Relaxed two-coloring of cubic graphs

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Abstract

We show that any graph of maximum degree at most 3 has a two-coloring such that one color-class is an independent set, while the other color-class induces monochromatic components of order at most 750. On the other hand, for any constant C, we exhibit a 4-regular graph such that the deletion of any independent set leaves at least one component of order greater than C. Similar results are obtained for coloring graphs of given maximum degree with $k + \ell$ colors such that k parts form an independent set and ℓ parts span components of order bounded by a constant. A lot of interesting questions remain open.

1 Introduction

A coloring of the vertices of a graph is called *proper* if the endpoints of any edge are assigned different colors. The greedy algorithm, the simplest coloring procedure producing a proper coloring, guarantees the existence of a proper coloring of any d-regular graph with $d+1$ colors. The clique K_{d+1} shows that the number $d+1$ of required colors is best possible for d-regular graphs.

In this paper we consider a relaxation of proper coloring by allowing "errors" of certain controlled kind. The following definition will be convenient. We say that a coloring of a graph is *C-relaxed* if all monochromatic components have order at most C. With this definition, 1-relaxed is equivalent to proper coloring. It is easy to see that any graph of maximum degree at most 3 has a 2-relaxed two-coloring. Alon, Ding, Oporowski and Vertigan [4] proved that every graph of maximum degree 4 has a 57-relaxed two-coloring. They also gave a construction of a 6-regular graph for arbitrary C which does *not* admit a C-relaxed two-coloring. Haxell, Szabó and Tardos [9] established that it is always possible to find even a 6-relaxed two-coloring of a graph of maximum degree 4 and proved that every graph of maximum degree 5 has a C-relaxed two-coloring with some constant C (In fact, $C < 20000$).

Earlier work related to relaxed colorings focused on special kinds of graphs, like line graphs of cubic graphs [7, 10]. These works culminated in the result of Thomassen [13], who proved that there exists a two-coloring of the edges of any cubic graph such that not only every monochromatic component is bounded, but is a *path* of length at most five.

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In this paper we are concerned about the asymmetric version of the relaxation of proper two-coloring. Namely, we allow larger components in only one of the color-classes, the other one has to be an independent set. Obviously any 2-regular graph has a two-coloring where one of the color-classes is an independent set, and the other induces monochromatic components of order at most 2. Our main theorem claims that a similar statement holds for graphs of maximum degree 3 as well.

Theorem 1. *Let G be a graph of maximum degree at most 3. There exists a partition of the vertex set of G into subsets I and B where I is an independent set and every component of $G[B]$ is of order at most 750.*

We prove Theorem 1 in Section 2. After some initial simplification we break the graph G into two pieces: one containing vertices which do not participate in any triangle, the other containing vertices from triangles. We solve our problem separately for each piece in Subsections 2.1 and 2.2, respectively. To handle triangle-free graphs we apply a useful lemma from [9] about *matching transversals*. For graphs whose vertex set is the union of vertex disjoint triangles we utilize the theorem of Thomassen about certain edge-two-coloring of cubic graphs. In Subsection 2.3 we finish the proof of Theorem 1 by combining the two-coloring of the two pieces through a series of modifications. This process is quite technical but finally yields the bound of 750 on the component-order.

To complement Theorem 1 we prove in Section 3 that a similar statement cannot hold for 4-regular graphs.

Theorem 2. *For any constant C there exists a 4-regular graph G such that for any independent set $I \subseteq V(G)$, $G[V(G) \setminus I]$ has a component of order larger than C.*

In Section 3 we investigate relaxed colorings of graphs with more than two colors. For this we need the following definition. A graph G is called *C-relaxed (k, ℓ)-colorable* if there exists a C-relaxed $(k + \ell)$-coloring of G such that each of the first k color-classes are independent sets. A set of graphs $\mathcal{G}$ is called *(k, ℓ)-colorable* if there exists an absolute constant C such that every member $G \in \mathcal{G}$ admits a C-relaxed (k, ℓ)-coloring. Obviously $(k, 0)$-colorability is the same as the usual k-colorability. The main result of [9] could be formulated as the family of 5-regular graphs is $(0, 2)$-colorable. Our main results state that cubic graphs are $(1, 1)$-colorable, while 4-regular graphs are not.

In [9] the maximum degree condition for $(0, k)$-colorability is investigated. It is shown that there exists a constant $\delta > 0$ such that for large ℓ, $3 + \delta < \Delta(0, \ell)/\ell < 4$, where $\Delta(k, \ell)$ is the smallest integer such that the family of graphs with maximum degree $\Delta(k, \ell)$ is *not* (k, ℓ)-colorable. One of the outstanding questions of the topic is to determine the asymptotics of $\Delta(0, \ell)/\ell$. In Section 3 we give general bounds on the maximum degree which guarantees (k, ℓ)-colorability.

Finally, in Section 4 several of the intriguing open problems are gathered.

1.1 Notation, terminology.

The *length* of a path is the number of edges contained in it. The *order* of a graph is its number of vertices. The *size* of a graph is the number of its edges. $\Delta(G)$ of a graph G denotes the maximum degree in G. By *component* we always mean connected component. If X is the subset of vertices in some graph G, then $G[X]$ denotes the subgraph of G induced by X. For a vertex $v \in X$ we say that v is an X*-vertex*, and for a component C of $G[X]$ we say that C is an X*-component* (assuming that the reference to G is implied). For two subsets X and Y of the vertex set of some graph G and an edge e with one endpoint in X and other endpoint in Y we say that e is an XY*-edge*. The degree of a vertex in a graph G is denoted by $d_G(v)$. Sometimes $d_X(v)$ is used to denote the number of neighbors of v in some subset X; in this case there should be no confusion about the underlying graph. For a subset X we denote by $\Gamma(X)$ the set of vertices which have a neighbor in X. For a vertex v we write $\Gamma(v)$ instead of $\Gamma(\{v\})$ and for a subgraph H we write $\Gamma(H)$ instead of $\Gamma(V(H))$.

2 Two-coloring cubic graphs

In this section we prove our main theorem.

First we justify a couple of simplifying assumptions. If $\Delta(G) \leq 3$, no two triangles in G share exactly one vertex. Two triangles sharing an edge form a *diamond*. We argue that, without loss of generality, we can assume that our graph is diamond-free. Indeed, let D be a diamond in G and let G' be the graph obtained from G by deleting the vertices of D. By induction (on the number of diamonds) we obtain a partitioning of G' into sets I' and B' satisfying the properties of Theorem 1. Let the two vertices of D sharing the common edge be denoted by v_1, v_2, the remaining two vertices are denoted by u_1, u_2 and the unique neighbor of u_i outside of D by u_i' (u_i' might not exist). Now let us define a partition of $V(G)$ into sets I and B by letting $I' \subseteq I$ and $B' \subseteq B$ and putting u_i into I if and only if u_i' is in B'. The vertices v_1, v_2 are put into B regardless. Since u_1 is not adjacent to u_2, I is independent by definition. Also, the vertices of D put into B are separated from B' by a vertex of I, thus the largest newly introduced component of $G[B]$ could be the diamond itself. Thus the order of the largest component of $G[B]$ is $\max\{C', 4\}$, where C' is the order of the largest component of $G[B']$.

By a similar argument, we can assume that G does not contain two triangles connected by two edges (see Fig. 1).

