Deciding Relaxed Two-Colorability — a Hardness Jump

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Abstract. A coloring is proper if each color class induces connected components of order one (where the *order* of a graph is its number of vertices). Here we study relaxations of proper two-colorings, such that the order of the induced monochromatic components in one (or both) of the color classes is bounded by a constant. In a (C_1, C_2)-*relaxed coloring* of a graph G every monochromatic component induced by vertices of the first (second) color is of order at most C_1 (C_2, resp.). We are mostly concerned with $(1, C)$-relaxed colorings, in other words when/how is it possible to break up a graph into small components with the removal of an independent set.

We prove that every graph of maximum degree at most three can be $(1, 22)$-relaxed colored and we give a quasilinear algorithm which constructs such a coloring. We also show that a similar statement cannot be true for graphs of maximum degree at most 4 in a very strong sense: we construct 4-regular graphs such that the removal of any independent set leaves a connected component whose order is linear in the number of vertices.

Furthermore we investigate the complexity of the decision problem (Δ, C)-AsymRelCol: Given a graph of maximum degree at most Δ, is there a $(1, C)$-relaxed coloring of G? We find a remarkable hardness jump in the behavior of this problem. We note that there is not even an obvious monotonicity in the hardness of the problem as C grows, i.e. the hardness for component order $C + 1$ does not imply directly the hardness for C. In fact for $C = 1$ the problem is obviously polynomial-time decidable, while it is shown that it is NP-hard for $C = 2$ and $\Delta \geq 3$.

For arbitrary $\Delta \geq 2$ we still establish the monotonicity of hardness of (Δ, C)-AsymRelCol on the interval $2 \leq C \leq \infty$ in the following strong sense. There exists a critical component order $f(\Delta) \in \mathbb{N} \cup \{\infty\}$ such that the problem of deciding $(1, C)$-relaxed colorability of graphs of maximum degree at most Δ is NP-complete for every $2 \leq C < f(\Delta)$, while deciding $(1, f(\Delta))$-colorability is trivial: every graph of maximum degree Δ is $(1, f(\Delta))$-colorable. For $\Delta = 3$ the existence of this threshold is shown despite the fact that we do not know its precise value, only $6 \leq f(3) \leq 22$. For any $\Delta \geq 4$, (Δ, C)-AsymRelCol is NP-complete for arbitrary $C \geq 2$, so $f(\Delta) = \infty$.

We also study the symmetric version of the relaxed coloring problem, and make the first steps towards establishing a similar hardness jump.

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1 Introduction

A function from the vertex set of a graph to a k-element set is called a *k-coloring*. The values of the function are referred to as *color*s. A coloring is called *proper* if the value of the function differs on any pair of adjacent vertices. Proper coloring and the *chromatic number* of graphs (the smallest number of colors which allow a proper coloring) are among the most important concepts of graph theory. Numerous problems of pure mathematics and theoretical computer science require the study of proper colorings and even more real-life problems require the calculation or at least an estimation of the chromatic number. Nevertheless, there is the discouraging fact that the calculation of the chromatic number of a graph or the task of finding an optimal proper coloring are both intractable problems, even fast approximation is probably not possible. This is one of our motivations to study relaxations of proper coloring, because in some theoretical or practical situations a small deviation from proper is still acceptable, while the problem could become tractable. Another reason for the introduction of relaxed colorings is that in certain problems the use of the full strength of proper coloring is an "overkill". Often a weaker concept suffices and provides better overall results.

In this paper we study various relaxations of proper coloring, which allow the presence of some small level of conflicts in the color assignment. Namely, we will allow vertices of one or more color classes to participate in one conflict or, more generally, let each conflicting connected component have at most C vertices, where C is a fixed integer, not depending on the order of the graph. Most of our results deal with the case of relaxed two-colorings.

To formalize our problem precisely we say that a two-coloring of a graph is *(C_1, C_2)-relaxed* if every monochromatic component induced by the vertices of the first color is of order at most C_1, while every monochromatic component induced by the vertices of the second color is of order at most C_2. Note that $(1, 1)$-relaxed coloring corresponds to proper two-coloring.

In the present paper we deal with the two most natural cases of relaxed two-colorings. We say *symmetric relaxed coloring* when $C_1 = C_2$ and *asymmetric* relaxed coloring when $C_1 = 1$. Symmetric relaxed colorings were first studied by Alon, Ding, Oporowski and Vertigan [1] and implicitly, even earlier, by Thomassen [18] who resolved the problem for the line graph of 3-regular graphs initiated by Akiyama and Chvátal [2]. Asymmetric relaxed colorings were introduced in [5].

