Odd independent transversals are odd

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Dedicated to Béla Bollobás on the occasion of his 60th birthday

Abstract

We put the final piece into a puzzle first introduced by Bollobás, Erdős and Szemerédi in 1975. For arbitrary positive integers n and r we determine the largest integer $\Delta = \Delta(r, n)$, for which any r-partite graph with partite sets of size n and of maximum degree less than Δ has an independent transversal. This value was known for all even r. Here we determine the value for odd r and find that $\Delta(r, n) = \Delta(r-1, n)$. Informally this means that the addition of an oddth partite set does not make it any harder to guarantee an independent transversal.

In the proof we establish structural theorems which could be of independent interest. They work for *all* $r \geq 7$, and specify the structure of slightly sub-optimal graphs for *even* $r \geq 8$.

1 Introduction

Let G be a graph, and suppose the vertex set of G is partitioned into r parts $V(G) = V_1 \cup \ldots \cup V_r$. An *independent transversal* of G is an independent set in G containing exactly one vertex from each V_i. Let $\Delta = \Delta(r, n)$ be the largest integer such that any such G has an independent transversal whenever $|V_i| = n$ for each i, and the maximum degree $\Delta(G)$ satisfies $\Delta(G) < \Delta$. Define $\Delta_r = \lim_{n\to\infty} \Delta(r, n)/n$, where the limit is easily seen to exist. Clearly any edges of G that lie inside the classes V_i are irrelevant as far as the functions $\Delta(r, n)$ and Δ_r are concerned, so for simplicity we will consider only r-partite graphs.

The problem of determining the functions $\Delta(r, n)$ and Δ_r was raised and first studied by Bollobás, Erdős and Szemerédi [7] in 1975. This question is a very basic one, and it has come up in the study of various other combinatorial parameters such as linear arboricity and strong chromatic number. Throughout the years, continuing work on these problems has been done by several researchers [4, 5, 11, 8, 9, 15, 2, 6, 14] and steady progress has been made.

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Trivially $\Delta(2,n) = n$, thus $\Delta_2 = 1$. Graver (c.f. [7]) showed $\Delta_3 = 1$. In their original paper, Bollobás, Erdős and Szemerédi [7] proved that

$$\frac{2}{r} \leq \Delta_r \leq \frac{1}{2} + \frac{1}{r-2},$$

thus establishing $\mu = \lim_{r\to\infty} \Delta_r \leq 1/2$. They conjectured $\mu = 1/2$. Alon [4] was the first to separate μ from 0 by showing $\Delta_r \geq 1/(2e)$ for every r using the Local Lemma. This was improved to $\Delta_r \geq 1/2$ in [9], which settled the conjecture of [7] and established $\mu = 1/2$. Despite the significant progress on the asymptotic behaviour of Δ_r, knowledge about the exact values of Δ_r, even for very small values of r, was very sparse. Until very recently, the value of Δ_r was known only for $r = 2, 3, 4, 5$ [11]. The argument of Jin for 4- and 5-partite graphs, showing $\Delta_4 = \Delta_5 = 2/3$, is intricate and seems difficult to generalize.

Besides proving lower bounds for $r = 4$ and 5, Jin [11] also gave promising examples (with low maximum degree and no independent transversal), when r is a power of 2. (Later Yuster [15] also found the same construction.) Jin in fact conjectured that these examples provide the extremum for every r, i.e. $\Delta_{2^j} = \Delta_{2^j+i}$ for any $i \leq 2^j - 1$. Recently Alon [6] observed that the method of [9], which gives $\mu = 1/2$, actually implies the slightly stronger bound $\Delta_r \geq \frac{r}{2(r-1)}$. This implies Jin's construction [11] is optimal for powers of 2 and then one has $\Delta_r = \frac{r}{2(r-1)}$. For other integers r, Alon gave improvements on the constructions of Jin, thus disproving his conjecture in general.

Very recently, a construction matching the $\frac{r}{2(r-1)}$ lower bound was found [14] — but only for an even number of parts. After this discovery, taking into account that $\Delta_2 = \Delta_3$ and $\Delta_4 = \Delta_5$, it was natural to conjecture that $\Delta_{2t} = \Delta_{2t+1}$ for every t. Here we confirm this intuition by determining not only Δ_r, but all the values $\Delta(r,n)$ for every n when r is odd.

