Positional games on random graphs

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Abstract

We introduce and study Maker/Breaker-type positional games on random graphs. Our main concern is to determine the threshold probability $p_{\mathcal{F}}$ for the existence of Maker's strategy to claim a member of $\mathcal{F}$ in the unbiased game played on the edges of random graph $G(n,p)$, for various target families $\mathcal{F}$ of winning sets. More generally, for each probability above this threshold we study the smallest bias b such that Maker wins the $(1\colon b)$ biased game. We investigate these functions for a number of basic games, like the connectivity game, the perfect matching game, the clique game and the Hamiltonian cycle game.

1 Introduction

(Un)biased positional games. Let X be a finite nonempty set and $\mathcal{F} \subseteq 2^X$. The pair $(X, \mathcal{F})$ is a *positional game* on X. The game is played by two players Maker and Breaker, where in each move Maker claims one previously unclaimed element of X and then Breaker claims one previously unclaimed element of X. Maker wins if he claims all the elements of some set in $\mathcal{F}$, otherwise Breaker wins. The set X will be referred to as the board, and the set $\mathcal{F}$ as the set of winning sets. Whenever there is no confusion about what the board is, we may refer to the game $(X, \mathcal{F})$ as just $\mathcal{F}$.

Unless otherwise stated, we assume that Maker starts the game. We note, however, that the asymptotic statements discussed in the paper are not influenced by which player makes the first move. For technical reasons we still have to talk about games in which Breaker starts. So in order to avoid confusion, the positional game with board X and set of winning sets $\mathcal{F}$ in which Breaker makes the first move is denoted by $(\widehat{X}, \mathcal{F})$.

The set of all positional games could be partitioned into two classes. The game $(X, \mathcal{F})$ is called a *Maker's win* if Maker has a winning strategy, that is, playing against an arbitrary strategy Maker can occupy a member of $\mathcal{F}$. Clearly, if $(X, \mathcal{F})$ is *not* a Maker's win, then Breaker is able to prevent any opponent from occupying a winning set. Such a positional game is called a *Breaker's win.*

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Typical, well-studied examples of such positional games are played on the edges of a complete graph, i.e. $X = E(K_n)$. Maker's goal usually is to build a graph theoretic structure – like a spanning tree, a perfect matching, a Hamiltonian cycle, or a clique of fixed size. It turns out that all these games are won easily by Maker if n is sufficiently big, so in order to make things more fair (if such thing exists; actually no game of perfect information is *fair* as the winner—in theory—is known in the beginning of the game) one could give Breaker extra power by allowing him to claim more than 1 edge in each move.

If X is a finite nonempty set, $\mathcal{F} \subseteq 2^X$ and a, b are positive integers (possibly functions of the board size), then the 4-tuple $(X, \mathcal{F}, a, b)$ is a *biased* $(a\colon b)$ *game*. In a biased $(a\colon b)$ game, Maker claims a elements (instead of 1) and Breaker claims b elements (instead of 1) in each move. Recall, that unless otherwise stated Maker starts the game. The biased game in which Breaker starts is denoted by $(\widehat{X}, \mathcal{F}, a, b)$. Note that a is always the bias of Maker, independently from who is the first player to move.

For a family $\mathcal{F}$ the smallest integer $b_{\mathcal{F}}$ is sought (and sometimes found; see [1, 2, 3, 4, 5, 8]) for which Breaker wins the $(1 : b_{\mathcal{F}})$ game.

In the *connectivity game* Maker's goal is to build a connected spanning subgraph; i.e. in this game the family of winning sets is the family $\mathcal{T} = \mathcal{T}_n$ of all spanning trees on n vertices. Chvátal and Erdős proved [8] that $b_{\mathcal{T}} = \Theta(\frac{n}{\log n})$.

Beck [1] established $b_{\mathcal{H}} = \Theta(\frac{n}{\log n})$, where $\mathcal{H} = \mathcal{H}_n$ is the family of all Hamiltonian cycles on n vertices.

For the family $\mathcal{K}_k = \mathcal{K}_{k,n}$ of all k-cliques on n vertices, Bednarska and Łuczak [4] showed that $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$. More generally, they proved that in the game in which Maker's goal is to claim an arbitrary fixed graph G, the threshold bias is $\Theta(n^{1/m'(G)})$. (Here $m'(G)$ is the maximum of $\frac{e(H)-1}{v(H)-2}$ over all subgraphs H of G with at least 3 vertices.)

Playing on a random board. In the present paper we introduce another approach to even out the advantage Maker has in a $(1\colon 1)$ game, by randomly reducing the board size and keeping only those winning sets which survive this thinning intact.

Definition 1 *Let* $(X, \mathcal{F}, a, b)$ *be a biased game.* Random game $(X_p, \mathcal{F}_p, a, b)$ *is a probability space of games where each* $x \in X$ *is independently included in* X_p *with probability* p*, and* $\mathcal{F}_p = \{W \in \mathcal{F} : W \subseteq X_p\}$.

Apart from the trivial case $\emptyset \in \mathcal{F}$, Breaker surely wins when $p = 0$. On the other hand, the unbiased version of all the graph games that we consider are (easy) Maker's wins, when $p = 1$ and the board is sufficiently large. For any other probability p, $0 < p < 1$, we cannot be sure who (Maker or Breaker) wins the random game $\mathcal{F}_p$. The best we can conclude is that Maker (or Breaker) wins a.s. (almost surely), i.e. the probability that Maker (Breaker) wins tends to 1 if the board size tends to infinity. (So we actually talk about an infinite family of probability spaces of games . . .)

Let $(X, \mathcal{F})$ be a particular sequence of games, where $\emptyset \notin \mathcal{F}$, the board size tends to infinity, and $(X, \mathcal{F}, 1, 1)$ is won by Maker provided $|X|$ is big enough.

The first natural question to ask is: What is the threshold probability $p_{\mathcal{F}}$ at which an almost sure Breaker's win turns into an almost sure Maker's win. More precisely we would like to determine $p_{\mathcal{F}}$ for which

- $\Pr[(X_p, \mathcal{F}_p, 1, 1)$ is a Breaker's win$] \to 1$ for $p = o(p_{\mathcal{F}})$, and

- $\Pr[(X_p, \mathcal{F}_p, 1, 1)$ is a Maker's win$] \to 1$ for $p = \omega(p_{\mathcal{F}})$.

Such a threshold $p_{\mathcal{F}}$ exists [6], since being a Maker's win is an *increasing property.*

The main goal of this paper is to establish a connection between the natural threshold values, $b_{\mathcal{F}}$ and $p_{\mathcal{F}}$, corresponding to the two different weakenings of Maker's power: bias and random thinning, respectively. We find that there is an intriguing reciprocal connection between these two thresholds in a number of well-studied games on graphs.

Recall the notations $\mathcal{T}$, $\mathcal{H}$, and $\mathcal{K}_k$, and let us denote by $\mathcal{M}$ the set of all perfect matchings on the graph K_n.

Theorem 1 *For positional games, played on $E(K_n)$, we have*

(i) $p_{\mathcal{T}} = \frac{\log n}{n}$,

(ii) $p_{\mathcal{M}} = \frac{\log n}{n}$,

(iii) $\frac{\log n}{n} \leq p_{\mathcal{H}} \leq \frac{\log n}{\sqrt{n}}$,

(iv) $n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{K}_k} \leq n^{-\frac{2}{k+1}}$, *for every integer $k \geq 4$ and every constant $\varepsilon > 0$.*

(v) $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$.

For the connectivity game $\mathcal{T}$ an even more precise statement is true. In Corollary 20 we observe that Maker starts to win a.s. at the very moment when the last vertex of a random graph process picks up its second incident edge.

More generally, for every p we would like to find the smallest bias $b^p_{\mathcal{F}}$ such that Breaker wins the random game $(X_p, \mathcal{F}_p, 1, b^p_{\mathcal{F}})$ a.s. Note that by definition $b_{\mathcal{F}} = b^1_{\mathcal{F}}$. Another trivial observation is that $b^p_{\mathcal{F}} = 0$ provided p is less than the threshold for the appearance of the first element of $\mathcal{F}$ in the random graph.

We obtain the following.