Definition 1. (i) *Let $\mathcal{T}$ denote the class of all graphs G, with $\Delta(G) \leq 3$, where every vertex $v \in V(G)$ is* contained *in exactly one triangle of G and there are no subgraphs isomorphic to either two triangles sharing an edge or two triangles connected by two edges.*

(ii) *Let $\mathcal{Z}$ denote the class of all triangle-free graphs G, with $\Delta(G) \leq 3$.*

By the above observations, we can assume in the proof of Theorem 1 that there is a partition

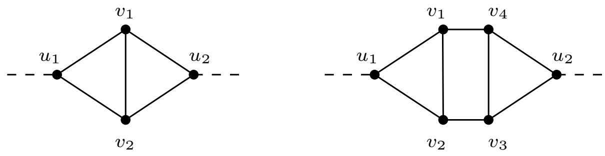

Figure 1: A diamond and two triangles connected by at least two edges

(Z, T) of the vertex set of G such that $G[T] \in \mathcal{T}$ and $G[Z] \in \mathcal{Z}$. We simply put the vertices contained in a triangle into T and define $Z = V(G) \setminus T$.

2.1 Relaxed two-coloring of triangle-free graphs

In this subsection we prove that Theorem 1 (with a better constant) holds if G is triangle-free.

Lemma 1. *For any $G \in \mathcal{Z}$, there exists a partition of the vertex set into I and B where I is an independent set and no component of $G[B]$ has order larger than* 6*.*

Proof. As a first approximation let us take a max-cut (U_1, U_2) (i.e., there is no other partition with more edges going across), with $|U_1|$ minimal (among all max-cuts).

Since (U_1, U_2) is a max-cut, every vertex has degree at most one within its own part. That is $G[U_1]$ and $G[U_2]$ consist of disjoint edges and isolated vertices. Eventually, our goal is to select one of the endpoints of each edge in $G[U_1]$ and move it to the other side such a way, that we do not create too large components.

First we make a few observations about the impossibility of certain configurations. For $i = 1, 2$ and $j = 0, 1$ let $U_{i,j} = \{x \in U_i : d_{U_i}(x) = j\}$. For $i \in \{1, 2\}$, we denote by i' the other element of $\{1, 2\}$, i.e., $i' \in \{1, 2\}$ and $i' \neq i$.

Proposition 1. *Let $x \in U_{i,1}$ and $x', x'' \in U_{i',1}$, for some $i = 1, 2$. Then x is not adjacent to both x' and x''.*

Proof. Switching the sides of x, x', x'' increases the number of edges in the cut and thus contradicts the maximality of (U_1, U_2). □

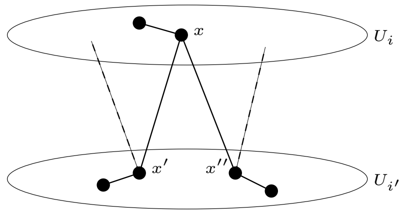

Figure 2: Configuration 1.

Proposition 2. *Let $x \in U_{2,0}$ and $x', x'', x''' \in U_{1,1}$. Then x is not adjacent to all of x', x'', and x'''.*

Proof. Switching the sides of x, x', x'', x''' would not decrease the number of edges in the cut, but would decrease the cardinality of $|U_1|$, a contradiction. □

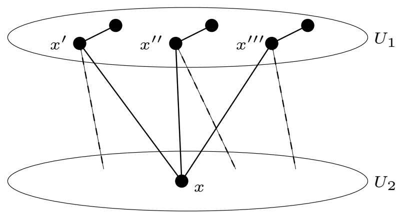

Figure 3: Configuration 2.

Proposition 3. *Let $x \in U_{i,0}$, $x', x'' \in U_{i,1}$ and $y', y'' \in U_{i',1}$, for some $i = 1, 2$. Then it is not possible that x is adjacent to both y' and y'', y' is adjacent to x', and y'' is adjacent to x''.*

Proof. Switching the sides of x, x', x'', y', y'' increases the number of edges in the cut and thus contradicts the maximality of (U_1, U_2). □

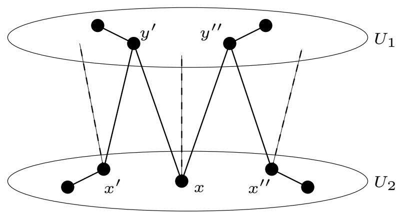

Figure 4: Configuration 3.

Note that Propositions 1, 2 and 3 fail to be true if G contains triangles.

We define an auxiliary graph H on the vertex set $V(H) = U_{1,1}$. Two vertices x and y of H are adjacent if they have a neighbor in the same component of $G[U_2]$.

Claim: $\Delta(H) \leq 2$

Proof. Let $x \in V(H)$. By the definition of $V(H)$, x has at most two neighbors in U_2. Let y be one of them.

If y is an isolated vertex of $G[U_2]$, then by Proposition 2, y has at most one more neighbor (besides x) in $U_{1,1}$. So y does not account for more than one H-neighbor of x. If y is in an edge-component of $G[U_2]$, let w be its unique neighbor in U_2. By Proposition 1 y has no other neighbor in $U_{1,1}$ but x. Similarly, w has at most one neighbor in $U_{1,1}$. So y is responsible for at most one H-neighbor of x.

We showed that each of the (at most two) U_2-neighbors of x can produce at most one H-neighbor for x. That is the degree of x in H is at most 2. □

Now we use the following lemma from [9].

Lemma 2. *[9, Corollary 4.3] Let H be a a graph with $\Delta(H) \leq 2$. Suppose that $V(H)$ is partitioned into subsets of size two, $V(H) = W_1 \cup \ldots \cup W_m$, $|W_i| = 2$ for $i = 1, \ldots, m$. Then there exists a "matching transversal", i.e., a subset $T \subseteq V(H)$ of the vertices such that $|W_i \cap T| = 1$ for every $i = 1, \ldots m$ and $\Delta(G[T]) \leq 1$.*

Since $\Delta(H) \leq 2$, we can apply Lemma 2 for H, with the edges of $G[U_1]$ as W_i. (Remember $G[V(H)]$ is a perfect matching!) We select a matching transversal T and move it over; That is we define $I = U_1 \setminus T$ and $B = U_2 \cup T$.

Clearly I is an independent set.

How large could a component be in $G[B]$? Note that T is an independent set in G and since T induces a matching in H, any component of $G[B]$ can contain at most two vertices from T.

If a component of $G[B]$ contains exactly one vertex of T, then its size is at most 5.

Suppose now that a component C of $G[B]$ contains two vertices $t_1, t_2 \in T$. There must be a component C' of $G[U_2]$ in which both t_1 and t_2 has a neighbor.

Since both t_1 and t_2 have at most two neighbors in U_2, C contains at most three components of $G[U_2]$.

If there are at most two components of $G[U_2]$ in C, then the cardinality of C is at most 6.

Assume now that there are three components C', C_1, C_2 of $G[U_2]$ glued together in C. By Proposition 1, neither t_1 nor t_2 is adjacent to two components of order two. So if $|C'| = 2$, then $|C_1| = |C_2| = 1$ and thus $|C| = 6$.

Similarly, if $|C'| = |C_i| = 1$ for some $i = 1, 2$, then $|C| = 6$.

Finally, the case $|C'| = 1$ and $|C_1| = |C_2| = 2$ is impossible because of Proposition 3.