Related relaxations of proper colorings. There are several other types of coloring concepts related to our relaxation of proper coloring.

In a series of papers Škrekovski [17], Havet and Sereni [8], and Havet, Kang, and Sereni [9] investigated the concept of *improper colorings* over various families of graphs. A coloring is called (k, l)-improper if none of the at most k colors induces a monochromatic component containing vertices of degree larger than l. Hence in an improper coloring the amount of error is measured in terms of the *maximum degree* of monochromatic components rather than in terms of their order.

Linial and Saks [15] studied low diameter graph decompositions, where the quality of the coloring is measured by the *diameter* of the monochromatic components. Their goal was to color graphs with as few colors as possible such that each monochromatic connected component has a small diameter.

Haxell, Pikhurko and Thomason [11] study the *fragmentability* of graphs introduced by Edwards and Farr [7], in particular for bounded degree graphs. A graph is called (α, f)-fragmentable if one can remove α fraction of the vertices and end up with components of order at most f. For comparison, in a $(1, C)$-relaxed coloring one must remove an independent set and end up with small components.

The problems. We study relaxed colorings from two points of view, extremal graph theory and complexity theory, and find that these points eventually meet for asymmetric relaxed colorings. We also make the first steps for a similar connection in the symmetric case. To demonstrate our problems, in the next few paragraphs we restrict our attention to asymmetric relaxed colorings; the corresponding questions are asked and partially answered for symmetric relaxed colorings, but there our knowledge is much less satisfactory.
On the one hand there is the purely graph theoretic question:

> For a given maximum degree Δ what is the smallest component order $f(\Delta) \in \mathbb{N} \cup \{\infty\}$ such that every graph of maximum degree Δ is $(1, f(\Delta))$-relaxed colorable?

On the other hand, for fixed Δ and C one can study the computational complexity question:

> What's the complexity of the decision problem: Given a graph of maximum degree Δ, is there a $(1, C)$-relaxed coloring?

Obviously, for the critical component order $f(\Delta)$ which answers the extremal graph theory question, the answer is *trivial* for the complexity question: every instance is a YES-instance. Note also, that for $C = 1$ the complexity question is polynomial-time solvable, as it is equivalent to testing whether a graph is bipartite.

In this paper we investigate the complexity question in the range between 1 and the critical component order $f(\Delta)$. We establish the monotonicity of the hardness of the problem in the interval $C \geq 2$ and prove a very sharp "hardness jump". By this we mean that the problem is NP-hard for every component order $2 \leq C < f(\Delta)$, while, of course, the problem becomes trivial (i.e. all instances are "YES"-instances) for component order $f(\Delta)$. It is maybe worthwhile to note that at the moment we do not see any *a priori* reason why the hardness of the decision problem should even be monotone in the component order C, i.e. why the hardness of the problem for component order $C + 1$ should imply the hardness for component order C. In fact the problem is obviously polynomial-time decidable for $C = 1$, while for $C = 2$ we show NP-completeness.

The other main contribution of the paper concerns the extremal graph theory question and obtains significant improvements over previously known bounds and algorithms. This result becomes particularly important in light of our NP-hardness results, as the exact determination of the place of the jump from NP-hard to trivial gets within reach.

To formalize our theorems we need further definitions. Let us denote by (Δ, C)-AsymRelCol the decision problem whether a given graph G of maximum degree at most Δ allows a $(1, C)$-relaxed coloring. Analogously, let us denote by (Δ, C)-SymRelCol the decision problem whether a given graph G of maximum degree at most Δ allows a (C, C)-relaxed coloring. Note here that both $(\Delta, 1)$-AsymRelCol and $(\Delta, 1)$-SymRelCol is simply testing whether a graph of maximum degree Δ is bipartite.

The asymmetric problem. For $\Delta = 2$ already $(2, 2)$-AsymRelCol is trivial. For $\Delta = 3$, it was shown in [5] that every cubic graph admits a $(1, 189)$-relaxed coloring, making $(3, 189)$-AsymRelCol trivial. In the proof the vertex set of the graph was partitioned into a triangle-free and a triangle-full part, then the parts were colored separately, finally the two colorings were assembled amid some technical difficulties. In our first main theorem we greatly improve on this result by using a different approach, which avoids the separation. Our method also implies a quasilinear time algorithm (as opposed to the $\Theta(n^7)$ algorithm implicitly contained in [5]). One still has to deal with the inconveniences of triangles, but the obtained component order is much smaller.