Theorem 1.1 *For every integer $n \geq 1$ and $r \geq 2$ odd,*

$$\Delta(r,n) = \Delta(r-1,n) = \left\lceil \frac{(r-1)n}{2(r-2)} \right\rceil .$$

In particular for every r odd we have

$$\Delta_r = \frac{r-1}{2(r-2)}.$$

The construction of [14] determined $\Delta_6 = 3/5$, so we will concentrate on the case in which $r \geq 7$. Our argument consists of two parts. First we establish a structural theorem about minimal counterexamples. It was known that every r-partite graph with parts of size n and no independent transversal must have maximum degree at least $\frac{r}{2(r-1)}n$. What we prove here is that (for $r \geq 7$) any graph without an independent transversal, even if its maximum degree is a bit more then the threshold $\frac{r}{2(r-1)}n$ (but not more than $\frac{r-1}{2(r-2)}n$) is the vertex-disjoint union of $r-1$ complete bipartite graphs together with some extra edges. Moreover if the graph is minimal with respect to not having an independent transversal then it *is* the disjoint union of $r-1$ complete bipartite graphs. This theorem, Theorem 3.7, is valid for any number of parts (≥ 7) and for an even number parts it proves that any near-extremal example has this structure. We remark that this is in accordance with the known independent transversal-free examples: The graphs of [11], [15], and [14] are all $r-1$ copies of $K_{r/2,r/2}$. It is worthwhile to note though that the extremal examples are not unique, at least not for powers of 2. Although the graphs of Jin [11] and Yuster [15] are the same as the ones in [14], the partitions are very different.

In the second part of our proof we show that if r is odd, G is the union of $r-1$ complete bipartite graphs and $\Delta(G) < \frac{r-1}{2(r-2)}n$, then G has an independent transversal.

The organization of the paper is the following. In Section 2 *induced matching configurations* are introduced, which are the basic structural tool of our proof. The technical Theorem 2.2 is applied in three different contexts throughout our paper, not always for the original graph with its vertex partition. In Section 3 we still deal with r-partite graphs where r is not necessarily odd and prove our structural theorems for independent transversal-free graphs. In Section 4 we prove Theorem 1.1. Here we finally make use of the fact that r is odd in the sense that an integer is odd if and only if every tree on r vertices could be considered a rooted tree in which every subtree not containing the root has strictly fewer than half of the vertices.

Throughout the paper, the neighborhood of a vertex v is denoted by $N(v)$. For a set $T \subseteq V(G)$ we write $N_T(v) = N(v) \cap T$ and $G[T]$ for the subgraph of G induced by T. We denote the degree of a vertex v by $deg(v) = |N(v)|$ and write $deg_T(v) = |N_T(v)|$. We say that a vertex v is *dominated* by a set T if $N_T(v) \neq \emptyset$. If $N_T(v) = \emptyset$, then we say that v is *independent* of T.

2 Induced matching configurations

As in the introduction, we consider r-partite graphs G, and to avoid trivialities we assume each part is nonempty. The notion of independent transversal naturally presupposes that the vertex partition $V(G) = V_1 \cup \ldots \cup V_r$ is fixed. However, for simplicity we will not refer explicitly to the vertex partition when the term independent transversal is used, unless there is a danger of confusion. By a *partial* independent transversal of G we mean an independent set U in G of size less than r, such that $|V_i \cap U| \leq 1$ for each i. Sometimes for emphasis we will refer to an independent transversal as a *complete* independent transversal. Often we will use the abbreviation IT. The aim of this section is to introduce and prove the existence of *induced matching configurations*, the basic structure employed in our proof.

Let G be an r-partite graph with vertex partition $V(G) = V_1 \cup \ldots \cup V_r$. A class V_i is called *active* for a subset $I \subseteq V(G)$ of the vertices if $V_i \cap I \neq \emptyset$, and $S(I)$ denotes the set of active classes of I. The *class-graph* $\mathcal{G}_I$ of a subset $I \subseteq V(G)$ of the vertices is obtained from $G[I]$ by contracting all the vertices of $V_i \cap I$ into one vertex, which, with slight abuse of notation, we also call V_i. Thus the vertex set of $\mathcal{G}_I$ is $S(I)$.

