G_y) = \nu(G_y)/2 = 12/2 = 6 = \nu(G)/2 = \eta(G)$. Hence no subset S of $V(G)$ is good.

13.3   Connectedness of 3-partite 3-graphs

For 3-graphs $\mathcal{H}$, Theorem 2.2 gives

$$\eta\left(\mathcal{M}\left(\mathcal{H}\right)\right) \geq \frac{\nu(\mathcal{H})}{3}.$$

Using our characterization, we can show that the Ryser-extremal 3-graphs are far from tight for this theorem. For a Ryser-extremal 3-partite 3-graph we can improve the bound to the following:

Proposition 13.2. *If $\mathcal{H}$ is a home-base hypergraph, then*

$$\eta\left(\mathcal{M}\left(\mathcal{H}\right)\right) \geq \frac{2}{3}\nu(\mathcal{H}).$$

It is not difficult to show that this bound is tight. The proof of Proposition 13.2 can be found in [20].

Since Proposition 13.2 is a strengthening of Theorem 2.2 when $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$, one could ask for the best possible extension of it when the ratio τ/ν is different from 2. To make this precise, let us define the function $f : [1,2] \to \mathbb{R}$ by

$$f(x) = \inf\left\{\frac{\eta\left(\mathcal{M}\left(\mathcal{H}\right)\right)}{\nu(\mathcal{H})} : \mathcal{H} \text{ is a 3-partite 3-graph}, \tau(\mathcal{H}) \geq x\nu(\mathcal{H})\right\}.$$

We then have that for any 3-partite 3-graph $\mathcal{H}$ with $\tau(\mathcal{H}) = x\nu(\mathcal{H})$ it holds that

$$\eta\left(\mathcal{M}\left(\mathcal{H}\right)\right) \geq f(x)\nu(\mathcal{H}).$$

Clearly f is monotone increasing and bounded below by $1/3$, by Theorem 2.2. Since Proposition 13.2 is tight, we have $f(2) = 2/3$, while there are easy examples showing $f(1) = 1/3$. One could speculate whether there is a linear lower bound on f interpolating these two extremes, so that $f(x) \geq x/3$. This would be very interesting, as it would imply Ryser's Conjecture for 4-partite 4-graphs by a straightforward generalization of Aharoni's argument for 3-partite 3-graphs. Unfortunately, this does not turn out to be the case, as there is a violation of this bound for $x = 4/3$:

Proposition 13.3. *There is a 3-partite 3-graph $\mathcal{H}$ with $\tau(\mathcal{H}) = 4$ and $\nu(\mathcal{H}) = 3$ such that $\eta\left(\mathcal{M}\left(\mathcal{H}\right)\right) = 1$.*

Proof. Let $\mathcal{H}$ be the 3-partite 3-graph on the vertices $\{1,2,3\} \times \{1,2,3,4\}$ with vertex classes given by the first coordinate. The edges are two intersecting matchings of size 3 and a matching of size 2 which interesects every edge of the first two matchings. The first matching is $\{(1,1),(2,2),(3,3)\}$, $\{(1,2),(2,3),(3,1)\}$, and $\{(1,3),(2,1),(3,2)\}$; the second is $\{(1,2),(2,4),(3,3)\}$, $\{(1,3),(2,2),(3,4)\}$, and $\{(1,4),(2,3),(3,2)\}$; and the two remaining edges are $e = \{(1,2),(2,2),(3,2)\}$, and $e' = \{(1,3),(2,3),(3,3)\}$. It is not hard to check that $\tau(\mathcal{H}) = 4$ and $\nu(\mathcal{H}) = 3$. Since e and e' intersect all the other edges, they do not participate in a simplex of $\mathcal{M}(\mathcal{H}))$ with any other edge. Therefore labeling the two vertices of S^0 with e and $f \neq e, e'$ is not fillable. Hence $\eta\left(\mathcal{M}\left(\mathcal{H}\right)\right) < 2$.

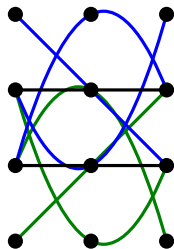**Figure 10:** A 3-partite 3-graph $\mathcal{H}$ with $\tau(\mathcal{H}) = 4$, $\nu(\mathcal{H}) = 3$, and $\eta(\mathcal{M}(\mathcal{H})) = 1$.

$\square$

This shows that $f(x) = 1/3$ for $x \in [1, 4/3]$. It can also be shown that $f(x) \geq x/5$ for every $x \in [1, 2]$, but this only represents an improvement when $x \in (\frac{5}{3}, 2)$ (see [20]). As far as we know the following could be true.

Conjecture 3. *For every $x \in [1, 2]$ we have $f(x) \geq x/4$.*

To approach Ryser's Conjecture for 4-graphs, we seem to need a much better understanding of the potential link 3-graphs, in particular those with $\tau(\mathcal{H}) > \nu(\mathcal{H})$. We believe the function f will be a useful tool for this purpose, even though the extension of Aharoni's argument, at least in its most straightforward version, does not succeed due to the fact that $f(4/3) = 1/3$.

13.4   Open Problems on the tightness of Theorem 2.2

The main purpose of Theorem 2.5 in our paper was to characterize those *bipartite* graphs for which Theorem 2.2 is tight when $r = 2$ and in turn give a structural description of the link graphs of Ryser-extremal 3-graphs. Nevertheless we find the underlying extremal graph theoretic problem, relating the connectedness and the dimension of the independence complex of the line graph, interesting in its own right. In particular, several interesting questions remain open concerning the tightness of Theorem 2.2 and the characterization of its extremal examples.