Theorem 2 *There exist constants C_1, C_2, C_3, such that*

(i) $b^p_{\mathcal{T}} = \Theta(p b_{\mathcal{T}}) = \Theta\left(p \frac{n}{\log n}\right)$, *provided* $p \geq C_1 \frac{1}{b_{\mathcal{T}}}$,

(ii) $b^p_{\mathcal{M}} = \Theta(p b_{\mathcal{M}}) = \Theta\left(p \frac{n}{\log n}\right)$, *provided* $p \geq C_2 \frac{1}{b_{\mathcal{M}}}$,

(iii) $\Omega\left(p\frac{\sqrt{n}}{\log n}\right) \le b^p_{\mathcal{H}} \le O\left(p\frac{n}{\log n}\right)$, *provided* $p \ge C_3\frac{\log n}{\sqrt{n}}$,

(iv) There exists $c_k > 0$, *such that* $b^p_{\mathcal{K}_k} = \Theta\left(pb_{\mathcal{K}_k}\right) = \Theta\left(pn^{\frac{2}{k+1}}\right)$, *provided* $p = \Omega\left(\frac{\log^{c_k} n}{b_{\mathcal{K}_k}}\right)$.

One can see that $b^p_{\mathcal{F}}$ is of order $p/p_{\mathcal{F}} = pb_{\mathcal{F}}$ for the connectivity game and the perfect matching game, provided $p \ge Cp_{\mathcal{F}}$ for some constant C. In particular for these games $p_{\mathcal{F}} = \Theta(1/b_{\mathcal{F}})$. In part (iv) of Theorem 2, generalizing the arguments of Bednarska and Łuczak [4] we show that one can estimate $b^p_{\mathcal{K}_k}$ up to a constant factor, for all probabilities down to a polylogarithmic factor away from the critical probability $1/b_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$. On the other hand Theorem 1 part (v) shows that in the case $k = 3$ we cannot get arbitrarily close to probability $1/b_{\mathcal{K}_k}$, since Maker *can win* even for probabilities below $1/b_{\mathcal{K}_3} = n^{-1/2}$.

Nevertheless we think the Hamiltonian cycle game behaves "nicely", i.e. the same way as the connectivity game and the perfect matching game.

Conjecture 1 *Let $\mathcal{H}$ be the set of Hamiltonian cycles in K_n. There exists a constant C such that*

$$b^p_{\mathcal{H}} = \Theta\left(p\frac{n}{\log n}\right), \text{ provided } p \ge C\frac{\log n}{n}.$$

In particular,

$$p_{\mathcal{H}} = \frac{\log n}{n}.$$

Observe that the validity of the conjecture would mean that in a random graph with edge probability $p \ge C\frac{\log n}{n}$ Maker could build a Hamiltonian cycle. So Pósa's Theorem (which only proves the existence of a Hamiltonian cycle) would be true constructively even if an adversary is playing against us.

The paper is organized as follows. In Section 2 we prove a general criterion for Breaker's win in a different, auxiliary random game. In Section 3, the analysis of four biased random games is presented. In particular, in Subsections 3.1, 3.2, 3.3 and 3.4 we look at the connectivity game, the Hamiltonian cycle game, the perfect matching game and the clique game, respectively. In Section 4 we analyze more precisely a couple of (1: 1) games – the connectivity game (Subsection 4.1) and the clique game (Subsection 4.2). Finally, in Section 5 we give a collection of open questions and conjectures.

Notation. For a graph G, $e(G)$ and $v(G)$ denote the number of edges and vertices (respectively) of G, $\delta(G)$ denotes the minimum degree of G, and $E(G)$ and $V(G)$ denote the sets of edges and vertices (respectively). If $C \subseteq V(G)$ and $v \in V(G)$, then $N_C(v)$ denotes the set of neighbors of v in C. The logarithm $\log n$ in this paper is always of natural base. For functions $f(n), g(n) \ge 0$, we say that $f = O(g)$ if there are constants C and K, such that $f(n) \le Cg(n)$ for $n \ge K$; $f = \Omega(g)$ if $g = O(f)$; $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$; $f = o(g)$ if $f(n)/g(n) \to 0$ when $n \to \infty$; $f = \omega(g)$ if $g = o(f)$.

2 A criterion

One of few general, but still very applicable results to decide the winner of biased positional games is the biased version of the Erdős–Selfridge Theorem [10, 2]. It provides a criterion for Breaker to win, applicable on any game.

Theorem 3 *(Beck, [2]) If*

$$\sum_{A\in\mathcal{F}}(1+b)^{-|A|/a}<1,$$

then Breaker has a winning strategy in the $(\widehat{X},\mathcal{F},a,b)$ game.

If Maker plays the first move then the 1 on the right hand side of the criterion is to be replaced by the fraction $\frac{1}{1+b}$.

We will also need the following extension.

Theorem 4 *([2, 4]) If for a positive integer c we have*

$$\sum_{A\in\mathcal{F}}(1+b)^{-|A|/a}<c\frac{1}{1+b},$$

then Breaker has a winning strategy in the $(X,\{\cup_{B\in F}B : F\in\binom{\mathcal{F}}{c}\},a,b)$ game.

In this section we give an adaptation of the first criterion which proves to be very useful in dealing with positional games on a random board. We need the following technical definition.

Definition 2 *Let $(X,\mathcal{F},a,b)$ be a biased game.* Random game $(X_p,\mathcal{F}_p^{\cap},a,b)$ with induced set of winning sets *is a probability space of games, where X_p is defined as in Definition 1 and $\mathcal{F}_p^{\cap}=\{W : \exists F\in\mathcal{F},\ W=F\cap X_p\}$.*

The following statement is the randomized version of Theorem 3. It is stated for the biased $(b\colon 1)$ game in which Breaker is the first player, because this is the version we will need in our applications.

Theorem 5 *Let $\mathcal{F}$ be a set of winning sets on X with*

$$\sum_{A\in\mathcal{F}}2^{-\frac{|A|}{b}}<1 \tag{1}$$

(i.e. the condition of the Erdős–Selfridge Theorem holds for the $(\widehat{X},\mathcal{F},b,1)$ game), and

$$\lim_{n\to\infty}\min_{A\in\mathcal{F}}\frac{|A|}{b}=\infty. \tag{2}$$

If p and $\delta>0$ are chosen so that $p>\frac{4\log 2}{\delta^2 b}$ holds, then the game $(\widehat{X}_p,\mathcal{F}_p^{\cap},(1-\delta)pb,1)$ is a Breaker's win a.s.

Proof. For each $A \in \mathcal{F}$ and its corresponding set $A' \in \mathcal{F}_p^{\cap}$ we have $\mathbf{E}[|A'|] = p|A|$. If all winning sets $A' \in \mathcal{F}_p^{\cap}$ have size at least $(1-\delta)p|A|$, then

$$\sum_{A' \in \mathcal{F}_p^{\cap}} 2^{-\frac{|A'|}{(1-\delta)pb}} \leq \sum_{A \in \mathcal{F}} 2^{-\frac{(1-\delta)p|A|}{(1-\delta)pb}} = \sum_{A \in \mathcal{F}} 2^{-\frac{|A|}{b}} < 1.$$

Using the Erdős–Selfridge theorem we obtain that Breaker wins the $(\widehat{X}_p, \mathcal{F}_p^{\cap}, (1-\delta)pb, 1)$ game, provided $|A'| \geq (1-\delta)p|A|$ for all $A' \in \mathcal{F}_p^{\cap}$.

Next we check that this condition holds almost surely. Using a Chernoff bound, we obtain that

$$\Pr\left[\exists A \in \mathcal{F} : \, |A'| \leq (1-\delta)p|A|\right] \leq \sum_{A \in \mathcal{F}} e^{-\frac{\delta^2 p|A|}{2}}.$$

If we denote $\min_{A \in \mathcal{F}} \frac{|A|}{b}$ by m_n, then we have

$$\sum_{A \in \mathcal{F}} e^{-\frac{\delta^2 p|A|}{2}} \leq \sum_{A \in \mathcal{F}} 2^{-2\frac{|A|}{b}} \leq \sum_{A \in \mathcal{F}} 2^{-m_n} 2^{-\frac{|A|}{b}} < 2^{-m_n} \to 0,$$

and therefore all winning sets $A' \in \mathcal{F}_p^{\cap}$ have size at least $(1-\delta)p|A|$ a.s.