Theorem 1. *Any graph G with $\Delta(G) \leq 3$ is $(1, 22)$-relaxed colorable, i.e.*

$$f(3) \leq 22.$$

Moreover there is an $O(n \log^4 n)$ algorithm which finds such a 22-relaxed coloring.

A lower bound of 6 on $f(3)$ was established in [5].

In our next theorem we show that $(3, C)$-AsymRelCol exhibits the promised hardness jump.

Theorem 2. *For the integer $f(3)$ we have that*
(i) $(3, C)$-AsymRelCol is NP-complete for every $2 \leq C < f(3)$;
(ii) any graph G of maximum degree at most 3 is $(1, f(3))$-relaxed colorable.

In [5] it was shown that for any $\Delta \geq 4$ and positive C, (Δ, C)-AsymRelCol never becomes "trivial", i.e. for every finite C there is a "NO" instance, so $f(4) = \infty$. We show here however that the monotonicity of the hardness of $(4, C)$-AsymRelCol still exists for $C \geq 2$.

Theorem 3. *$(4, C)$-AsymRelCol is NP-complete for every $2 \leq C < f(4) = \infty$.*

Obviously, this implies that (Δ, C)-AsymRelCol is NP-complete for every $\Delta > 4$ and $2 \leq C < f(\Delta) = \infty$.

Remark. Let $f(\Delta, n)$ be the smallest integer f such that every n-vertex graph of maximum degree Δ is $(1, f)$-relaxed colorable. Then $f(\Delta) = \sup f(\Delta, n)$. While $f(3)$ is finite, our graph G_k on Figure 2 provides a simple example for $f(4)$

being non-finite in a strong sense: in any asymmetric relaxed coloring of G_k there is a monochromatic component whose order is linear in the number of vertices. This is in sharp contrast with the examples of [1, 5] where the monochromatic component order is only logarithmic in the number of vertices. It would be interesting to determine the exact asymptotics of the function $f(4,n)$; we only know of the trivial upper bound $f(4,n) \leq \frac{3}{4}n$ and the lower bound $f(4,n) \geq \frac{2}{3}n$ because of G_k.

Combining arguments of [5] and the present paper we are able to prove a tight upper bound on the component order for graphs of maximum degree 3 in which every vertex is contained in a triangle.

Theorem 4. *Let G be a graph of maximum degree 3, in which every vertex is contained in a triangle. Then G has a $(1,6)$-relaxed coloring.*

The proof of this theorem will appear in the full version of the paper [4]. An example in [5] shows that the component order 6 is best possible. We note that a 6-relaxed coloring of triangle-free graphs was already proved in [5].

The symmetric problem. Investigations about relaxed vertex colorings were originally initiated for the symmetric case by Alon, Ding, Oporowski and Vertigan [1]. They showed that any graph of maximum degree 4 has a two-coloring such that each monochromatic component is of order at most 57. This was improved by Haxell, Szabó and Tardos [10], who showed that a two-coloring is possible even with monochromatic component order of 6, and such a $(6,6)$-relaxed coloring can be constructed in polynomial time (the algorithm of [1] is not obviously polynomial). In [10] it is also proved that the family of graphs of maximum degree 5 is $(17617, 17617)$-relaxed colorable. This coloring is using the Local Lemma and it is not known whether there is a constant C and a polynomial-time algorithm which constructs a (C,C)-relaxed coloring of graphs of maximum degree 5. Alon et al. [1] showed that a similar statement cannot be true for the family of graphs of maximum degree 6, as for every constant C there exists a 6-regular graph G_C such that in any two-coloring of $V(G_C)$ there is a monochromatic component of order larger than C.

For the problem (Δ, C)-SymRelCol we make progress in the direction of establishing a sudden jump in hardness. By taking a max-cut one can easily see that $(3,C)$-SymRelCol is trivial already for $C=2$, so the first interesting maximum degree is $\Delta = 4$. From the result of [10] mentioned earlier it follows that $(4,6)$-SymRelCol is trivial. Here we show that $(4,C)$-SymRelCol is NP-complete for $C=2$ and $C=3$. We do not know about the hardness of the problem for $C=4$ and $C=5$. Again, we do not know any *direct* reason for the monotonicity of the problem. I.e., at the moment it is in principle possible that $(4,4)$-SymRelCol is in P while $(4,5)$-SymRelCol is again NP-complete.