What happens with the characterization if one leaves out the restriction of bipartiteness? The graph G consisting of a triangle and a hanging edge is an example of a non-bipartite graph which is tight for Theorem 2.2. Indeed, $\nu(G) = 2$ while the line graph is K_4 minus an edge, having a disconnected independence complex, so $\eta(\mathcal{M}(G)) = 1$. It would be very interesting to obtain a full characterizations of those graphs G which are tight for Theorem 2.2.

Another natural direction is to consider hypergraphs with uniformity higher than 2. It is not difficult to see that Theorem 2.2 is also best possible for every $r > 2$. Just take a matching of size mr and add m edges that intersect r different matching edges each. However, a characterization of those r-graphs for which $\eta(\mathcal{M}(\mathcal{H})) = \frac{\nu(\mathcal{H})}{r}$ is still outstanding; the case of r-partite r-graphs already being very interesting.

A related question concerns the relationship of Theorem 2.2 to Ryser's Conjecture for $r > 2$. We have just completed the proof that a graph is tight for Theorem 2.2 if and only if it is the link graph of a Ryser-extremal 3-graph. Is this equivalence or at least one of its directions true for $r > 2$?

Finally, Theorem 2.2 has a chance to be best possible only for graphs whose matching number is even. It would be interesting to prove a characterization of 2-graphs with an *odd* matching number and having a line graph with connectedness as small as possible (in terms of the matching number). Is there a CP-decomposition-type characterization of all (bipartite) graphs G with $\nu(G) = 2k + 1$ and $\eta(\mathcal{M}(G)) = k + 1$?

References

[1] A. Abu-Kazneh, J. Barát, A. Pokrovskiy, T. Szabó, *A family of extremal hypergraphs for Ryser's conjecture*, submitted.

[2] A. Abu-Kazneh and A. Pokrovskiy, *Intersecting extremal constructions in Ryser's Conjecture for r-partite hypergraphs*, J. Combin. Math. Combin. Comput. **103** (2017), 81-104.

[3] M. Adamaszek and J. A. Barmak, *On a lower bound for the connectivity of the independence complex of a graph*, Discrete Mathematics **311** (2011), 2566-2569.

[4] R. Aharoni, *Ryser's conjecture for tri-partite 3-graphs*, Combinatorica **21** (2001), no. 1, 1-4.

[5] R. Aharoni, J. Barát and I. Wanless, *Multipartite hypergraphs achieving equality in Ryser's Conjecture*, Graphs and Combinatorics **32** (2016), no. 1, 1-15.

[6] R. Aharoni and E. Berger, *The intersection of a matroid and a simplicial complex*, Trans. Amer. Math. Soc. **358** (2006), 4895-4917.

[7] R. Aharoni, E. Berger, and R. Ziv, *Independent systems of representatives in weighted graphs*, Combinatorica **27** (2007), no. 3, 253-267.

[8] R. Aharoni, M. Chudnovsky, and A. Kotlov, *Triangulated spheres and colored cliques*, Discrete and Computational Geometry **28** (2) (2002), 223-229.

[9] R. Aharoni and P. Haxell, *Hall's theorem for hypergraphs*, J. Graph Theory **35** (2000), no. 2, 83-88.

[10] E. Berger and R. Ziv, *A note on the cover number and independence number in hypergraphs*, Discrete Mathematics **308** (2008) 2649-2654.

[11] A. Björner, *Topological methods*, in Handbook of Combinatorics, Vol. 1, 2 (Elsevier, Amsterdam 1995) 1819-1872.

[12] N. Francetić, S. Herke, B. D. McKay, I. M. Wanless, *On Ryser's Conjecture for Linear Intersecting Multipartite Hypergraphs*, European J. Combinatorics **61** (2017), 91-105.

[13] Z. Füredi, *Maximum degree and fractional matchings in uniform hypergraphs*, Combinatorica **1** (1981), 155-162.

[14] P. Haxell, *Independent transversals and hypergraph matchings - an elementary approach*, in Recent Trends in Combinatorics (A. Beveridge, J. Griggs, L. Hogben, G. Musiker, P. Tetali, eds), IMA Volume in Mathematics and its Applications, Springer 2016, 215-233.

[15] P. Haxell and A. D. Scott, *On Ryser's conjecture*, The Electronic Journal of Combinatorics 19.1 (2012), paper 23.

[16] L. Lovász, *On minimax theorems of combinatorics*, Matematikai Lapok **26** (1975), 209-264 (in Hungarian).

[17] T. Mansour, C. Song, and R. Yuster, *A comment on Ryser's conjecture for intersecting hypergraphs*, Graphs and Combinatorics **25** (2009), 101-109.

[18] R. Meshulam, *Domination numbers and homology*, J. Combin. Theory Ser. A 102 (2003), 321-330.

[19] J. Milnor, *Construction of Universal Bundles, II*, Ann. of Math **63**, 430-436.

[20] L. Narins, Ph.D Thesis, FU Berlin, *Extremal Hypergraphs for Ryser's Conjecture*, Freie Universität Berlin, 2014.

[21] H. Ryser, *Neue probleme der kombinatorik*, Vorträge über Kombinatorik Oberwolfach, Mathematisches Forschungsinstitut Oberwolfach, Colloquia Mathematica Societatis János Bolyai, 1967, 69-91.

[22] E. Sperner, *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Abh. Math. Sem. Univ. Hamburg **6** (1928), 265-272.

[23] T. Szabó and G. Tardos, *Extremal problems for transversals in graphs with bounded degree*, Combinatorica **26** (2006), 333-351.

[24] Zs. Tuza, *Some special cases of Ryser's conjecture*, manuscript, 1979.

[25] Zs. Tuza, *Ryser's conjecture on transversals of r-partite hypergraphs*, Ars Combinatoria **16** (1983), 201-209.