Concluding, we proved that all components of $G[B]$ are of order at most 6. □

2.2 Relaxed two-coloring of graphs from $\mathcal{T}$

In this subsection we prove Theorem 1 for graphs from $\mathcal{T}$. In fact, we show a stronger statement which comes in handy in Subsection 2.3, when we put together the relaxed two-coloring of a general graph G of maximum degree at most 3 from the relaxed two-colorings of $G[Z]$ and $G[T]$. Instead of partitioning $G[T]$ into two parts, we partition into three. The extra part X provides some flexibility: vertices of X could later be placed arbitrarily into either I or B. Note that the degree of every vertex in $G[T]$ is either 2 or 3. Let V_i be the set of vertices of degree i. The flexible part X lies completely in V_2; these are the vertices which could have a neighbor in the triangle-free part Z.

From now on, in this subsection, G denotes a graph from $\mathcal{T}$.

Lemma 3. *For any $G \in \mathcal{T}$, there exists a partition of the vertex set $V(G)$ into three sets I, B and X such that*

(i) $I \subseteq V_3$, $X \subseteq V_2$, $I \cup X$ *is an independent set and no component of* $G[B \cup X]$ *is larger than* 21.

(ii) *every component of* $G[B \cup X]$ *contains at most three vertices from* $B \cap V_2$, *all of which are contained in the same triangle. Any component of* $G[B \cup X]$ *containing exactly one vertex from* $B \cap V_2$ *is of order at most* 8, *and each component containing two or three vertices from* $B \cap V_2$ *is fully contained in one triangle.*

Definition 2. *Let $T_1, \ldots, T_m$ be the triangles of a graph $G \in \mathcal{T}$. The* triangle-graph $T(G)$ *of G is defined on the vertex set $V(T(G)) = \{T_1, \ldots, T_m\}$ where two vertices are adjacent if the corresponding triangles are connected by an edge.*

By the definition of $\mathcal{T}$, $T(G)$ is a simple graph.

A vertex in V_3 is denoted by $v_{i,j}$ if it is contained in triangle T_i and there is an edge incident to it which goes to a vertex of triangle T_j.

In order to prove our lemma, we need the following statement, due to Thomassen.

Theorem 3. *[13, Theorem 2.] Let H be a graph of maximum degree at most 3. Then the edge set of H has a red/blue coloring and an orientation of the edges such that*

(i) *each monochromatic component is a directed path of length at most* 5, *and*

(ii) *each vertex of degree* 2 *is either an interior vertex of a monochromatic directed path or the endpoint of a monochromatic directed path of length at most* 3.

We apply Thomassen's theorem to the triangle-graph $T(G)$ and obtain an edge-coloring c_t and edge-orientation o_t of $T(G)$. Given c_t, we can define a canonical red/blue coloring c of the *vertices* in V_3, namely let the color of vertices $v_{i,j}, v_{j,i}$ be the same as the color of the edge T_iT_j. With this definition, a monochromatic path of length i in $T(G)$ corresponds to a monochromatic path of length $2i - 1$ in G. Thus c is a vertex-two-coloring of $G[V_3]$ where each monochromatic component is a path of length at most 9.

Using o_t, we can define a canonical orientation o of those edges of G which go within the monochromatic paths of c. Let the orientation of the edge $v_{i,j}v_{j,i}$ be defined to be the same as the orientation of the edge T_iT_j. Edge $v_{i,j}v_{i,k}$ within triangle T_i is oriented if and only if both of its endpoints received the same color, in which case it is oriented such that the path $v_{j,i}, v_{i,j}, v_{i,k}, v_{k,i}$ becomes directed.

This vertex-coloring c and partial edge-orientation o of $G[V_3]$ can now be transformed into a partition (I, B, X) of G satisfying Lemma 3.

The interior vertices of the monochromatic directed path components of G will be called *interior vertices*, while the starting and ending vertices of the monochromatic directed path components will be called *extremal vertices*. Every vertex in V_3 is either an interior or extremal vertex, but not both.

A triangle is called a *degree-i* triangle if the corresponding vertex in the triangle-graph has degree i. A degree-3 triangle always contains two interior vertices and one extremal vertex. A degree-2 triangle either contains two extremal or two interior vertices. A degree-1 triangle always contains one extremal vertex.

We are now ready to define our partition.

Let I contain the starting vertices of all the monochromatic directed paths. Let I also contain the end vertices of the monochromatic paths unless the path is of length 1 or it ends in a degree-2 triangle. This definition ensures that all vertices selected to I are of degree 3, i.e., $I \subseteq V_3$.

The (unique) vertex of degree 2 of a degree-2 triangle is put into X if none of the other two vertices of the triangle were put into I. Thus $X \subseteq V_2$ by definition.

The rest of the graph belongs to B. Observe that all interior vertices of monochromatic paths are in B, together with the endpoints of monochromatic paths of length one and the endpoints of monochromatic paths which end at a degree-2 triangle. Besides these, some vertices from V_2 are also in B. The unique vertex of degree 2 of a degree-2 triangle K is in B if and only if a path starts at K. Also, all vertices of degree 2 from degree-1 and degree-0 triangles are in B.

Is $I \cup X$ an independent set? Its definition ensures that X is an independent set and there is no edge between X and I. Now let us consider I and observe that I contains only extremal vertices. There cannot be two adjacent extremal vertices in two different triangles unless they are the starting- and end-vertices of the same path of length one; these are not both in I by definition. Also, any triangle contains at most one vertex from I because any degree-3 triangle contains exactly one extremal vertex (Theorem 3(i)), and no degree-2 triangle contains more than one starting vertex (Theorem 3(ii)). We can conclude that $I \cup X$ is independent.

In the following, we look at why the components of $G[B \cup X]$ are of bounded order. A monochromatic directed path component of length i in G gives rise to paths in $G[B]$ of length either $i-2$ or $i-1$, depending whether both extremal vertices were put in I or just the starting vertex. We call such a path a *B-path*. Every B-path is of length at most 8. A component of $G[B \cup X]$ could possibly contain several B-paths, so we investigate how B-paths could "glue" to each other and to vertices of V_2 to form the $(B \cup X)$-components.

Case (i). In a degree-3 triangle there are always two interior vertices which belong to one particular B-path P. The third vertex w of the triangle is either in I (i.e., there is no "gluing") or it is the endpoint of a monochromatic path of length one. In the latter case w is in B and glued to P as a monochromatic path of length 0.

Case (ii). What can happen in a degree-2 triangle K? Let u_1, u_2, w be the vertices of K, where $u_1, u_2 \in V_3$ and $w \in V_2$. If u_1 and u_2 are both interior vertices, then they are in the same B-path and w is in X glued to it. Otherwise u_1 and u_2 are the two extremal vertices of different monochromatic paths P_1 and P_2, respectively. If one of these paths, say P_1, has a starting vertex in K, then $u_1 \in I$. Then u_2 is the endpoint of P_2, thus $u_2, w \in B$, i.e., w is glued to P_2. If both paths P_1 and P_2 have endpoints in K, then one of them, say P_1, was obtained from a monochromatic path of $T(G)$ of length at most 3 (by Theorem 3(ii)) and the other was obtained from a monochromatic path of $T(G)$ of length at most 5 (by Theorem 3(i)). In this case w is contained in X. Only in this case two B-paths of length at least 1 are glued into the same $(B \cup X)$-component. The length of P_1 is at most 4, while the length of P_2 is at most 8, so altogether they form a B-component of order at most 14.

Case (iii). In a degree-1 triangle the two vertices of degree 2 could be glued to the end of a monochromatic path of length one.

In conclusion, two B-paths of length at least 1 could only be glued at their endpoint (when they both end at the same degree-2 triangle). Their other vertices might be glued to the vertices in their triangles, but not more. Hence the 14 vertices of the two glued B-path of the last case of Case (*ii*) could still pick up 7 more vertices from their triangles, which adds up to 21.