Theorem 5. *The problems $(4,2)$-SymRelCol and $(4,3)$-SymRelCol are NP-complete.*

The proof of the theorem appears in the full version of the paper [4].

Related work. Similar hardness jumps of the k-SAT problem with limited occurrences of each variable were shown by Tovey [19] for $k = 3$ and Kratochvíl, Savický and Tuza [14] for arbitrary k. Let k, s be positive integers. A Boolean formula in conjunctive normal form is called a (k, s)-formula if every clause contains *exactly* k distinct variables and every variable occurs in *at most* s clauses. Tovey showed that every $(3, 3)$-formula is satisfiable while the satisfiability problem restricted to $(3, 4)$-formulas is NP-complete. Kratochvíl, Savický and Tuza [14] generalized this by establishing the existence of a function $f(k)$, such that every $(k, f(k))$-formula is satisfiable while the satisfiability problem restricted to $(k, f(k) + 1)$-CNF formulas is NP-complete. By a standard application of the Local Lemma they obtained $f(k) \geq \left\lfloor \frac{2^k}{ek} \right\rfloor$. After some development [14, 16] the most recent upper estimate on $f(k)$ is only a log-factor away from the lower bound and is due to Hoory and Szeider [12]. Recently new bounds were also obtained on small values of the function $f(k)$ [13]. Observe that the monotonicity of the hardness of the satisfiability problem for (k, s)-formulas is given by definition.

Notation. The order of a graph G is defined to be the number of vertices of G. Similarly, the order of a connected component C of G is the number of vertices contained in C. A graph G is r-regular if all its vertices have degree r. A graph G is called k-edge-connected if there is no edge-cut (a subset of the edges of G that disconnects G) of size at most $k - 1$.

The subgraph of a graph G induced by a vertex set $U \subseteq V(G)$ is denoted throughout by $G[U]$. Connected components in an induced subgraph $G[U]$ are called U-components and neighbors of a vertex $v \in V(G)$ in the induced subgraph $G[U]$ are called U-neighbors.

2 Trivial $(3, C)$-AsymRelCol – bounding $f(3)$

Proof (of Theorem 1.). In this section and the next one we simplify our notation by saying *C-relaxed coloring* instead of $(1, C)$-relaxed coloring.

All graphs we consider have maximum degree three. The main part of the proof is to establish the statement for 2-edge-connected 3-regular graphs. One can then easily extend this argument to arbitrary graphs of maximum degree 3. More details will be included in the full version of the paper [4].

Lemma 1. *Every* 2*-edge-connected,* 3*-regular graph has a vertex partition* $I \cup X \cup B = V(G)$ *such that*

(i) $I \cup X$ *induces a graph where each* I*-vertex has degree* 0 *and each* X*-vertex has degree* 1*.*

(ii) *No triangle contains two vertices from* X*.*

(iii) *Every* B*-component is of order at most* 6*.*

Observe that it is easy to argue that without loss of generality G is diamond-free, where a *diamond* is a graph consisting of two triangles sharing an edge. Hence in the proof we consider only graphs where no two triangles intersect.

First let us see how Lemma 1 implies Theorem 1 for 2-edge-connected 3-regular graphs. Let I, X, B be such as promised by Lemma 1. We do a postprocessing in two phases, during which we distribute the vertices of X between I and B. For each adjacent pair vw of vertices in X we put one of them to B and the other into I. When this happens we say that we *distributed* the X-edge vw. In the first phase some vertices contained in B will be moved to I, but once a vertex is in I, it stays there during the rest of the postprocessing.

For the first phase let us say that a vertex v is *ready for a change* if $v \in B$ and all its neighbors are in $B \cup X$. Once we find a vertex v ready for a change we move v to I, and distribute the X-edges it is adjacent to by moving all X-neighbors of v into B (and their X-neighbors into I). We iteratively make this change until we find no more vertex ready for a change, at which point the first phase ends. Property (ii) ensures that the rules of our change are well-defined. It is not possible that an X-neighbor of v is instructed to be placed in B, while it could also be the X-neighbor of another X-neighbor of v which would instruct it to be in I. After each change the property (i) stays true simply because some of the edges in X had their endpoints distributed one into I and one into B.