□

3 Games

3.1 Connectivity game

The first game we study is a random version of the biased connectivity game $(E(K_n), \mathcal{T}, 1, b)$ on a complete graph on n vertices K_n. Maker's goal is to build a spanning, connected subgraph, i.e. $\mathcal{T}$ is the set of all spanning trees on n vertices.

It is obvious that $p_{\mathcal{T}} = \Omega(\frac{\log n}{n})$, since for lower probabilities the random graph is a.s. not connected, and Breaker wins even if he does not claim any edges.

First we generalize this for arbitrary probability p by providing Breaker with a strategy to isolate a vertex. One of our main tools is the following winning criterion of Chvátal and Erdős on games with disjoint winning sets.

Theorem 6 *[8] In a biased $(b\colon 1)$ game with k disjoint winning sets of size s Maker wins if*

$$s \leq (b-1) \sum_{i=1}^{k-1} \frac{1}{i}. \tag{3}$$

Corollary 7 *In a biased $(b\colon 2)$ game with $k+1$ disjoint winning sets of size at most s Maker wins if*

$$s \leq \left(\left\lfloor \frac{b}{2} \right\rfloor - 1 \right) \sum_{i=1}^{k-1} \frac{1}{i}.$$

Proof of Corollary. Recall that as a default Maker starts the game in the Theorem and the Corollary as well. Now Theorem 6 obviously remains true (i.e. Maker wins) even if Breaker starts, provided there are $k+1$ disjoint winning sets instead of k. This implies that when Breaker starts, the bias is $(2b:2)$, there are $k+1$ winning sets and (3) holds, then Maker still wins. Indeed, since the winning sets are disjoint, after Breaker's move Maker can just pretend to play a $(b:1)$ game and answer with his first b moves to one of the two selections of Breaker, and answer with his second b moves to the other move of Breaker, both according to the $(b:1)$ strategy. Now the Corollary follows, since starting instead of being second player cannot hurt Maker. □

Theorem 8 *There exists $K_0 > 0$ so that for arbitrary $p \in [0,1]$ and $b \geq K_0 p \frac{n}{\log n}$ Breaker, playing the $(1:b)$ game on the edges of random graph $G(n,p)$, can achieve that Maker's graph has an isolated vertex a.s.*

Proof. Let us fix $b = \lfloor K_0 pn/\log n \rfloor$, where K_0 is a constant to be determined later. Note that we can assume $p > \log n/2n$, since otherwise the random graph does have an isolated vertex a.s., thus Breaker achieves his goal without having to play any moves.

We present a strategy for Breaker to claim all the edges incident to some vertex of $G(n,p)$. If successful, this strategy prevents Maker from building a connected subgraph. Similar strategy was introduced by Chvátal and Erdős [8] for solving the problem on the complete graph.

Let C be an arbitrary subset of the vertex set of cardinality $\lfloor n/\log n \rfloor$. Breaker will claim all the edges incident to some vertex $v \in C$ (thus preventing Maker from claiming any edge incident to v). We would like to use the game from Corollary 7, with the winning sets being the $\lfloor n/\log n \rfloor$ stars of size at most $n-1$ whose center is in C. Since these stars are not necessarily disjoint, formally we will talk about ordered pairs of vertices: the winning sets are denoted by $W_v = \{(v,u) : u \in V\}$, $v \in C$. We call this game *Box.* To avoid confusion with Maker and Breaker of the game from Theorem 8, the players from Corollary 7 will be called BoxMaker and BoxBreaker. Recall that in Box the bias is $(b:2)$.

Breaker will utilize the strategy of BoxMaker from Corollary 7 to achieve his goal. How? He will play a game of Box in such a way that a win for BoxMaker automatically implies a win for Breaker. When Maker selects an edge uv, Breaker interprets it as BoxBreaker claimed the elements (u,v) and (v,u) in Box. Whenever Breaker would like to make a move, he looks at the current move of BoxMaker in Box, and takes those edges which correspond to the b ordered pairs BoxMaker selected. If he is supposed to select an edge which has already been selected by him, he selects an arbitrary unoccupied edge. Note that the above strategy never calls for Breaker to select an edge which has already been selected by Maker.

It is also obvious, that if BoxMaker wins Box, then Breaker occupied all incident edges of a vertex from C.

In order to apply Corollary 7 it is enough then to show that the size $d(v)$ of each winning set is appropriately bounded from above, i.e. for each $v \in C$ we have $d(v) \leq \frac{K_0}{8} pn \leq \left(\left\lfloor \frac{b}{2} \right\rfloor - 1\right) \sum_{i=1}^{k-1} \frac{1}{i}$ a.s.

Indeed, using a Chernoff bound and a large enough K_0, we obtain that for every $v \in C$

$$\Pr\left[d(v) > \frac{K_0}{8}pn\right] \le e^{-\frac{K_0 pn}{8}} \le n^{-\frac{K_0}{16}}.$$

Therefore we have

$$\Pr\left[\exists v \in C : d(v) > \frac{K_0}{8}pn\right] \le n \cdot n^{-\frac{K_0}{16}} \to 0,$$

provided K_0 is large enough. Then Corollary 7 guarantees BoxMaker's win, thus Breaker's win a.s., and the proof of Theorem 8 is complete. □

Next we give a winning strategy for Maker in the connectivity game, thus determining the threshold bias $b^p_{\mathcal{T}}$ up to a constant factor.

Obviously, Breaker wins if and only if he claims all the edges of a cut, i.e. all the edges connecting some set of vertices with its complement. In order to win Maker has to claim one edge in each of the cuts. This observation enables us to formulate the connectivity game in a different way, where winning sets are cuts and roles of players are exchanged – Breaker wants to occupy a cut and Maker wants to prevent Breaker from doing so. To avoid confusion we refer to the players of this "cut-game" by CutMaker and CutBreaker.

This new point of view enables us to give Maker a winning strategy using Theorem 5, which is a criterion for CutBreaker's win. Observe, that in this "cut-game" CutBreaker (alias Maker) only cares about occupying the existing edges of a cut, that's why we are going to look at the family $\mathcal{F}_p^{\cap}$ instead of $\mathcal{F}_p$.

Theorem 9 *There exists $k_0 > 0$, so that for $p > \frac{32 \log n}{n}$ and $b \le k_0 p \frac{n}{\log n}$ Maker wins the random connectivity game $(E(K_n)_p, \mathcal{T}_p, 1, b)$ a.s.*

Proof. For $b_0 = \frac{\log 2}{2} \cdot \frac{n}{\log n}$ we are going to prove that the conditions of Theorem 5 are satisfied if $\mathcal{F}$ is the set of all cuts in a complete graph with n vertices.

On one hand, Beck [2] showed $\sum_{k=1}^{n/2} \binom{n}{k} 2^{-\frac{k(n-k)}{b_0}} \to 0$, which means that condition (1) holds in this setting.

On the other hand, for a cut $A \in \mathcal{F}$ we have $|A| \ge n-1$ which implies condition (2). If we set $\delta = 1/2$ we can apply Theorem 5 which gives that $(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{4} p \frac{n}{\log n}, 1)$ is a CutBreaker's win a.s. The statement of the theorem immediately follows. □

Theorem 8 and Theorem 9 together imply part (i) of both Theorem 1 and 2.

3.2 Hamiltonian cycle game

Here we investigate the random version of the $(1\!:\!b)$ biased game $(E(K_n), \mathcal{H}, 1, b)$ on the complete graph K_n, where $\mathcal{H}$ is the set of all Hamiltonian cycles. Maker's goal is to occupy all edges of a Hamiltonian cycle, while Breaker wants to prevent that. Breaker can obviously win when Maker is not able to claim a connected graph and thus from Theorem 8 we obtain the following corollary.

Corollary 10 *There exists* $H_0 > 0$ *so that for every* $p \in [0,1]$ *and* $b \geq H_0 p \frac{n}{\log n}$ *Breaker wins the random Hamiltonian cycle game* $(E(K_n)_p, \mathcal{H}_p, 1, b)$ *a.s.*

The next Theorem describes Maker's strategy.