A set of vertices I is called an *induced matching configuration* (*IMC*), if $G[I]$ is a perfect matching and the graph $\mathcal{G}_I$ is a tree on r vertices. In particular every class is active, $|I| = 2(r-1)$, and $G[I]$ has at most one edge joining each pair of classes (since a tree has no multiple edges). The following

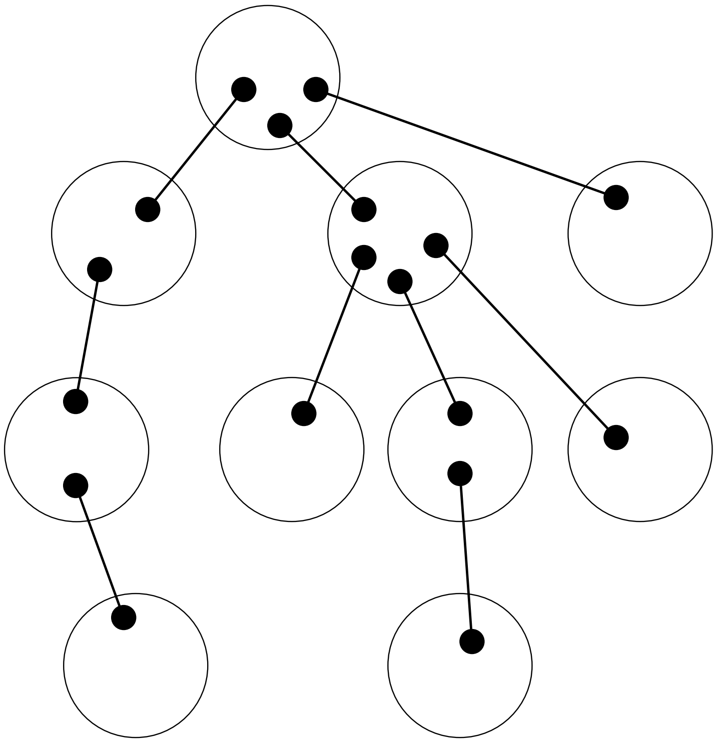

Figure 1: An induced matching configuration

lemma is a simple but important observation. It describes circumstances under which a complete independent transversal could be obtained from an IMC.

Lemma 2.1 *Let G be an r-partite graph with vertex partition $V(G) = V_1 \cup \cdots \cup V_r$, and let I be an IMC in G. For any index $i \in [r]$, there is a partial independent transversal $T_i^I \subseteq I$ of G, such that $T_i^I \cap V_j = \emptyset$ if and only if $j = i$.*

Moreover, for any vertex v not dominated by I, there exists an independent transversal T_v^I containing v.

Proof. Consider the rooted tree produced from $\mathcal{G}_I$ by selecting V_i as the root. For every $j \neq i$ include in T_i^I the (unique) element of $I \cap V_j$ whose neighbour in I is in the parent class of V_j in $\mathcal{G}_I$.

For the second part, let $v \in V_i$ be a vertex not dominated by I. Then $T_v^I := T_i^I \cup \{v\}$ is an independent transversal. $\square$

The main theorem of this section, Theorem 2.2, gives certain technical information about vertex-partitioned graphs that do not have independent transversals. In particular, we show that they *do* have induced matching configurations if their maximum degree is not too large. The proof of Theorem 2.2 we give here is based on the proof given in [10] that $\Delta_r \geq 1/2$.

Before we state Theorem 2.2 we need to establish some definitions and notation. Let G be an r-partite graph with vertex partition $V(G) = V_1 \cup \cdots \cup V_r$ that does not have an independent transversal. For a subset $A \subseteq \{V_1, \ldots, V_r\}$ let $\mathcal{T}_A$ be the set of partial independent transversals T which satisfy $|T \cap V_i| = 1$ iff $V_i \in A$. For a partial independent transversal T and a vertex $v \notin T$, we denote by $C(v,T)$ the vertex set of the component of $G[\{v\} \cup T]$ that contains v (so $G[C(v,T)]$ is always a star with center v).