Crucially, at the end of the first phase every B-component is a path. As a result of one change no two B-components are joined, possibly a vertex u from X which changed its color to B is now stuck to an old B-component. In case this happens both of the other neighbors of u are in I (and stay there).

Let C be a B-component after the first phase. We claim that all vertices adjacent to C are in I except possibly two: one-one at each endpoint of C. By (iii) there is an at most 6-long path C' in C which used to be in a B-component before the first phase. So we can distinguish three cases in terms of how many X-neighbors can C have besides its I-neighbors.

Observations. After the first phase every B-component is one of the following:
(a) C is either a path of length at most 6 with one X-neighbor at each of its endpoints, or
(b) C is a path of length at most 7 with one X-neighbor at one of its endpoints, or
(c) C is a path of length 8 with no X-neighbors.

In the second phase we distribute the vertices that are still in X between I and B in such a way that the connected components in $G[B]$ don't grow too much. This is done by finding a *matching transversal* in an auxiliary graph H. The graph H is defined on the vertices of X, $V(H) = X$. There is an edge between two vertices u and v in H iff u and v are incident to the same component of $G[B]$.

Claim. $\Delta(H) \leq 2$.

Proof. Let us pick an arbitrary vertex y from $X = V(H)$. We aim to show that each of the two edges e_1, e_2 that are not incident to another X-vertex is "responsible" for at most one neighbor of y in H. That is, the component in $G[B]$ incident to y via such an e_1 or e_2 is incident to at most one other vertex from X.

Indeed, by the Observation above each B-component is a path, possibly adjacent to X-vertices through its endpoints, but not more than to two.

The following Lemma guarantees a transversal inducing a matching.

Lemma 2 ([10], Corollary 4.3). *Let G be a graph with $\Delta(G) \leq 2$ together with a vertex partition $\mathcal{P} = \{P_1, \ldots, P_m\}$ into 2-element subsets. Then there is a transversal T ($(T \cap P_i) \neq \emptyset$, for all $i \in \{1, \ldots, m\}$) with $\Delta(G[T]) \leq 1$.*

We remark that the proof of this Lemma in [10] involves a linear time algorithm which constructs the transversal.

We apply Lemma 2 and find such a transversal T of H on the partition defined by the edges in $G[X]$, $\mathcal{P} = E(G[X])$.

Now the second phase of our postprocessing consists of moving all vertices of T into B and moving $X \setminus T$ into I. Since $\Delta(H(T)) \leq 1$ we connect at most three connected components $Q_1$and Q_2 and Q_3 of $G[B]$ by moving an edge $\{u, v\}$ of H into B, with u incident to Q_1 and Q_2 and v incident to Q_2 and Q_3. Obviously, Q_1 and Q_3 are incident to at least one vertex of H (u and v respectively) and Q_2 is incident to at least two vertices from H (u and v) before moving the vertices of T. According to the Observation above the largest B-component created this way is of order at most $7 + 1 + 6 + 1 + 7 = 22$. Lemma 1(i) guarantees that I is independent so the defined coloring is 22-relaxed.

□

It remains to show that the partition of $V(G)$, promised in Lemma 1 indeed exists and can be found in time $O(n \log^4 n)$. The complete proof of Lemma 1 is relegated to the full version of this paper [4].

Let us here only informally discuss the main ideas of the algorithm that partitions the vertex set of G and denote it by PA(G).

In a first step the algorithm PA(G) finds a perfect matching M in G. Thus $G - M$ consists of disjoint cycles only. Moreover M is chosen such that $G - M$ is triangle-free. Such a matching can be found in $O(n \log^4 n)$ time, see Biedl, Bose, Demaine and Lubiw [6]. (Note that the algorithm in [6] only yields some perfect matching. In order to obtain a perfect matching M such that $G - M$ is triangle-free we first contract all triangles in G yielding a new graph G'. Then we apply the algorithm to G' instead and get a perfect matching M' of G'. We observe that this perfect matching M' can easily be extended to a perfect matching M of G where each triangle of G contains exactly one edge of M. Thus $G - M$ is triangle-free.) This is in fact the bottleneck of our algorithm all other parts are done in linear time. The unique neighbor of a vertex v in M is called the *partner* of v.