Theorem 11 *There exists* $h_0 > 0$, *so that for* $p > \frac{32 \log n}{\sqrt{n}}$ *and* $b \leq h_0 p \frac{\sqrt{n}}{\log n}$ *Maker wins the random Hamiltonian cycle game* $(E(K_n)_p, \mathcal{H}_p, 1, b)$ *a.s.*

Proof. Maker wins, if at the end of the game the subgraph G_M (containing the edges claimed by Maker) has connectivity $\kappa(G_M)$ greater or equal than independence number $\alpha(G_M)$. Indeed, from the criterion of Chvátal and Erdős for Hamiltonicity [9], we obtain that G_M then contains a Hamiltonian cycle.

We show that Maker, using only his odd moves, can ensure that the connectivity of his graph at the end of the game is greater then $k = \sqrt{n}/2$ and, using his even moves, can make the independence number at the end of the game smaller then $k = \sqrt{n}/2$. In other words we will look at two separate games where in each of them Maker plays one move against Breaker's $2b$ moves. This is a correct strategy, because moves of Maker made in one of these games cannot hurt him in the other.

We first look at the odd Maker's moves. To ensure that $\kappa(G_M) \geq k$, Maker has to claim one edge in every cut of a graph obtained from the initial graph by removing some k vertices. More precisely, we are going to prove the conditions of Theorem 5 for the biased $(b' : 1)$ game, where $b' = \frac{\log 2}{2} \cdot \frac{\sqrt{n}}{\log n}$ and

$$\mathcal{F} = \left\{ \{v_1 v_2 : v_1 \in V_1, v_2 \in V_2\} : V(K_n) = V_0 \dot\cup V_1 \dot\cup V_2,\ |V_0| = k,\ V_1, V_2 \neq \emptyset \right\}.$$

That is, Maker plays the role of "CutBreaker" by trying to break all the cuts in $\mathcal{F}$.

Since the size of each of the sets in $\mathcal{F}$ is at least $n - k - 1$ we have

$$\lim_{n\to\infty} \min_{A \in \mathcal{F}} \frac{|A|}{b'} = \lim_{n\to\infty} \frac{2 \log n (n - \sqrt{n}/2 - 1)}{\log 2 \sqrt{n}} = \infty,$$

and the condition (2) holds. Next, we have

$$\begin{aligned}
\sum_{A \in \mathcal{F}} 2^{-\frac{|A|}{b'}} &= \sum_{i=1}^{\frac{n-k}{2}} \binom{n}{i} \binom{n-i}{k} 2^{-\frac{i(n-i-k)}{b'}} \\
&< \sum_{i=1}^{k} n^{2k} 2^{-\frac{n-k-1}{b'}} + \sum_{i=k+1}^{\frac{n-k}{2}} 2^{2n - \frac{k(n-2k)}{b'}} \\
&< k \cdot n^{-\sqrt{n}} + n \cdot n^{-n} \to 0,
\end{aligned}$$

which gives the condition (1). Therefore, CutBreaker (alias Maker) wins the game $(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{4} p \frac{\sqrt{n}}{\log n}, 1)$ a.s., provided $p \geq \frac{32 \log n}{\sqrt{n}}$.

In the other part of the game using even moves Maker has to ensure that $\alpha(G_M) \leq k = \sqrt{n}/2$. That is going to be true if Maker manages to claim at least one edge in every clique of k elements. To prove that it is possible we again use Theorem 5 for a biased $(b'\colon 1)$ game with the same value of $b' = \frac{\log 2}{2} \cdot \frac{\sqrt{n}}{\log n}$. But now $\mathcal{F}$ is the family of the edgesets of all cliques of size k and Maker will play the role of "CliqueBreaker" in this game.

We have

$$\lim_{n\to\infty} \min_{A\in\mathcal{F}} \frac{|A|}{b'} = \lim_{n\to\infty} \frac{2\log n \binom{\frac{\sqrt{n}}{2}}{2}}{\log 2\sqrt{n}} = \infty,$$

and the condition (2) is satisfied. It remains to prove that the condition (1) holds.

$$\begin{aligned}\sum_{A\in\mathcal{F}} 2^{-\frac{|A|}{b'}} &= \binom{n}{k} 2^{-\frac{\binom{k}{2}}{b'}} < \left(\frac{ne}{k} 2^{-\frac{k-1}{2b'}}\right)^k \\ &< 2^{-\sqrt{n}} \to 0.\end{aligned}$$

Therefore, CliqueBreaker wins the game $(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{4} p \frac{\sqrt{n}}{\log n}, 1)$ a.s., provided $p \geq \frac{32\log n}{\sqrt{n}}$.

Putting the two parts of the game together we have that Maker wins $(E(K_n)_p, \mathcal{H}_p, 1, \frac{1}{16} p \frac{\sqrt{n}}{\log n})$ a.s. $\square$

Combining the statements of Corollary 10 and Theorem 11 we obtain part (iii) of both Theorems 1 and 2.

3.3 Perfect matching game

The upper and lower bounds obtained in the previous subsection for the threshold bias of the random Hamiltonian cycle game are not tight. We firmly believe that our strategy for Maker in that game is not optimal. The game we consider next is simpler for Maker, and for that we are able to obtain bounds optimal up to a constant factor.

Recall that $\mathcal{M}$ is the set of all perfect matchings on K_n. We will assume that n is even. In the game $(E(K_n), \mathcal{M}, 1, b)$ Maker's goal is to occupy all edges of a perfect matching, while Breaker wants to prevent that.

The following theorem provides the winning strategy in the random perfect matching game for Maker.

Theorem 12 *There exists $m_0 > 0$, so that for $p > 64\frac{\log n}{n}$ and $b \leq m_0 p \frac{n}{\log n}$ Maker wins the random perfect matching game $(E(K_n)_p, \mathcal{M}_p, 1, b)$ a.s.*

Proof. We can show that Maker can win in a slightly harder game. More precisely, if the set of vertices of K_n is partitioned into two sets A and B of equal size before the game starts, we are going to show that Maker can claim a perfect matching with edges going only between A and B.

For disjoint sets $X, Y \subset V(K_n)$, we define $E(X,Y)$ to be the set of edges between X and Y. Let $\mathcal{F}$ be a family of sets of edges,

$$\mathcal{F} = \{E(X,Y) : \emptyset \neq X \subset A, \emptyset \neq Y \subset B, |X| + |Y| = \frac{n}{2} + 1\}.$$

Suppose that at the end of the game Maker has not claimed all edges of any perfect matching between A and B. Hall's necessary and sufficient condition for existence of a perfect matching implies that there exist sets $X_0 \subset A$ and $Y_0 \subset B$ such that $|X_0| > |Y_0|$ and all edges in $E(K_n)_p \cap E(X_0, B \setminus Y_0)$ were claimed by Breaker.

Therefore, in order to win, Maker has to claim at least one edge in each of the sets from $\mathcal{F}$, i.e. the game $(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, b, 1)$, which we call *Hall*, should be a HallBreaker's win.

To prove that HallBreaker wins we are going to use Theorem 5. We set $\delta = 1/2$ and $b_0 = \frac{\log 2}{4} \cdot \frac{n}{\log n}$.

First we show that condition (1) holds. We have

$$\begin{aligned}
\sum_{k=1}^{n/2} \binom{n/2}{k} \binom{n/2}{n/2-k+1} 2^{-\frac{k(n/2-k+1)}{b_0}} &< 2 \sum_{k=1}^{\lfloor n/4 \rfloor} \binom{n/2}{k}^2 2^{-\frac{k(n/2-k+1)}{b_0}} \\
&< 2 \sum_{k=1}^{\lfloor n/4 \rfloor} \left(e^{2\log(n/2) - 2\log n}\right)^k \\
&= 2 \sum_{k=1}^{\lfloor n/4 \rfloor} \left(\frac{1}{4}\right)^k < 1.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \min_{A\in\mathcal{F}} \frac{|A|}{b_0} > \lim_{n\to\infty} \log n = \infty,$$

the condition (2) is also satisfied and we can apply Theorem 5 proving that HallBreaker wins the random game $(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{8} p \frac{n}{\log n}, 1)$ a.s., provided $p > 64 \log n/n$.