To prove the existence of an IMC we need to deal with the more general definition of a feasible pair (which is slightly different and stronger than the one in [10]). Despite looking akwardly complicated, the following definition captures a relatively simple concept (see Figure 2). Below by a *nontrivial* star we mean a star with at least 2 vertices. We call the pair (I,T) *feasible* if

(a) $I \subseteq V(G)$ and T is a partial independent transversal of maximum size,

(b) $S(I \cap T) = S(I) \cap S(T)$,

(c) $G[I]$ is a forest, whose components are the $|W|$ vertex disjoint nontrivial stars $G[C(v,T)]$, with $v \in W$, where $W = I \setminus T$,

(d) (tree property) the graph $\mathcal{G}_I$ is a tree on the vertex set $S(I)$,

(e) (minimality property) there is no $v_0 \in W$ and $T' \in \mathcal{T}_{S(T)}$ with $T' \cap W = \emptyset$ such that $|C(v_0,T')| < |C(v_0,T)|$, but $C(v,T') = C(v,T)$ for $v \in W - \{v_0\}$.

Feasible pairs always exist, as $(\emptyset, T)$ is feasible if T is any partial transversal of maximum size (we consider the empty graph to be a tree).

The following theorem not only establishes the existence of an IMC, but it is used to derive our structural theorem, when it is applied for an auxiliary vertex-partitioned graph different from G.

Theorem 2.2 *Let G be an r-partite graph with vertex partition $V(G) = V_1 \cup \ldots \cup V_r$, and suppose G does not have an independent transversal of these classes. Let (I_0, T_0) be a feasible pair for G. Then there exists a feasible pair (I,T) in G such that*

(i) $I_0 \subseteq I$, $|S(I)| \geq 2$, and $T \cap V_i = T_0 \cap V_i$ for every $V_i \in S(I_0)$,

(ii) I dominates all vertices in the active classes $\bigcup\{V_i : V_i \in S(I)\}$.
In particular all vertices in $\bigcup\{V_i : V_i \in S(I)\}$ are dominated by $|I| \leq 2|S(I)| - 2$ vertices, and each class $V_i \in S(I)$ contains at least one of these dominating vertices from I.

If in addition $r \geq 3$, $|V_i| = n$ for $i = 1, \ldots, r$, and the maximum degree Δ of G satisfies $\Delta < \frac{r-1}{2(r-2)}n$, then

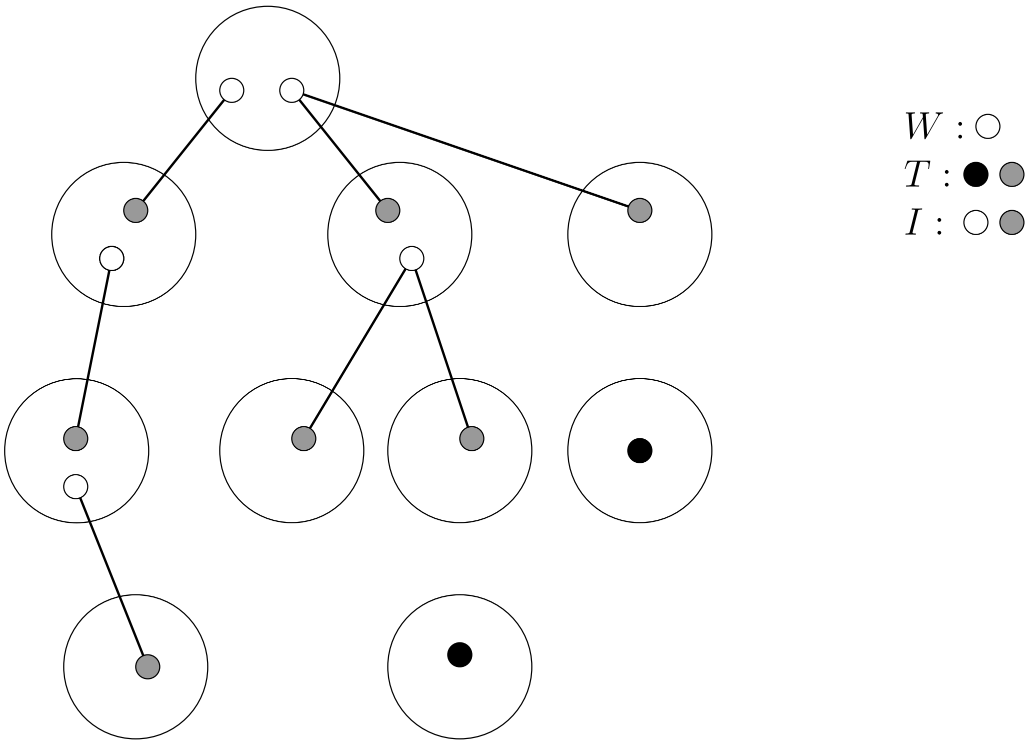

Figure 2: A feasible pair

(iii) I is an IMC in G, that dominates every vertex of G.