Next, PA(G) colors iteratively all vertices of G, one cycle of $G - M$ after another, by traversing each cycle in a predefined direction. As a default PA(G) tries to color the vertices of a cycle with the colors I and B alternatingly. Its original goal is to create a proper two-coloring this way. Of course there are several reasons which will prevent PA(G) from doing so. One main obstacle is when the partner of the currently processed vertex v is already colored, and it is done so with the same color we just gave to v. If the conflict would be in color I

then the algorithm resolves this by changing both v and its partner to X. The algorithm generally decides not to care if the conflict is in B.

Of course there is a complication with this rule when the partner is within the same triangle as v, since Lemma 1 does not allow two X-vertices in the same triangle. This and other anomalies (like the coloring of the last vertex of a cycle when the first and next-to-last vertex have distinct colors) are handled in the full version of this paper [4] by a (hopefully) well-designed set of exceptions in place.

After having colored cycle C the algorithm immediately proceeds with the cycle containing the partner v of the last vertex colored in C unless v is already colored. Otherwise the algorithm looks for vertices in C with an uncolored partner by stepping backwards along the order in which the vertices of C have been colored and eventually starts to color such a partner. If none of the vertices of C have an uncolored partner the algorithm starts with a vertex whose partner is colored.

3 Hard $(3, C)$-AsymRelCol

Proof (of Theorem 2 and 3). For a C-relaxed coloring we denote the color class forming an independent set by I and the color class spanning components of order at most C by B.

Definition 1. *Let $C \geq 2$ and $\Delta \geq 1$ be integers. A graph G is called (Δ, C)-*forcing *with forced vertex $f \in V(G)$ if*
(i) $\Delta(G) \leq \Delta$ and f has degree at most $\Delta - 1$,
(ii) G is C-relaxed colorable, and
(iii) f is contained in I for every C-relaxed two-coloring of G.

Lemma 3. *For any non-negative integer Δ and integer $C \geq 2$ the decision problem (Δ, C)-AsymRelCol is NP-complete provided a (Δ, C)-forcing graph exists.*

The proof is detailed in the full version of the paper [4]. In the proof we establish a reduction from the 3-SAT Problem using appropriate gadgets built from (Δ, C)-forcing graphs.

3.1 $(3, C)$-forcing graphs

All graphs we consider in this subsection have maximum degree at most three. Let $\mathcal{G}_C$ denote the family of graphs of maximum degree at most three that are not C-relaxed two-colorable.

Lemma 4. *For all $C \geq 2$, if $\mathcal{G}_C \neq \emptyset$ then there is a $(3, C)$-forcing graph.*

Proof. Let us assume first that $C \geq 6$. By a lemma of [5] we can assume that any member of $\mathcal{G}_C$ contains a triangle.

Lemma 5 ([5]). *Any triangle-free graph of maximum degree at most 3 has a 6-relaxed coloring.*

Let us fix a graph $G \in \mathcal{G}_C$ which is minimal with respect to deletion of edges. Let T be a triangle in G (guaranteed by Lemma 5) with $V(T) = \{t_1, t_2, u\}$ and $e = \{u, v\}$ be the unique edge incident to u not contained in T. We split e into e_1, e_2 with $e_1 = \{u, f\}$ and $e_2 = \{f, v\}$ and denote this new graph by H. H is C-relaxed colorable since the minimality of G ensures that $G - e$ has a C-relaxed coloring while the non-C-relaxed-colorability of G ensures that the colors of u and v are the same on *any* C-relaxed coloring of $G - e$. So *any* C-relaxed coloring χ of $G - e$ can be extended to a C-relaxed coloring of H by coloring f to the opposite of the color of u and v. Moreover any such extension is unique. If $\chi(u) = \chi(v) = I$, then obviously $\chi(f) = B$. If $\chi(u) = \chi(v) = B = \chi(f)$ and χ is a C-relaxed coloring of H, then χ restricted to $V(G)$ is a C-relaxed coloring of G, a contradiction.

Thus in any C-relaxed coloring χ_H of H, $(\chi_H(u), \chi_H(f), \chi_H(v))$ is either (I, B, I) or (B, I, B).

We denote by v_1, v_2 the neighbors of t_1 and t_2, respectively, not contained in T (might be $t_1 = t_2$). See also Figure 1. Suppose the vertices (u, f, v) of H can be colored with (I, B, I). But then $\chi_H(t_1) = \chi_H(t_2) = B$.