This immediately implies that Maker wins $(E(K_n)_p, \mathcal{M}_p, 1, b)$ a.s. $\square$

Theorem 8 ensures a win for Breaker in the perfect matching game, if $b > K_0 pn/\log n$. This, together with the above Theorem 12 proves part (*ii*) of Theorems 1 and 2.

3.4 Clique game

Here we look at the random version of the $(1\colon b)$ biased clique game $(E(K_n), \mathcal{K}_k, 1, b)$ on a complete graph K_n, where $\mathcal{K}_k$ is the set of all cliques of constant size k. Maker's goal is to occupy all edges of a clique of size k while Breaker wants to prevent that.

The deterministic clique game was extensively studied by Bednarska and Łuczak in [4]. They proved a more general result by determining the order of the threshold bias for the whole family of games in which Maker's goal is to claim an arbitrary fixed graph H. In this section, we will largely rely on the constructions and ideas from their paper.

If $\{F_1, \dots, F_t\}$ is a family of k-cliques having two common vertices, and $e_i \in E(F_i)$, $i = 1, \dots, t$ are distinct edges, then we call the graph $\cup_{i=1}^t F_i$ a *t-2-cluster* and the graph $\cup_{i=1}^t (F_i - e_i)$ a *t-fan*. If furthermore the k-cliques have three vertices in common, then a t-2-cluster is called a *t-3-cluster* and a

t-fan is called a *t-flower*. A t-fan or a t-2-cluster is said to be *simple*, if the pairwise intersections (of any two k-cliques) have size exactly 2.

In order to prevent Maker to occupy a clique K_k, Breaker will play two auxiliary games. In the first one he prevents Maker from occupying a 3-cluster of constant size.

Lemma 13 *There exists $t = t(k)$, so that for $\varepsilon = \frac{1}{2(k+2)}$, $p = \omega(n^{-\frac{2}{k+1}})$ and $b > pn^{\frac{2(1-\varepsilon)}{k+1}}$ Breaker wins the game $(E(K_n)_p, t\text{-3-clusters}, 1, b)$ a.s.*

Proof. To apply Theorem 3, it is enough to check that there exists t such that for the random variable

$$Y := \sum_{t\text{-3-cluster } C \text{ in } G(n,p)} (1+b)^{-e(C)},$$

$Y < \frac{1}{b+1}$ holds a.s.

We have

$$\mathbf{E}[Y] = \sum_{t\text{-3-cluster } C \text{ in } K_n} \left(\frac{p}{1+b}\right)^{e(C)}.$$

Let $b_1 = \frac{b+1}{p} - 1$. In [4], it is shown that there exists t for which

$$\sum_{t\text{-3-cluster } C \text{ in } K_n} \left(\frac{p}{1+b}\right)^{e(C)} \leq K_0 \frac{1}{b_1^{1+k_0}},$$

where $k_0, K_0 > 0$ are constants depending on k. This implies $\mathbf{E}[Y] = o\left(\frac{1}{b+1}\right)$, and by Markov inequality we get that $Y < \frac{1}{b+1}$ a.s. □

During a game, a t-fan (or t-flower) is said to be *dangerous* if all the t edges missing from the cliques that make up the t-fan are present in the graph on which the game is played, but not yet claimed by any of the players. Note that if at any moment of the game $(E(K_n)_p, t\text{-3-clusters}, 1, b)$ Maker claimed a dangerous $(b+1)t$-flower, then he could win since he could claim a t-3-cluster in his next t moves by simply claiming missing edges, one by one. Hence, Lemma 13 implies the following.

Corollary 14 *There exists $t = t(k)$ so that for $\varepsilon = \frac{1}{2(k+2)}$ and $p = \omega(n^{-\frac{2}{k+1}})$, Breaker playing a $(1{:}\,pn^{\frac{2(1-\varepsilon)}{k+1}})$ game on edges of random graph $E(K_n)_p$ can make sure that Maker does not claim a dangerous $\left(pn^{\frac{2(1-\varepsilon)}{k+1}}t\right)$-flower at any moment of the game.*

Next we deal with the second auxiliary game of Breaker; in this game he prevents the appearance of too many simple b^{ε}-fans.

Lemma 15 *There exists $C_0 > 0$, such that for $\varepsilon_1 = \frac{1}{6(k+2)}$, $p \geq n^{-\frac{2}{k+1}} \log^{1/\varepsilon_1} n$, $b > C_0 pn^{\frac{2}{k+1}}$ and $s = b^{\varepsilon_1}$ Breaker wins the game $(E(K_n)_p$, unions of $\frac{1}{2}\binom{b}{s}$ simple s-fans, $1, b/2)$ a.s.*

Proof. Let $c_s(n)$ be the number of simple s-2-clusters contained in K_n, and let X_s be the random variable counting the number of simple s-2-clusters contained in $G(n,p)$. Using the first moment method we get

$$\Pr[X_s \geq \mathbf{E}[X_s]\log n] \leq \frac{1}{\log n} \longrightarrow 0,$$

and using this, a.s. we have that

$$\begin{aligned}
&\sum_{\substack{\text{dangerous simple}\\ s\text{-fan } C \text{ in } G(n,p)}} (1+b/2)^{-e(C)} \\
\leq\;& \sum_{\substack{\text{simple } s\text{-2-cluster } K\\ \text{in } G(n,p)}} \binom{k}{2}^s (1+b/2)^{-s(\binom{k}{2}-2)-1} \\
\leq\;& \binom{k}{2}^s \log n \cdot c_s(n) p^{s(\binom{k}{2}-1)+1} 2^{sk^2} b^{-s(\binom{k}{2}-2)-1} \\
\leq\;& \log n \cdot C_1^s \binom{n}{2} \frac{\binom{n}{k-2}^s}{s!} \left(\frac{p}{b}\right)^{s(\binom{k}{2}-1)+1} b^s \\
\leq\;& n^3 \cdot C_1^s n^{(k-2)s} \left(\frac{1}{C_0 n^{\frac{2}{k+1}}}\right)^{s(k+1)(k-2)/2+1} \frac{b^s}{s!} \\
\leq\;& n^3 \cdot \left(\frac{C_1}{C_0^{\binom{k}{2}-1}}\right)^s \left(\frac{1}{C_0 n^{\frac{2}{k+1}}}\right) \frac{b^s}{s!} < \frac{1}{2}\binom{b}{s}\frac{1}{b+1},
\end{aligned}$$

where $C_1 = C_1(k)$ is a constant. The last inequality is valid since $p \geq n^{-\frac{2}{k+1}} \log^{1/\varepsilon_1} n$, and for C_0 large enough $\left(C_1/C_0^{\binom{k}{2}-1}\right)^s \leq n^{-5}$. This enables us to apply Theorem 4, and the statement of the lemma is proved. □

Now we are ready to state and prove the theorem ensuring Breaker's win in the clique game on the random graph. In the proof, we are going to use this result of Bednarska and Łuczak.

Lemma 16 *[4] For every $0 < \varepsilon < 1$ there exists b_0 so that every graph with $b > b_0$ vertices and at most $b^{2-\varepsilon}$ edges has at least $\frac{1}{2}\binom{b}{b^{\varepsilon/3}}$ independent sets of size $b^{\varepsilon/3}$.*

Theorem 17 *There exists $C_0 > 0$ so that for $p \geq n^{-\frac{2}{k+1}} \log^{6k+12} n$ and $b \geq C_0 p n^{\frac{2}{k+1}}$ Breaker wins the random clique game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, b)$ a.s.*

Proof. Breaker will use $b/2$ of his moves to defend "immediate threats", i.e. to claim the remaining edge in all k-cliques in which Maker occupied all but one edge. In order to be able to do this Breaker must ensure that he never has to block more than $b/2$ immediate threats, that is, there is no dangerous $b/2$-fan.

He will use his other $b/2$ moves to prevent Maker from creating a dangerous $(b/2)$-fan.