Proof. To prove the theorem we apply the following algorithm.

ALGORITHM

Input: The feasible pair (I_0, T_0).

Maintain: A feasible pair (I, T).

The set $W = I \setminus T$.

The set $S = S(I)$ of active classes for I.

The set $\mathcal{T} \subseteq \mathcal{T}_{S(T_0)}$ of transversals $T' \in \mathcal{T}_{S(T_0)}$, for which $T' \cap W = \emptyset$ and $C(v, T) = C(v, T')$ for every $v \in W$.

Initialization: $I := I_0, T := T_0$

Idea: Iteratively grow I and change T accordingly in the non-active classes of I.

Iteration: If $I = \emptyset$, select a vertex $w \in \cup_{V_i \notin S(T)} V_i$ and transversal $T' \in \mathcal{T}$, such that $deg_{T'}(w)$ is minimal. Update I by adding $C(w, T')$, update $T := T'$ and **iterate**.

If I dominates all vertices $w \in \cup_{V_i \in S} V_i$ in its active classes, then **stop** and **return** (I, T).

Otherwise select a vertex $w \in \cup_{V_i \in S} V_i$ and transversal $T' \in \mathcal{T}$, such that w is not dominated by I and $deg_{T'}(w)$ is minimal. Update I by adding $C(w, T')$, update $T := T'$ and **iterate**.

First we prove that the pair (I,T) maintained by the algorithm is feasible throughout. Suppose that (I,T) is feasible, and w and T' are as defined in the Iteration. (Note that $\mathcal{T} \neq \emptyset$ since $T \in \mathcal{T}$.) We claim that with $I' = I \cup C(w,T')$, (I',T') is feasible as well.

Suppose first that $I = \emptyset$. The conditions $(a),(b),(d),(e)$ are satisfied, by definition of w and T'. For condition (c) we must check that the star $G[C(w,T')]$ is nontrivial. This in fact is the case, because otherwise $T' \cup \{w\}$ would be an independent transversal of size larger than $|T|$.

Suppose now that $I \neq \emptyset$. Condition (a) holds since $|T'| = |T|$. For condition (b) note that $I' \cap T' = (I \cap T) \cup N_{T'}(w)$ and $I' = I \cup C(w,T')$. Then $S(I' \cap T') = S(I \cap T) \cup S(N_{T'}(w)) = (S(I) \cap S(T)) \cup S(N_{T'}(w)) = (S(I) \cup S(N_{T'}(w))) \cap (S(T) \cup S(N_{T'}(w))) = (S(I) \cup S(C(w,T')) \cap S(T') = S(I') \cap S(T')$.

For condition (c), consider the sets $C(v,T') = C(v,T)$ for $v \in W$; these are pairwise disjoint and of order at least 2. The last set $C(w,T')$ is disjoint from any set $C(v,T')$ $(v \in W)$, since w is independent of I and T' agrees with T on the active classes of I.

Now assume for contradiction that $|C(w,T')| = 1$, i.e. $deg_{T'}(w) = 0$. The class of w must contain a vertex of T', otherwise T' is not a maximum independent transversal, because w could be appended to it. Let u be the element of T' in the class of w. Since this class is active (we chose w from an active class) and T and T' agree on active classes, $u \in T$ as well. Then by (b) $u \in I$ and the degree of u in $G[I]$ is exactly one by (c). Let $v_0 \in W$ be its neighbour. Then $T'' = T' - \{u\} \cup \{w\}$ is a partial independent transversal in $\mathcal{T}_{S(T_0)}$ contradicting condition (e) of the feasibility of (I,T). Indeed, $C(v_0,T'') = C(v_0,T') - \{u\} = C(v_0,T) - \{u\}$, while $C(v,T'') = C(v,T') = C(v,T)$ for $v \in W - \{v_0\}$. Note also that $w \notin W$ because every $v \in W$ is dominated by a vertex of I (in $G[I]$ there is no isolated vertex). So $T'' \cap W = \emptyset$. Thus T'' provides the contradiction sought after, and this proves property (c) for the feasibility of (I',T').

For condition (d), it is enough to observe that w is in a class active for I, while all its T'-neighbours are in non-active classes. That is we obtain $\mathcal{G}_{I'}$ from $\mathcal{G}_I$ by appending $deg_{T'}(w)$ leaves.