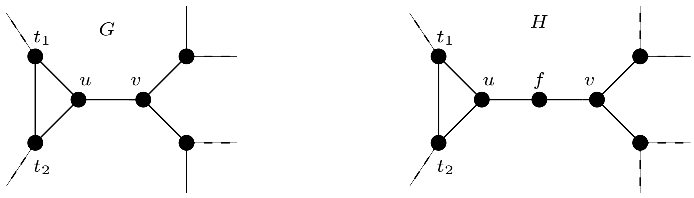

Fig. 1. Splitting $e = \{u, v\}$ into $e_1 = \{u, f\}$ and $e_2 = \{f, v\}$

[**Case** (*i*)**:**] If $\chi_H(v_1) = \chi_H(v_2) = I$ then we define a C-relaxed coloring χ_G for G as follows:
$\chi_G(x) = \chi_H(x)$ for all $x \in V(G) \setminus \{u\}$ and $\chi_G(u) = B$.
[**Case** (*ii*)**:**] Without loss of generality $\chi_H(v_1) = B$. We define a C-relaxed coloring χ_G for G as follows:
$\chi_G(x) = \chi_H(x)$ for all $x \in V(G) \setminus \{t_1, u\}$, $\chi_G(t_1) = I$, and $\chi_G(u) = B$. Indeed, the B-component containing t_2 did not increase, since $\chi_G(t_1) = \chi_G(v) = I$ and in H $\chi_H(t_1) = B$.

This contradicts the fact that G is not C-relaxed two-colorable. Thus in any C-relaxed coloring of H the vertices (u, f, v) are colored (B, I, B). The vertex f is contained in I and is of degree 2, hence H is a $(3, C)$-forcing graph with forced vertex f.

In the full version of this paper [4] we provide explicit constructions of $(3, C)$-forcing graphs with $2 \leq C \leq 5$. $\square$

Note that $(3, C)$-AsymRelCol is obviously trivial for all C with $\mathcal{G}_C = \emptyset$, so Theorem 2 follows immediately from Lemma 4 and Lemma 3. $\square$

3.2 $(4, C)$-forcing graphs

Lemma 6. *For all $\Delta \geq 4$ and all $C \geq 2$ there is a (Δ, C)-forcing graph.*

The graph $G_k - \{v_{1,1}, v_{1,2}\}$ is $(4, 2k-2)$-forcing. A proof can be found in the full version of this paper [4]. Combining Lemma 6 and Lemma 3 concludes the

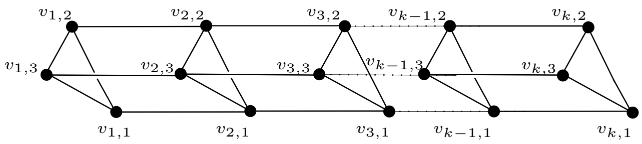

Fig. 2. G_k with one B-component of order $2k$

proof of Theorem 3. □

4 Summarizing Overview and Open Problems.

It would be interesting to determine exactly the critical monochromatic component order $f(3)$ from where the problem $(3, C)$-AsymRelCol becomes trivial.

We conjecture that there is a sudden jump in the hardness of the problem $(4, C)$-SymRelCol. Such a result would particularly be interesting, since here the determination of the critical component order is even more within reach (between 4 and 6.) As a first step one could try to prove the monotonicity of the problem.

The similar problem is wide open for graphs with maximum degree 5: Does SymRelCol exhibit a monotone behavior for $C \geq 2$? Is there a "jump in hardness"? Is there a constant C and a polynomial-time algorithm which finds a (C, C)-coloring of graphs of maximum degree 5? We only know the existence of such colorings.

For colorings with more than two colors we know much less. Even the graph theoretic questions about interesting maximum degrees are open. We list here three of the most important questions: Is there a constant C such that every graph with maximum degree 9 can be three-colored such that every monochromatic component is of order at most C? The answer is "yes" for graphs with maximum degree 8 and "no" for graphs of maximum degree 10 (see [10]). Is there a constant C such that every graph of maximum degree 5 can be red/blue/green-colored such that the set of red vertices and the set of blue vertices are both independent while every green monochromatic component is of order at most C? The answer is "yes" for graphs with maximum degree 4 and "no" for graphs of maximum degree 6 (see [5]). Determine asymptotically the largest Δ_k for which there exists a constant C_k such that every graph of maximum degree Δ_k can be k-colored such that every monochromatic component is of order at most C_k. The current bounds are $3 < \Delta_k/k \leq 4$ (see [10]).

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