From Corollary 14 we get that Breaker can prevent Maker from claiming a dangerous f-flower (where $f = tpn^{\frac{2(1-\varepsilon)}{k+1}}$, $\varepsilon = \frac{1}{2(k+2)}$ and t is a positive constant) using less than $b/4$ edges per move. On the other hand, from Lemma 15 we have that if C_0 is large enough Breaker can prevent Maker from claiming $\frac{1}{2}\binom{b/2}{s}$ simple s-fans using $b/4$ edges per move, where $s = (b/2)^{\varepsilon/3}$.

Suppose that Maker managed to claim a dangerous $(b/2)$-fan. We define an auxiliary graph G' with the vertex set being the set of all $b/2$ k-cliques of this dangerous fan, and two k-cliques being connected with an edge if they have at least 3 vertices in common. Since there is no dangerous f-flower in Maker's graph, the degree of each of the vertices of the graph G' is at most fk and therefore $e(G') < \frac{bfk}{2} \leq \left(\frac{b}{2}\right)^{2-\varepsilon}$. On the other hand, the number of independent sets in G' of size s cannot be more than $\frac{1}{2}\binom{b/2}{s}$, since each of the independent sets in G' corresponds to a simple s-fan in Maker's graph.

Since the last two facts are obviously in contradiction with Lemma 16, Maker cannot claim a dangerous $b/2$-fan and the statement of the theorem is proved. □

To prove the theorem for Maker's win, we need the following lemma which is a slight modification of a result from [4]. Let $G(n, M)$ denote the graph obtained by choosing a graph on n vertices with M edges uniformly at random.

Lemma 18 *There exists $0 < \delta_k < 1$, such that for $M = 2\lfloor n^{2-2/(k+1)} \rfloor$ a.s. each subgraph of $G(n, M)$ with $\lfloor (1-\delta_k)M \rfloor$ edges contains a copy of K_k.*

Proof. For $0 < \delta_k < 1$, we call a subgraph F of K_n bad, if F has M edges and it contains a subgraph F' with $\lfloor (1-\delta_k)M \rfloor$ edges that does not contain a copy of K_k. In [4], it is proved that there exist constants $0 < \delta_k < 1$ and $c_1' > 0$ such that the number of bad subgraphs of K_n is bounded from above by

$$e^{-c_1' M/6} \binom{\binom{n}{2}}{M} = o(1) \binom{\binom{n}{2}}{M}.$$

□

Using the last lemma we can prove a theorem for Maker's win in the random clique game.

Theorem 19 *There exists $c_0 > 0$ so that for $p > \frac{1}{c_0} n^{-\frac{2}{k+1}}$ and $b \leq c_0 pn^{\frac{2}{k+1}}$ Maker wins the random clique game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, b)$ a.s.*

Proof. We will follow the analysis of the random Maker's strategy proposed in [4], looking at $G(n, M')$, where $M' = p\binom{n}{2}$. We will prove that the k-clique game on $G(n, M')$ is a Maker's win a.s., which implies that the same is true on $G(n, p)$, as being a Maker's win is a monotone property [7, Chapter 2].

In each of his moves Maker chooses one of the edges of $G(n, M')$ that was not previously claimed by him, uniformly at random. If the edge is free he claims it and we call that a successful Maker's move. If the edge was already claimed by Breaker, then Maker skips his move (e.g. claims an arbitrary free edge, and that edge we will not encounter for the future analysis).

Let $0 < \delta_k < 1$ be chosen so that the conditions of Lemma 18 are satisfied. We look at the course of game after $M = 2\lfloor n^{2-2/(k+1)} \rfloor$ moves.

By choosing $c_0 \leq \delta_k/12$, we have

$$\begin{aligned} M &\leq \frac{\delta_k}{6c_0} \lfloor n^{2-2/(k+1)} \rfloor \\ &\leq \frac{\delta_k}{2} \frac{1}{b+1} p \binom{n}{2}. \end{aligned}$$

That means that only at most $\delta_k/2$ fraction of the total number of elements of the board $E(G(n, M'))$ is claimed (by both players) after move M. Therefore, the probability that the edge randomly chosen in Maker's mth move, $m \leq M$, is already claimed by Breaker is bounded from above by $\delta_k/2$. That means that Maker has at least $(1 - \delta_k)M$ successful moves a.s.

Since in each of his moves Maker has chosen edges uniformly at random (without repetition) from $E(G(n, M'))$, the graph containing edges chosen by Maker in his first M moves (both successful and unsuccessful) actually is a random graph $G(n, M)$. Applying Lemma 18, we get that the graph containing edges claimed by Maker in his successful moves contains a clique of size k a.s., which means that a.s. there exists a non-randomized winning strategy for Maker. □

Combining the statements of Theorem 17 and Theorem 19 we obtain part (iv) of Theorem 2.

4 Unbiased games

4.1 Connectivity one-on-one

A theorem of Lehman enables us to determine the threshold probability $p_{\mathcal{T}}$ with extraordinary precision. Namely, Lehman [11] proved that the unbiased connectivity game is won by Maker (now as a second player!) if and only if the underlying graph contains two edge-disjoint spanning trees. The threshold for the appearance of two edge-disjoint spanning trees was determined exactly by Palmer and Spencer [12].

To formulate the consequence of these two results we need the concept of *graph process*. Let $e_1, \ldots e_m$ be the edges of K_n, where $m = \binom{n}{2}$. Choose a permutation $\pi \in S_m$ uniformly at random and define an increasing sequence of subgraphs (G_i) where $V(G_i) = V(K_n)$ and $E(G_i) = \{e_{\pi(1)}, \ldots, e_{\pi(i)}\}$. It is clear that G_i is an n-vertex graph with i edges, selected uniformly at random from all n-vertex graphs with i edges.

Given a particular graph process (G_i) and a graph property $\mathcal{P}$ possessed by K_n, the *hitting time* $\tau(\mathcal{P}) = \tau(\mathcal{P}, (G_i))$ is the minimal i for which G_i has property $\mathcal{P}$.

The consequence of the theorems of Lehman, and Palmer and Spencer is that the very moment the last vertex receives its second adjacent edge, the unbiased connectivity game is won by Maker a.s. More precisely, the following is true.

Corollary 20 *For the unbiased connectivity game we have that a.s.*

$$\tau(\textit{Maker wins } \mathcal{T}) = \tau(\exists \textit{ two edge-disjoint spanning trees}) = \tau(\delta(G) \geq 2).$$

In particular, for edge-probability $p = (\log n + \log\log n + g(n))/n$, where $g(n)$ tends to infinity arbitrarily slowly, Maker wins the unbiased connectivity game a.s., while if $g(n) \to -\infty$, then Breaker wins a.s.

Remark. The assumption that Maker is the second player is just technical, for the sake of smooth applicability of Lehman's Theorem. If Maker is the first player, then from the proof of Lehman's Theorem one can infer that Maker wins if and only if the base graph contains a spanning tree and a spanning forest of two components, which are edge-disjoint. This property has the same sharp threshold as the presence of two edge-disjoint spanning trees, and the hitting time should be the same when the next to last vertex receives its second incident edge.

4.2 k-cliques one-on-one

Let us fix k and let $(F_1, \dots, F_s)$ be a sequence of k-cliques. Then $F = \cup_{i=1}^{s} F_i$ is called an *s-bunch* if $V(F_i) \setminus (\cup_{j=1}^{i-1} V(F_j)) \neq \emptyset$ and $|V(F_i) \cap (\cup_{j<i} V(F_j))| \geq 2$, for each $i = 2, \dots, s$. Recall that an s-bunch in which the pairwise intersection of any two cliques is the same two vertices, was called a *simple s-2-cluster*. Let us denote the simple s-2-cluster by C_s.

For a graph G, the *density* of G is defined as $d(G) = \frac{e(G)}{v(G)}$, and the *maximum density* of G is defined as $m(G) = \max_{H \subseteq G} d(H)$. A graph G with $m(G) = d(G)$ is called *balanced*. The maximum density of a graph G determines the threshold probability for the appearance of G in the random graph. More precisely, (i) if $p = o(n^{-1/m(G)})$, then $G(n,p)$ does not contain G a.s., and (ii) if $p = \omega(n^{-1/m(G)})$, then $G(n,p)$ does contain G a.s.

We need two properties of simple s-2-clusters and s-bunches.