For part (*ii*) of our lemma we first observe that if a $(B \cup X)$-component C contains some vertex of a degree-0 or degree-1 triangle then C is fully contained in it. Now suppose $w \in B \cap V_2$ is a vertex of a degree-2 triangle K. Let $V(K) = \{w, u_1, u_2\}$. Since $w \notin X$, either u_1 or u_2 must be in I. Without loss of generality $u_1 \in I$ and u_1 is a starting vertex of a monochromatic B-path. Thus u_2 must be the endpoint of a B-path of length at most 4 (corresponding to a monochromatic path of $T(G)$ of length at most 3 by Theorem 3(*ii*)). By the discussion above, this B-path is not glued to any other B-path except maybe to vertices in the triangles it intersects. Besides K, there could be two other triangles intersected by P. If the third vertex in these triangles is of degree 2, then it is contained in X. Thus C does not contain any more vertices of $B \cap V_2$. Also, C cannot contain more than 8 vertices.

2.3 Putting things together

Proof of Theorem 1. As it was mentioned at the beginning of this section, we can assume that we find a partition of $V(G)$ into two parts (T, Z) with $G[T] \in \mathcal{T}$ and $G[Z] \in \mathcal{Z}$. Further, we define $\mathcal{T}_i$ as the set of triangles in $G[T]$ having i neighbors in Z. A vertex is called a *neighbor of triangle* K if it is adjacent to a vertex of K but not contained in K. (Since all triangles are disjoint, a neighbor of a triangle K has exactly one neighbor in K.)

Eventually, we try to achieve that any B-component of $G[T]$ is adjacent to at most one B-component of $G[Z]$. As it turns out, the principal hurdle to this are those triangles from $\mathcal{T}_2$ and $\mathcal{T}_3$ whose neighbors form an independent set. Hence, initially, we define a supergraph H of $G[Z]$ such that possibly *no* set of neighbors of a triangle from $\mathcal{T}_2 \cup \mathcal{T}_3$ is independent. We do this by iteratively adding edges to $G[Z]$. We might not be able to make all the neighbor-sets non-independent however, as we would also like to keep H triangle-free.

Formally, we define the graph $H \supseteq G[Z]$ starting from $G[Z]$ as follows:
We iteratively check for every triangle $K \in \mathcal{T}_3$ whether its three distinct neighbors (all in Z) form an independent set. If the answer is positive, we check whether there exist two of these three

neighbors, say u and v such that adding the edge $\{u, v\}$ to H does not introduce a triangle in H. If there is such a pair of neighbors of K, then we select one such pair and add the edge between them to H. We call this edge e_K. Otherwise, i.e., if for all three pairs the addition of the edge would create a triangle in H, we do not do anything.

Then, iteratively for every triangle $K \in \mathcal{T}_2$ we check whether the two neighbors of K in Z form an independent set. If the answer is positive, we add the edge between these two neighbors (and name it e_K) unless it introduces a triangle in H, in which case we do not change anything.

We define $\mathcal{D} \subseteq \mathcal{T}_3$ and $\mathcal{P} \subseteq \mathcal{T}_2$ to be the sets of triangles whose neighborhood in (the final) H forms an independent set. Observe that e_K is defined for some (but not necessarily all) triangles from $(\mathcal{T}_2 \setminus \mathcal{P}) \cup (\mathcal{T}_3 \setminus \mathcal{D})$. Note also that the function $K \to e_K$ is injective.

The following easy proposition is needed in most of our arguments.

Proposition 4. *For any vertex $v \in Z$,*

$$d_H(v) = d_G(v) - |\{K \in \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 : v \in \Gamma(K) \text{ and } v \notin e_K \text{ or } e_K \text{ is not defined}\}|.$$

In particular,

$$d_H(v) \leq 3 - |\{K \in \mathcal{P} \cup \mathcal{D} \cup \mathcal{T}_0 \cup \mathcal{T}_1 : v \in \Gamma(K)\}|$$

and $H \in \mathcal{Z}$.

Proof. The first statement is true because we subtract from $d_G(v)$ the number of those edges incident to v going to triangles *because of which* we did not introduce any new edge at v (note that any incident edge of v goes to *at most one* triangle since G is diamond-free). Now equality follows, since the function $K \to e_K$ is injective.

The bound on $d_H(v)$ then follows easily after noting that $d_G(v) \leq 3$ and that e_K exists only if $K \in (\mathcal{T}_2 \setminus \mathcal{P}) \cup (\mathcal{T}_3 \setminus \mathcal{D})$.

Hence $\Delta(H) \leq 3$. By definition, the above process did not introduce triangles to H, which implies $H \in \mathcal{Z}$. □

We denote by G^* the supergraph of G which contains all edges we added by the above procedure, formally $G^* = G \cup H$. Note that the maximum degree of G^* might be larger than 3. We still prove the existence of the appropriate relaxed coloring for G^* instead of G. Note that $G^*[Z] = H$ and $G^*[T] = G[T]$.

We choose a partition (I_Z, B_Z) on $G^*[Z]$ and a partition (I_T, B_T, X) on $G^*[T]$ fulfilling the conditions of Lemma 1 and Lemma 3, respectively. As a first approximation of our final partition (I, B) we define $(I^{(1)}, B^{(1)}) = (I_T \cup I_H, B_T \cup B_H)$.

$I^{(1)}$ is an independent set in G^* since by Lemma 3(i) the vertices in I_T have all their neighbors in T.

Note that eventually a B-component of $G^*[T]$ can attach to B-components of $G^*[Z]$ only through a vertex whose degree in T is 2. By Lemma 3(ii) all $B^{(1)}$-vertices of degree 2 of a

$B^{(1)} \cup X$-component C_T of $G^*[T]$ are contained in one triangle. We denote this special triangle of C_T containing all the degree 2 B-vertices of C_T by K_{C_T}. By the end of our proof vertices of X will be placed into the part different from their Z-neighbor. This will ensure that the component C_T has a chance to connect to some $B^{(1)}$-component of $G^*[Z]$ only through K_{C_T}. From Lemma 1 we know that any $B^{(1)}$-component C_Z of $G^*[Z]$ is of order at most 6 and from Lemma 3(ii) it follows that no attaching $B^{(1)} \cup X$-component of $G^*[T]$ is of order larger than 8. (We mean "attaching" only through $B^{(1)}$-vertices, as by the end, X-vertices will be placed such that they block attachment to B-components of $G^*[Z]$.)

The main problem is that currently a $B^{(1)}$-component of $G^*[T]$ could be adjacent to more than one $B^{(1)}$-component of $G^*[Z]$. Hence we do not have any control on the order of $B^{(1)}$-components of G^*. Our plan is to modify $(I^{(1)}, B^{(1)})$, resulting in partitions $(I^{(i)}, B^{(i)})$ with $i \in \{2,3,4,5\}$ such that $I^{(5)} \cup B^{(5)} = V(G)$, $I^{(5)} \cap B^{(5)} = \emptyset$, and any $B^{(5)}$-component C_T of $G^*[T]$ with $K_{C_T} \in (\mathcal{T}_3 \backslash \mathcal{D}) \cup \mathcal{T}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_0$ will attach to at most one $B^{(5)}$-component of $G^*[Z']$ where $Z' = Z \cup \left(\bigcup_{K \in \mathcal{D}} V(K)\right)$. Moreover $I^{(i)}$ remains an independent set throughout and we will control the order of $B^{(i)}$-components of $G^*[T]$, $G^*[Z]$ and eventually of $G^*[Z']$. Naturally, components C_T with $K_{C_T} \in (\mathcal{T}_2 \backslash \mathcal{P}) \cup \mathcal{T}_1 \cup \mathcal{T}_0$ do not pose any problem (we will return to this more formally later). In the first modification we handle components with $K_{C_T} \in \mathcal{P}$. In the second, we consider components with $K_{C_T} \in \mathcal{D}$. The third modification will resolve components C_T with $K_{C_T} \in \mathcal{T}_3 \setminus \mathcal{D}$, while the fourth modification determines the final placement of the elements of X.