Lemma 21 *For every positive integer s, C_s is balanced and has maximum density $m(C_s) = d(C_s) = \frac{k+1}{2} - \frac{k}{sk-2s+2}$.*

Proof. It is easy to check that $v(C_s) = s(k-2)+2$, $e(C_s) = s\binom{k}{2} - s + 1$, and thus $d(C_s) = \frac{e(C_s)}{v(C_s)} = \frac{k+1}{2} - \frac{k}{sk-2s+2}$.

Let T be a subgraph of C_s. We want to prove $d(T) \leq d(C_s)$. Since C_s is the union of k-cliques, $C_s = \cup_{i=1}^{s} F_i$, if we set $E_i = F_i \cap T$ we have that $T = \cup_{i=1}^{s} E_i$, and we can assume that each E_i is a clique of order $k_i \leq k$.

We can also assume that the two vertices in $\cap_{i=1}^{s} V(F_i)$ are in T, since otherwise their inclusion would increase the density. This implies $k_i \geq 2$ for $i = 1, \dots, s$.

Let us relabel the cliques in such a way that $E_i \neq F_i$ if and only if $i = 1, \dots, s_1$. Then

$$\frac{e(C_s)}{v(C_s)} \geq \frac{e(T)}{v(T)} = \frac{e(C_s) - \sum_{i=1}^{s_1} \left(\binom{k}{2} - \binom{k_i}{2}\right)}{v(C_s) - \sum_{i=1}^{s_1} (k - k_i)},$$

since

$$\frac{e(C_s)}{v(C_s)} < \frac{k+1}{2} \leq \frac{\sum_{i=1}^{s_1}(k-k_i)\,\frac{k+k_i-1}{2}}{\sum_{i=1}^{s_1}(k-k_i)}.$$

The last inequality is true since the last fraction is the weighted average of the numbers $(k+k_i-1)/2$, each of them being at least $(k+1)/2$. □

Lemma 22 *Let $s \geq 3$ be a positive integer. No s-bunch has smaller maximum density than the simple s-2-cluster.*

Proof. When $k = 3$, the s bunch is a union of triangles. Then any s-bunch has the same number of vertices as the simple s-2-cluster, while the number of edges, and thus the density is minimized for the simple s-2-cluster.

From now on let us assume that $k \geq 4$. Let $s \geq 3$, and let $(F_1, F_2, \ldots, F_s)$ be the sequence of k-cliques of an arbitrary s-bunch $B_s = \cup_{i=1}^{s} F_i$. For every $i \in \{2, 3, \ldots, s\}$, let $F_i' = \left(\cup_{j=1}^{i-1} F_j\right) \cap F_i$. Then, we have

$$\begin{aligned} d(B_s) &= \frac{s\binom{k}{2} - \sum_{i=2}^{s} e(F_i')}{sk - \sum_{i=2}^{s} v(F_i')} \\ &= \frac{e(C_s) - \sum_{i=2}^{s}(e(F_i') - 1)}{v(C_s) - \sum_{i=2}^{s}(v(F_i') - 2)} \\ &\geq \frac{e(C_s) - \sum_{i=2}^{s}(\binom{v(F_i')}{2} - 1)}{v(C_s) - \sum_{i=2}^{s}(v(F_i') - 2)} \\ &\geq \frac{e(C_s)}{v(C_s)}. \end{aligned}$$

In the last inequality the terms with $v(F_i') = 2$ disappear, and otherwise we use that $v(F_i') \leq k-1$ for every i, so $\frac{\binom{v(F_i')}{2}-1}{v(F_i')-2} \leq \frac{k}{2} \leq \frac{e(C_s)}{v(C_s)}$.

Hence, simple s-2-clusters have the smallest density among all s-bunches. For any s-bunch B_s and the simple s-2-cluster C_s we immediately obtain

$$m(B_s) \geq d(B_s) \geq d(C_s) = m(C_s),$$

and the lemma is proved. □

Remark. The previous lemma is of course true for $s = 1$, but not for $s = 2$.

As a consequence of the last two lemmas we get a strategy for Breaker in the $(1:1)$ clique game.

Let H be a graph and consider the auxiliary graph G_H with vertices corresponding to the k-cliques of H, two vertices being adjacent if the corresponding cliques have at least two vertices in common. Let $F_1, \ldots, F_s$ be the cliques corresponding to a connected component of G_H. Then the graph $\cup_{i=1}^{s} F_i$ is called an *s-collection* or just a *collection* of H. Note that the edgeset of any H is uniquely partitioned into sets N and $E(A_i)$, where N contains the edges which do not participate in a k-clique, while the A_i are the collections of H.

Theorem 23 *For every $k \geq 4$ and $\varepsilon > 0$, $p_{\mathcal{K}_k} \geq n^{-\frac{2}{k+1}-\varepsilon}$. For $k = 3$, we have that $p_{\mathcal{K}_3} \geq n^{-\frac{5}{9}}$.*

Proof. First we give a strategy for Breaker to win $\mathcal{K}_k$ if the game is played on the edgeset of a $(2k-4)$-degenerate graph L. Consider the ordering $v_1, \dots, v_{v(L)}$ of $V(L)$, such that $|N_{V_j}(v_{j+1})| \leq 2k-4$ for $j = 1, \dots, v(L)-1$, where $V_j = \{v_1, \dots, v_j\}$. Then Breaker's strategy is the following: if Maker takes an edge connecting v_{j+1} to V_j, then Breaker takes another one also connecting v_{j+1} to V_j. If there is no such edge available, then Breaker takes an arbitrary edge. Suppose for a contradiction that Maker managed to occupy a k-clique $v_{i_1}, \dots, v_{i_k}$ against this strategy, where $i_1 < \dots < i_k$. This is impossible, since Maker could have never claimed $k-1$ of the edges $v_j v_{i_k}$, $j < i_k$.

Let $E(K_n)_p = N \dot\cup E(A_1) \dot\cup \dots \dot\cup E(A_h)$ be the partition of the edges, such that N contains all edges that do not participate in any k-clique, and each A_i is a collection of k-cliques. (Corresponding to the connected components of the auxiliary graph $G_{G(n,p)}$ defined on the set of k-cliques of $G(n,p)$.)

Breaker can play the game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, 1)$ by playing separately on each of the sets $E(A_i)$. More precisely, whenever Maker claims an edge which is in some $E(A_i)$, Breaker can play according to a strategy restricted just to $E(A_i)$. Since, crucially, the edgeset of each k-clique is completely contained in exactly one of the $E(A_i)$, Maker can only win the game on $E(K_n)_p$ if he wins on one of the $E(A_i)$.

Now we are going to show that every collection A on $v(A) = v$ vertices contains a $\lceil \frac{v-2}{k-2} \rceil$-bunch. We take an arbitrary k-clique F_1 from A, and build a bunch recursively as follows. If we picked k-cliques $F_1, \dots, F_i$ then we choose F_{i+1} such that $|V(F_{i+1}) \cap (\cup_{j=1}^{i} V(F_j))| \geq 2$ and $V(F_{i+1}) \setminus (\cup_{j=1}^{i} V(F_j)) \neq \emptyset$. Note that this means that $\cup_{j=1}^{i+1} F_j$ is an $(i+1)$-bunch. Since the auxiliary graph G_A of the collection is connected we can keep doing this until $V(A) = \cup_{j=1}^{i_0} V(F_j)$ for some i_0. Knowing that $v(F_i) = k$ for all $i \leq i_0$, we have $i_0 \geq 1 + \frac{v-k}{k-2} = \frac{v-2}{k-2}$. So there exists an $\lceil \frac{v-2}{k-2} \rceil$-bunch which is a subgraph of A.

We first look at the case $k \geq 4$. Let $\varepsilon > 0$ be a constant. From Lemma 21 it follows that there exists an integer v such that for $s_0 = \lceil \frac{v-2}{k-2} \rceil$ we have $m(C_{s_0}) \geq \frac{k+1}{2} - \frac{k}{v} > \left(\frac{2}{k+1} + \varepsilon\right)^{-1}$. Then for $p = O(n^{-\frac{2}{k+1} - \varepsilon})$ it follows that there is no s_0-bunch in $G(n,p)$ a.s., since we have have that the first s_0-bunch that appears in the random graph is the one of the minimum maximum density, which, by Lemma 22, is the simple s_0-2-cluster. Note here that there is a constant (depending on k and ε) number of nonisomorphic s_0-bunches.

Since in $G(n,p)$ there are no s_0-bunches a.s., there are also no collections on v vertices a.s.

Finally, all the collections A_i are $(2k-4)$-degenerate a.s., since graphs which are not $(2k-4)$-degenerate have maximum density at least $\frac{2k-3}{2} \geq \frac{k+1}{2}$, provided $k \geq 4$. Note that we know already that a.s. all collections have order at most v and thus there are at most a constant (depending on k and ϵ) number of nonisomorphic non-$(2k-4)$-degenerate graphs.

This proves that Breaker has a winning strategy a.s., if $k \geq 4$ and $p = O(n^{-\frac{2}{k+1} - \varepsilon})$.

Next, we look at the case $k = 3$. As we saw, any collection of triangles on v vertices contains a $(v-2)$-bunch. Thus for $p = o(n^{-5/9})$, no v-collection with

$v \geq 15$ will appear in $G(n,p)$ a.s., since it would contain a 13-bunch, whose maximum density is at least $m(C_{13}) = 2 - \frac{3}{15}$. This observation makes the problem finite: one has to check who wins on collections up to 14 vertices.

Suppose that Maker can win the triangle game on some collection of triangles on $v \leq 14$ vertices and with maximum density less then $9/5$. Let A be a minimal such collection (Maker cannot win on any proper subcollection of A).

If there was a vertex $w \in V(A)$ with $d_A(w) \leq 2$, the minimality of A would imply that Breaker has a winning strategy on A. Indeed, Breaker plays according to his strategy on $A - w$, and as soon as Maker claims one edge adjacent to w Breaker claims the other edge adjacent to w (if that edge exists otherwise he does not move). This would mean that Breaker can win on A, a contradiction. Thus, $\delta_A \geq 3$.

Let B be a $(v-2)$-bunch contained in A, with $V(A) = V(B)$. Since $\delta_B = 2$, we have $e(A) \neq e(B)$. Then

$$2 - \frac{3}{v} = m(C_{v-2}) \leq \frac{e(B)}{v} < \frac{e(A)}{v} < \frac{9}{5},$$

and

$$2v - 3 = e(C_{v-2}) \leq e(B) < e(A) < \frac{9v}{5}.$$

It is easy to check that Maker cannot win the game on a graph with less then 5 vertices, thus $v > 4$, so $e(B) = e(C_{v-2})$ and $e(A) - e(B) = 1$.

Let $\{e\} = E(A) \setminus E(B)$, and let $T_1, \ldots, T_{v-2}$ be the sequence of triangles whose union is the $(v-2)$-bunch B. Since $e(B) = e(C_{v-2})$, for every $i = 2, \ldots, v-2$ we have that T_i has a common edge with $\cup_{j=1}^{i-1} T_j$. Then B must have at least 2 vertices of degree 2. From $\delta_{B \cup \{e\}} = \delta_A = 3$ we obtain that B has exactly two vertices b_1, b_2 with $d_B(b_1) = d_B(b_2) = 2$, and moreover $e = \{b_1, b_2\}$. Since e has to participate in at least one triangle of the collection A, b_1 and b_2 have to be connected with a 2-path in B, which is possible only if all $T_1, \ldots, T_{v-2}$ share a vertex. That means that A is a $(v-1)$-wheel and it is easy to see that Breaker can win the triangle game on a wheel of arbitrary size by a simple pairing strategy.

This contradiction proves that for $p = o(n^{-5/9})$, a.s. there is no triangle collection in $G(n,p)$ on which Maker can win, which means that Breaker a.s. wins the game on the whole graph. □

From Theorem 19 we get that Maker can win the game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, 1)$ for $p = \Theta(n^{-\frac{2}{k+1}})$ and thus we immediately obtain $p_{\mathcal{K}_k} = O(n^{-\frac{2}{k+1}})$. For the triangle game $\mathcal{K}_3$ a stronger upper bound can be found.

Proposition 24 *The game* $(E(K_n)_p, (\mathcal{K}_3)_p, 1, 1)$ *is a Maker's win a.s., provided* $p = \omega(n^{-\frac{5}{9}})$.

Proof. It is easy to check that Maker can claim a triangle in the (1:1) game if the board on which the game is played is K_5 minus an edge. Therefore, as soon as the graph $G(n,p)$ contains $K_5 - e$ a.s., the initial game can be won by Maker a.s. □

Theorem 19, Theorem 23 and Proposition 24 imply parts (*iv*) and (*v*) of Theorem 1.

5 Open questions

More sharp thresholds? We saw in the previous section that the connectivity game has a sharp threshold, and even more. We think that both the perfect matching game and the Hamiltonian cycle game have the same sharp threshold $\frac{\log n}{n}$, and maybe even more... It would be very interesting to decide whether the following conjectures are true.

Conjecture 2

$$\begin{aligned} (i) \quad & \tau(\textit{Maker wins } \mathcal{M}) = \tau(\delta(G) \geq 2), \\ (ii) \quad & \tau(\textit{Maker wins } \mathcal{H}) = \tau(\delta(G) \geq 4). \end{aligned}$$

Clique game/H-game. The exact determination of the threshold $p_{\mathcal{K}_k}$ for the k-clique game remains outstanding.

Problem 1 *Decide whether* $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$ *for* $k \geq 4$.

The arguments of Bednarska and Łuczak [4] could be extended to full generality to positional games on random graphs along the lines of Section 3.4. More precisely, the following is true. Let $\mathcal{K}_H$ be the family of subgraphs of K_n, isomorphic to H. Then for any fixed graph H there is a constant $c(H)$, such that

$$b^p_{\mathcal{K}_H} = \Theta\left(pb_{\mathcal{K}_H}\right) = \Theta\left(pn^{-1/m'(H)}\right),$$

provided $p \geq \Omega\left(\frac{\log^{c(H)} n}{n^{1/m'(H)}}\right)$.

Concerning the one-on-one game, it would be desirable to determine those graphs for which an extension of the low-density Maker's win, à la Proposition 24, exists.

Problem 2 *Characterize those graphs H for which there exists a constant $\epsilon(H) > 0$, such that the unbiased game $\mathcal{K}_H$ is a.s. a Maker's win if $p = n^{-1/m'(H)-\epsilon(H)}$.*

For such graphs the determination of the threshold $p_{\mathcal{K}_H}$ is a finite problem, in a way similar to the case $H = K_3$.

Relationships between thresholds. It is an intriguing task to understand under what circumstances the following is true.

Problem 3 *Characterize those games $(X, \mathcal{F})$ for which*

$$p_{\mathcal{F}} = \frac{1}{b_{\mathcal{F}}}.$$

More generally, characterize the games for which

$$b^p_{\mathcal{F}} = \Theta\left(pb_{\mathcal{F}}\right),$$

for every $p = \omega\left(\frac{1}{b_{\mathcal{F}}}\right)$.

This is not true in general as the triangle game shows. What is the reason it is true for the connectivity game and the perfect matching game? Is it because the appearance of these properties has a sharp threshold in $G(n,p)$? Or because the winning sets are not of constant size?

Problem 4 *Suppose* $p_{\mathcal{F}} = 1/b_{\mathcal{F}}$. *Is it true that for every* $p \geq p_{\mathcal{F}}$, $b^p_{\mathcal{F}} = \Theta(pb_{\mathcal{F}})$*?*

It would be very interesting to relate the thresholds $b_{\mathcal{F}}$ and $p_{\mathcal{F}}$ to some thresholds of the family $\mathcal{F}$ in the random graph $G(n,p)$ (or, more generally, in the random set X_p). It seems to us that if the family $\mathcal{F}_p$ is quite dense and well-distributed in X, then Maker still wins the $(1\colon 1)$ game.

Problem 5 *Characterize those games* $(X, \mathcal{F})$ *for which there exists a constant* K*, such that for any probability* p *with* $\Pr\left[\min_{x\in X_p} |\{F \in \mathcal{F}_p : x \in F\}| > K\right] \longrightarrow 1$*, we have* $p_{\mathcal{F}} = O(p)$ *and/or* $b_{\mathcal{F}} = \Omega(1/p)$*.*

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