For a vertex $v \in B^{(i)}$ (or $I^{(i)}$) we will say that the *status of v is B (or I)* and the status of a vertex from $B^{(i)}$ is *opposite* to the status of a vertex from $I^{(i)}$. We also say that the status of a vertex from X is *neutral*.

By Proposition 4, the neighbors t_1' and t_2' ($\in Z$) of any triangle $K \in \mathcal{P}$ have degree at most two in $G^*[Z]$. Moreover t_1' and t_2' must have at least one common neighbor in $G^*[Z]$ because the addition of the edge $t_1't_2'$ would have created a triangle in H. Let us arbitrarily choose one of these common neighbors for every triangle $K \in \mathcal{P}$ and denote it by $w(K)$ and let $N(K) = \{t_1', t_2'\}$ (see Fig. 5). We define $W = \{w(K)\}_{K \in \mathcal{P}}$.

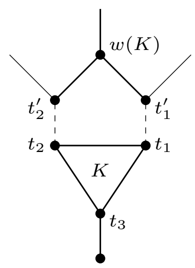

Figure 5: A triangle $K \in \mathcal{P}$ with $N(K) = \{t_1', t_2'\}$ and $w(K)$

In the following, we define $(I^{(2)}, B^{(2)})$ by switching the current status of some vertices of Z. We will not change the status of any vertex from T.

Modification 1: As a first step we set the status of each vertex from W to B. Then, iteratively for every triangle $K \in \mathcal{P}$, we set the status of a vertex $v \in N(K)$ to B if its neighbor in $G^*[Z \backslash \{w(K)\}]$ exists and has status I. Otherwise we set the status of v to I, provided $v \notin W$. Finally, let $I^{(2)}$ ($B^{(2)}$) be the set of those vertices whose status is I (B).

Claim 1. *Every $B^{(2)}$-component C_T of $G^*[T]$ with $K_{C_T} \in \mathcal{P}$ connects to at most one $B^{(2)}$-component of $G^*[Z]$. $I^{(2)}$ is an independent set in G^*. No $B^{(2)}$-component of $G^*[Z]$ is of order larger than* 30.

Note that C_T is also a $(B^{(2)} \cup X)$-component of $G^*[T]$ and it is contained in K_{C_T}.

Proof. Either $N(K_{C_T}) \cap I^{(2)} \neq \emptyset$ or both vertices of $N(K_{C_T})$ are contained in the same $B^{(2)}$-component on Z, since $w(K_{C_T}) \in B^{(2)}$. This proves the first part of the claim.

Clearly $I^{(2)}$ is an independent set in G^*. Indeed, by construction $I^{(2)} \cap Z$ is independent, while the vertices $I^{(2)} \cap T = I_T$ do not have a neighbor in Z by Lemma 3(i).

To bound the $B^{(2)}$-components of $G^*[Z]$ we need a few observations. Recall that W is the collection of all $w(K)$ with $K \in \mathcal{P}$, and let us define $N = \cup_{K \in \mathcal{P}} N(K) \setminus W$ and $R = Z \setminus W \setminus N$.

Observations. Let C be a $B^{(2)}$-component of $G^*[Z]$. Then

(1) every $v \in N \cap C$ is a pendant vertex of C and its lone neighbor in C is contained in W.

(2) There are no two incident WW-edges in C.

(3) Each W-vertex is adjacent to at most one $R \cup W$-vertex in C.

(4) If C contains a WW-edge or no vertex from R, then $|C| \leq 6$.

Proof. (1) Let $v \in N(K)$ where $K \in \mathcal{P}$. Then v has at most 2 neighbors in $G^*[Z]$ by Proposition 4, and one of those neighbors is certainly $w = w(K)$. In case v has a second neighbor u in Z, then just after having set the status of v in Modification 1, its status is opposite to the status of u. We will show that this remains true even at the end of Modification 1. Assume it does not hold anymore. The status of v did not change again, since its degree in $G^*[Z]$ is 2 and thus v is incident to at most one triangle from $\mathcal{P}$. (Hence v is scheduled only once for a possible change of its status.) Thus u must have gotten a new status according to the state of its unique neighbor in $G^*[Z \backslash \{w(K')\}]$, with $K' \in \mathcal{P}$ and $u \in N(K')$, if it exists. This vertex is necessarily v, since $v \notin W$ and u has at most one neighbor in $Z \backslash W$. This yields a contradiction, since u and v already had opposite status, so the status of u did not change.

(2) Assume there are two incident WW-edges in C. Let us look at a vertex $w \in W$ adjacent to two other vertices $u, v \in W$. Let $w = w(K)$, for some triangle $K \in \mathcal{P}$. Since $d_H(w) \leq 3$, without loss of generality $u \in N(K)$, so u is incident to $K \in \mathcal{P}$. Therefore u has degree at most

2 in $G^*[Z]$ by Proposition 4. So if $u = w(K')$ then $w \in N(K')$ and w is of degree at most 2 in $G^*[Z]$ as well. But then $v \in N(K)$, and has degree two, too. If $v = w(K'')$ then $w \in N(K'')$, so w is the neighbor of two distinct triangles from $\mathcal{P}$. This, by Proposition 4, implies $d_H(w) \leq 1$, a contradiction.

(3) A vertex $w(K) \in W$ has two neighbors u_1, u_2 in $N(K)$. By the previous observation, at least one of these neighbors, say u_1, is in N. Then either $u_2 \in W$ or $u_2 \in N$. In the first case the degree of $w(K)$ in $G^*[Z]$ is two by considerations similar to the one in the proof of (2). If $u_2 \in N$, then $w(K)$ can have at most one neighbor in $W \cup R$ since the maximum degree of H is at most 3.

(4) The first case reduces to the second since the WW-edge forbids any vertex from R in C by Observations (3) and (1). If $R \cap C = \emptyset$, then by Observations (1) and (2) $|C|$ is at most the number of vertices in a tree with at most 2 inner vertices, thus at most 6. □

Let now C be a component of $G^*[B^{(2)} \cap Z]$. We will bound its order. Recall that the order of each component $G^*[B^{(1)} \cap Z]$ is at most 6, and changes could have happened only to the status of $W \cup N$-vertices.

By Observation (4) we can assume that there are no WW-edges in C and there exists a vertex $v \in R \cap C$. Let C' be the largest $R \cap B^{(1)}$-component containing v. We claim that C is then contained in the 2-neighborhood of C'. In other words, every vertex of C has a neighbor which has a neighbor in C'. By Observation (1) the vertices of C' are only adjacent to W-vertices in C. These vertices from W are obviously part of the 1-neighborhood of C'. By Observation (3) none of these W-vertices can be connected to another R-vertex and by our assumption neither to another W-vertex, only to N-vertices. These N-vertices are part of the 2-neighborhood of C' and by Observation (1) they all are pendant in C.

Now the bound on the order of C easily follows. Since C' is connected and $\Delta(G^*[Z]) \leq 3$, the number of neighbors of C' in $Z \setminus V(C')$ is at most $|C'| + 2$. Each of these neighbors can have 2 more neighbors, so the 2-neighborhood of C' has order at most $|C'| + 3(|C'| + 2) = 4|C'| + 6 = 30$, since $|C'| \leq 6$ by Lemma 1. This proves the upper bound on $|C|$. □

In order to proceed to Modification 2, we need a few definitions. A *dragon* D_K of G^* is a triangle $K \in \mathcal{D}$, together with the three disjoint vertices $t'_1, t'_2, t'_3 \in \Gamma(K) \cap Z$ and the common (not necessarily distinct) neighbor(s) of every pair of vertices from t'_1, t'_2, t'_3 (see Fig. 6 for the three different kinds of dragons). All vertices but t_1, t_2 and t_3 of D_K are in Z, thus not contained in triangles. Let $Ext(D_K) = \{v \in V(D_K) \ : \ \text{there is a neighbor of } v \text{ in } Z \setminus V(D_K)\}$ be the *exterior* of dragon D_K. Informally, $Ext(D_K)$ are those vertices of D_K which have a chance to extend significantly a B-component of Z' beyond D_K. We will need the following basic properties of dragons and their exteriors. Recall the definition of the set W from Modification 1.

Proposition 5. *(0) For any edge $u, v \in V(D_K) \setminus V(K)$ of a dragon D_K, exactly one of the endpoints is adjacent to K.*

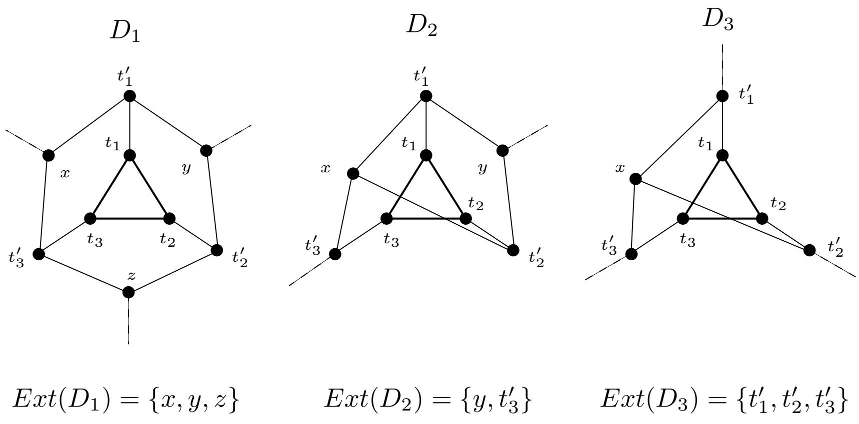

Figure 6: The three different dragons D_i and their exteriors $Ext(D_i)$.

(1) The set of vertices v with $d_{D_K}(v) = 2$ form an independent set for each dragon D_K. In particular, $Ext(D_K)$ is an independent set.

(2) Let $K, K' \in \mathcal{D}$ be distinct dragons with $V(D_K) \cap V(D_{K'}) \neq \emptyset$. Then $V(D_K) \setminus V(K) = V(D_{K'}) \setminus V(K')$ and $Ext(D_K) = Ext(D_{K'}) = \emptyset$.

(3) $W \cap (\bigcup_{D \in \mathcal{D}} Ext(D)) = \emptyset$

Proof. (0) If neither u nor v was adjacent to K, then they both were adjacent to two vertices of $\Gamma(K) \setminus V(K)$, so they would have a common neighbor in $\Gamma(K) \setminus V(K)$, implying $u, v \in T$, a contradiction. Hence one of them, say u, is a neighbor of K. Then v is not a neighbor of K, otherwise the neighborhood of K is not independent.

(1) Suppose there are two adjacent vertices $u, v \in D_K$ with $d_{D_K}(u) = d_{D_K}(v) = 2$. Using part (0), assume that u is a neighbor of K and v is not a neighbor of K. Thus, since $d_{D_K}(u) = 2$, v has to be the common neighbor of u and both of the two other vertices adjacent to K, contradicting that v is of degree 2 within D_K. For the second claim note that a vertex in the exterior of a dragon has a Z-neighbor outside the dragon, so it can have at most two neighbors within the dragon by Proposition 4.

(2) Obviously $V(K) \cap V(K') = \emptyset$. Suppose that $V(D_K) \setminus V(K) \neq V(D_{K'}) \setminus V(K')$. Then there exists $w \in V(D_K) \setminus V(K) \setminus V(D_{K'})$ which has a neighbor $v \in V(D_K) \cap V(D_{K'})$. By Proposition 4 each vertex in the intersection of D_K and $D_{K'}$ have at least one incident edge which is also in the intersection. Let the edge $\{u, v\}$ be in $V(D_K) \cap V(D_{K'})$. Since v has a Z-neighbor outside $D_{K'}$, Proposition 4 implies that v has degree at most two within $D_{K'}$. If v were adjacent to K, then Proposition 4 would even imply that $d_{D_{K'}}(v) \leq 1$, a contradiction. Thus, by part (0), u is adjacent to K. Then, by Proposition 4, the degree of u in $D_{K'}$ is two. This is in contradiction with part (1), since v was already shown to have degree two within $D_{K'}$. Hence $V(D_K) \setminus V(K) = V(D_{K'}) \setminus V(K')$.

By Proposition 4 only vertices v with $d_{D_K}(v) = 2$ could have a neighbor in $V(K')$. There are at most three of these in $V(D_K)$, so all of them must connect to a vertex in $V(K')$. Then, again by Proposition 4, these vertices cannot have a neighbor in $Z \setminus V(D_K)$, so $Ext(D_K) = \emptyset$.

(3) If $w(K')$ of a triangle $K' \in \mathcal{P}$ is contained in D_K, then at least one vertex of $N(K')$ is contained in D_K as well, since vertices from D_K have degree at most 1 to the outside of D_K. Such a vertex $u \in N(K')$ has degree two within D_K by Proposition 4, since u is the neighbor of a triangle from $\mathcal{P}$. Then by part (1), $w(K')$ cannot have degree two within D_K. So it cannot be contained in $Ext(D_K)$ either. □

In the following modification we will switch the current status of some vertices of Z to obtain $I^{(3)}$ and $B^{(3)}$. That is $I^{(3)} \cap T = I_T$ and $B^{(3)} \cap T = B_T$. Each of the vertices whose status is switched is contained in some dragon.

Modification 2: Iteratively, for every triangle $K \in \mathcal{D}$, we switch the status of each vertex of $Ext(D_K)$ to the opposite of its unique neighbor in $Z \setminus V(D_K)$. The status of all other vertices in $V(D_K)$ is set to B. Finally, let $I^{(3)}$ ($B^{(3)}$) be the set of vertices whose status is I (B).

Note that the status of t_1, t_2, and t_3 was B already before the modification. Recall that $Z' = \cup_{K \in \mathcal{D}} V(K) \cup Z$.

Claim 2. *$I^{(3)}$ is an independent set. A $B^{(3)}$-component of $G^*[Z']$ is of order at most 30. Every $B^{(3)}$-component C_T of $G^*[T]$ with $K_{C_T} \in \mathcal{P}$ connects to at most one $B^{(3)}$-component of $G^*[Z']$.*

Proof. According to Proposition 5 part (2) if two dragons intersect, then their exteriors are empty. So the status of any vertex changed at most once during Modification 2.

$I^{(2)}$ is independent. If a vertex's status is changed to I, then it is in $Ext(D_K)$ for some dragon D_K. The status of its unique neighbor in $Z \setminus V(D_K)$ is opposite by the construction, while the status of its neighbors in $Z \cap V(D_K)$ are also opposite because $Ext(D_K)$ is an independent set by Proposition 5 part (1) and the rest of the dragon is put into $B^{(3)}$. Thus $I^{(3)} \cap Z$ is an independent set. $I^{(3)} \cap T = I_T$ did not change, thus still independent and has no incident edges going to Z. In conclusion, $I^{(3)}$ is independent.

Let C be a $B^{(3)}$-component of $G^*[Z']$. If C intersects a dragon D_K which intersects another dragon $D_{K'}$, then C is equal to the union of D_K and $D_{K'}$ and its order is 12. If C intersects a dragon which does not intersect any other dragon, then C is contained in the dragon and its order is at most 9. Otherwise C is the subset of a $B^{(2)}$-component of $G^*[Z]$, and has order at most 30 by Claim 1.

For the last part of the claim it is enough to see that $W \cap I^{(3)} = \emptyset$. This is true because $W \cap I^{(2)} = \emptyset$ and by Proposition 5 part (3), $W \cap (\bigcup_{D \in \mathcal{D}} Ext(D)) = \emptyset$.

□

In the next modification we change the status of some vertices of T from B to neutral. Thus $I^{(3)} = I^{(4)}$ and $B^{(3)} \cap Z = B^{(4)} \cap Z$.

Modification 3: Iteratively, for every triangle $K \in \mathcal{T}_3 \setminus \mathcal{D}$, choose a vertex $v_K \in V(K)$ such that $\Gamma(K) \setminus \Gamma(v_K) \in E(H)$ and change the status of v_K to neutral. Note that by the definition of $\mathcal{D}$ such a vertex v_K necessarily exists. Finally, let $I^{(4)} = I^{(3)}$, let $B^{(4)}$ be the set of vertices with status B, and let X' be the set of those vertices whose status is neutral.

Claim 3. *$I^{(4)}$ is an independent set. Parts (i) and (ii) of Lemma 3 are still true with $G^*[T]$, $I^{(4)} \cap T$, $B^{(4)} \cap T$ and X' in place of G, I, B, and X, respectively. Every $(B^{(4)} \cup X')$-component C_T of $G^*[T]$ with $K_{C_T} \in (\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \setminus \mathcal{D})$ connects to at most one $B^{(4)}$-component of $G^*[Z']$ via a $B^{(4)}$-vertex of C_T.*

Proof. $I^{(4)} = I^{(3)}$, thus the first part follows from Claim 2.

The partition (I_T, B_T, X) was chosen to satisfy Lemma 3. The only difference between the partitions (I_T, B_T, X) and $(I^{(4)} \cap T, B^{(4)} \cap T, X')$ is that the status of one vertex of certain degree-0 triangles (using the notation of Subsection 2.2) is changed from B to neutral. All claims of Lemma 3 still easily hold.

Let C_T be a $(B^{(4)} \cup X')$-component of $G^*[T]$. If $K_{C_T} \in (\mathcal{T}_0 \cup \mathcal{T}_1)$, then there is at most one $B^{(4)}$-vertex of C_T with a neighbor in Z'. If $K_{C_T} \in \mathcal{T}_2 \setminus \mathcal{P}$, then there are two neighbors of C_T in Z' but these neighbors are adjacent. So there is at most one connecting $B^{(4)}$-component of $G^*[Z']$. If $K_{C_T} \in \mathcal{P}$, then we just refer to Claim 2 noting that $B^{(4)} \subseteq B^{(3)}$ and no changes happened in Z'. Finally, if $K_{C_T} \in \mathcal{T}_3 \setminus \mathcal{D}$, then $C_T = V(K_{C_T}) \setminus \{v_{K_{C_T}}\}$. The neighbors of the two vertices of C_T are adjacent in Z, thus C_T connects to at most one $B^{(4)}$-component of $G^*[Z']$. □

In the very last modification we set the final status of the vertices from X'. Thus the status of vertices in Z' do not change.

Modification 4: Iteratively, set the status of each vertex from X' to the opposite of the status of its unique neighbor in $G^*[Z]$ if it exists. Otherwise set its status to B. Define I (B) to be the set of vertices with status I (B).

Now we are ready to finish the proof of Theorem 1.

I is an independent set since it is the subset of $I^{(4)} \cup X'$, which is independent by Claim 3 (cf. Lemma 3(i)).

Let C_T be a B-component of $G^*[T']$, where $T' = V(G) \setminus Z'$. Then $C_T = C' \cup (C_T \cap X')$, where C' is a $B^{(4)}$-component of $G^*[T']$. By Modification 4, C_T is not connected to a B-component of $G^*[Z']$ via a vertex from $C_T \cap X'$ and by Claim 3 we know that C_T is connected to at most one B-component of $G^*[Z']$ through some vertex of C'.

Let C be an arbitrary B-component of G^* and denote $C_Z = C \cap Z'$. By the above, C_Z is connected, i.e., it is a B-component of $G^*[Z']$.

If $C_Z = \emptyset$, then, since $B \subseteq B^{(4)} \cup X'$, by Claim 3 (cf. Lemma 3(i)) C is of order at most 21.

Assume now that $C_Z \neq \emptyset$. By Claim 2, C_Z has at most 30 vertices. A vertex v of C_Z can connect to at most three B-components of $G^*[T']$. Claim 3 (cf. Lemma 3(ii)) then implies that no attaching B-component C_T of $G^*[T']$ is larger than 8.

We can immediately conclude that $|C| \leq |C_T|(3 \cdot |C_Z|) + |C_Z| \leq 8(3 * 30 + 30) = 750$.

3 Relaxed coloring with more than two colors

Recall the definition of $\Delta(k, \ell)$: it is the smallest integer Δ such that the family of graphs with maximum degree Δ is *not* (k, ℓ)-colorable.

Theorem 4. *Let $\ell > 0$. For any constant C there exists a graph of maximum degree $\Delta = 2(k + 2\ell - 1)$ which is not C-relaxed (k, ℓ)-colorable. That is $\Delta(k, \ell) \leq 2k + 4\ell - 2$.*

Proof. Erdős and Sachs [8] proved the existence of a $(k + 2\ell)$-regular graph G_C with girth $C + 1$, for an arbitrary integer C. Our construction is the line graph H of $G = G_C$. Denote by n the number of vertices of G and e the number of edges of G.

Obviously H is $2(k+2\ell-1)$-regular and has $e = \frac{(k+2\ell)n}{2}$ vertices. Suppose we have a C-relaxed (k, l)-coloring of H, and $V_1, \ldots, V_k, V_{k+1}, \ldots, V_{k+\ell}$ are the appropriate color-classes. Then either

(i) $\exists i \in \{1, \ldots, \ell\}$ with $|V_{k+i}| \geq n$, or

(ii) $\exists j \in \{1, \ldots, k\}$ with $|V_j| \geq \lfloor \frac{n}{2} \rfloor + 1$.

Case (i). The set V_{k+i} corresponds to n edges in G. G has n vertices, so some of these edges form a cycle K in G, whose length is at least $C + 1$. The vertices of H corresponding to these edges in K also form a cycle of the same length. In particular they induce a component of H with order at least $C + 1$.

Case (ii). The set V_j corresponds to $\lfloor \frac{n}{2} \rfloor + 1$ edges in G. That is two of these edges will share an endpoint. The two vertices corresponding to these two edges are adjacent in H, a contradiction to the independence of V_j. □

Theorem 2 is a special case of Theorem 4 with $k = \ell = 1$. The construction of Alon, Ding, Oporowski and Vertigan [4] is a special case with $k = 0, \ell = 2$.

In [9] the following theorem has been proved:

Theorem 5. *[9, Theorem 3.3] There exists a constant C such that the following holds. Given a graph of maximum degree $\Delta \geq 3$, it is possible to $\lceil (\Delta + 1)/3 \rceil$-partition the vertex set such that each part induces components of size at most C.*

This statement has an immediate implication for (k, ℓ)-colorings.

Corollary 1. *Let k, ℓ be nonnegative integers. The family of graphs of maximum degree at most $k + 3\ell - 1$ is (k, ℓ)-colorable. That is $\Delta(k, \ell) > k + 3\ell - 1$.*

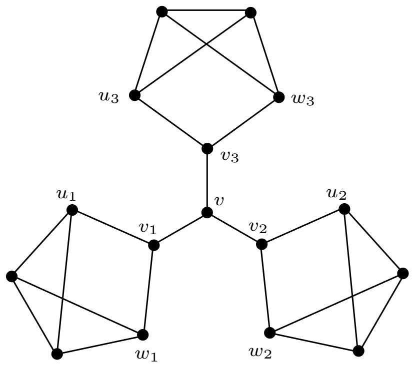

Figure 7: The triangle-graph of H

Proof. First suppose that $\ell > 1$. By a lemma from [12] one can partition the vertex set of G into $k+1$ classes $V_0 \cup V_1 \cup \ldots \cup V_k = V(G)$ such that $\Delta(G[V_i]) = 0$, for all $i \in [k]$ and $\Delta(G[V_0]) \leq \Delta - k$. Then we apply Theorem 5 to ℓ-partition V_0

Next we consider the case when $\ell = 1$. Again we apply the same lemma from [12] to partition the vertex set into k classes $V_1 \cup \ldots \cup V_k = V(G)$ such that $\Delta(G[V_i]) = 0$, for all $i \in \{2..k\}$ and $\Delta(G[V_1]) \leq 3$. Then we apply Theorem 1 to $(1, 1)$-color V_1. □

4 Remarks and Open Problems

4.1 Remarks

1. A more detailed analysis of the structure of B-components in $G[Z]$ yields a bound on the order of components in $G[B]$ of at most 189. A proof of this result can be found in [5]. Further modifications of the (I, B)-partition yield a bound of 96 on the order of B-components. For a sketch of the argument we refer again to [5].

2. The bound on the component order in Theorem 1 (and even the 96 mentioned above) is very likely far from optimal. The best we can show is a lower bound of 6 on the component order. Consider the graph H whose triangle-graph $T(H)$ is equal to the graph G in Figure 7. The removal of any independent set of H leaves at least one component of order at least 6. We omit the proof.

4.2 Open Problems

One would like to know more about the behavior of the function $\Delta(k, \ell)$ in general, or at least tighten the existing asymptotic gap. In the following, we discuss the most intriguing special cases.

Maximum degree condition for $(0, \ell)$-colorability. As we mentioned in the introduction the main theorem of [9] states that $\Delta(0, 2) = 6$. The value of $\Delta(0, 3)$ is not known and is certainly worth determining. It is known to be either 9 or 10 (see [9]). In other words, one has to decide whether there is a constant C such that it is possible to color the vertex set of any graph with maximum degree 9 by three colors such that every monochromatic component is bounded by C? Also in [9] it is shown that there exists $\delta > 0$ such that for large ℓ, $3+\delta < \Delta(0,\ell)/\ell < 4$. It would be of great interest to determine asymptotically $\Delta(0,\ell)$.

Maximum degree condition for $(k,1)$-colorability. Our main result in this paper states that $\Delta(1,1) = 4$. By the results of the last section the value of $\Delta(2,1)$ is either 5 or 6. Asymptotically, $\Delta(k,1)$ is between k and $2k$. We conjecture the lower bounds are (closer to) the truth.

How big components are needed? Of course the number 189 (or even 96 as noted in the previous remark) is an artifact of our proof and not the best possible result. It would be nice to determine the smallest possible component order we can have in Theorem 1 instead of 189. Our guess is that cubic graphs should be C-relaxed $(1,1)$-colorable with a one digit number C.

By a detailed analysis of the method of [9] one could prove that 5-regular graphs are C-relaxed $(0,2)$-colorable with $C = 17617$, but that is definitely far from the truth. The determination of the smallest possible C would be of interest but might be out of reach. Not so for 4-regular graphs; there the required maximum component size is between 4 and 6, it could very well be feasible to determine the smallest constant C such that every 4-regular graph is C-relaxed $(0,2)$-colorable.

Density version. Finally, let us generalize here a problem raised in [9]. A natural way to weaken the maximum degree condition is by rather bounding the maximum average degree of the graph, which allows a few very large degree vertices.

Let $\mu(G) = \max\{2|E(G[W])|/|W| : W \subseteq V(G)\}$. For non-negative integers k,ℓ what is the supremum value $\alpha(k,\ell)$ such that every graph G with $\mu(G) < \alpha(k,l)$ has a C-relaxed $(k+\ell)$-coloring with some constant C. Obviously $\alpha(k,\ell) \leq \Delta(k,\ell)$. In [9] the determination of $\alpha(0,2)$ was raised. The *wheel* graph shows that $\alpha(0,2) \leq 4$, while Kostochka [11] proved a lower bound of 3. The greedy coloring implies that $\alpha(k,0) = k$, for any k. We would be very much interested interested in the value of $\alpha(1,1)$.

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References

[1] J. Akiyama, G. Exoo, F. Harary, Coverings and packings in graphs II. Cyclic and acyclic invariants. *Math. Slovaca* 30 (1980), 405–417.

[2] J. Akiyama, V. Chvátal, A short proof of the linear arboricity for cubic graphs, *Bull. Liber. Arts Sci. NMS* 2 (1981).

[3] R. Aldred and N. Wormald, More on the linear k-arboricity of regular graphs, *Australas. J. Combin* 18(1998), 97–104.

[4] N. Alon, G. Ding, B. Oporowski, D. Vertigan, Partitioning into graphs with only small components, *Journal of Combinatorial Theory, (Series B)*, **87**(2003) 231-243.

[5] R. Berke, T. Szabó, Relaxed two-colorings of cubic graphs, *http://www.inf.ethz.ch/personal/berker/relcol.pdf.*

[6] J. Bermond, J. Fouquet, M. Habib, B. Péroche, On linear k-arboricity, *Discrete Math.* 52 (1984), 123–132.

[7] G. Ding, B. Oporowski, D. Sanders, D. Vertigan, preprint.

[8] P. Erdős, H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.Natur. Reihe* **12**(1963) 251–257.

[9] P.Haxell, T. Szabó, G. Tardos, Bounded size components — partitions and transversals, *Journal of Combinatorial Theory (Series B)*, **88** no.2. (2003) 281-297.

[10] B. Jackson, N. Wormald, On the linear k-arboricity of cubic graphs, *Discrete Math.* 162 (1996), 293–297.

[11] A. Kostochka, *personal communication.*

[12] L. Lovász, On decomposition of graphs, *Stud. Sci. Math. Hung* 1 (1966), 237–238.

[13] C. Thomassen, Two-colouring the edges of a cubic graph such that each monochromatic component is a path of length at most 5, *J. Comb. Th. Series B* 75 (1999), 100–109.