Now we check that condition (e) holds for (I',T'). Choose $v_0 \in W \cup \{w\}$, and suppose on the contrary that there exists $T'' \in \mathcal{T}_{S(T_0)}$ with $T'' \cap (W \cup \{w\}) = \emptyset$ such that $|C(v_0,T'')| < |C(v_0,T')|$ and $C(v,T'') = C(v,T')$ for every $v \in W \cup \{w\} \setminus \{v_0\}$. If $v_0 \in W$ then $|C(v_0,T'')| < |C(v_0,T')| = |C(v_0,T)|$ and $C(v,T'') = C(v,T') = C(v,T)$ for all $v \in W \setminus \{v_0\}$, contradicting condition (e) in the fact that (I,T) is feasible. If $v_0 = w$ then by definition $T'' \in \mathcal{T}$, since then $C(v,T'') = C(v,T') = C(v,T)$ for every $v \in W$. But $|C(w,T'')| < |C(w,T')|$ contradicts our choice of T'. Therefore no such v_0 can exist and we have verified that (I',T') is a feasible pair.

Thus the algorithm maintains and upon termination returns a feasible pair. Moreover the algorithm always terminates because in each step I increases in size.

To prove part (i) for the feasible pair (I,T) output by the algorithm, we note that the algorithm constructs (I,T) from (I_0,T_0) just by adding new vertices to I_0 and changing the transversal only outside of $S(I_0)$. Since I_0 is the union of nontrivial stars, $|S(I)| \geq |S(I_0)| \geq 2$ unless $I_0 = \emptyset$. If I_0 is empty, the first step of the algorithm adds a nontrivial star to it, thus $|S(I)| \geq 2$ in this case as well.

Part (ii) is immediately implied by the stopping rule for the algorithm. The statement about the domination of the active classes follows from the facts that $\mathcal{G}_I$ is a tree on $S(I)$ and that $G[I]$ contains no isolated vertices, so $|I| \leq 2|S(I)| - 2$.

For part (iii), assume each V_i has size n and $\Delta < \frac{r-1}{2(r-2)}n$. We claim that the algorithm can terminate only when every class is active, i.e. if $|S| = r$. Since I dominates every vertex in $\cup_{V_i \in S} V_i$, we know $|\cup_{V_i \in S} V_i| = |S|n \leq |I|\Delta \leq (2|S|-2)\Delta$. Therefore $\Delta \geq \frac{|S|}{2|S|-2}n$. But if $|S| \leq r-1$ then $\frac{|S|}{2|S|-2}n \geq \frac{r-1}{2r-4}n$, contradicting our assumption. Hence $|S| = r$.

To complete the proof it remains to show that I induces a matching in G, in other words each $C(v, T)$ with $v \in W$ has size exactly 2. If this is not the case then $|W| \leq |I \cap T| - 1$ and thus $|I| = |W| + |I \cap T| \leq 2|I \cap T| - 1 \leq 2|T| - 1 \leq 2r - 3$. Then as above $rn \leq |I|\Delta \leq (2r-3)\Delta$, since I now dominates the whole graph by part (ii). This implies $\Delta \geq \frac{r}{2r-3}n \geq \frac{r-1}{2r-4}$ for $r \geq 3$, a contradiction. Therefore I is an IMC. $\square$

3 Structural results

Our aim in this section is to show that if G is an r-partite graph with no independent transversal and Δ is not too large then G is the union of vertex-disjoint complete bipartite graphs, together with a few extra edges that can join two vertices in the same partite set, or cross between partite sets. Moreover if G also does not contain any *unnecessary* edges, where unnecessary means "not preventing an independent transversal", then G is precisely a union of vertex-disjoint complete bipartite graphs.

Let G be an r-partite graph with vertex partition $V_1 \cup \ldots \cup V_r$, and let I be an IMC in G. We define the sets A_v of vertices, which are uniquely dominated by I. More formally, for $v \in I$ let $A_v(I) := \{y \in V(G) : N(y) \cap I = \{v\}\}$. If there is no possibility of confusion we omit from the notation the reference to the IMC and write simply A_v.

Because the first few lemmas of this section will all have the same assumptions, to avoid repetition we define the following. (Here by $G[A, B]$ we mean the bipartite subgraph of G consisting of all edges of G joining A and B.)

Setup. Let G be an r-partite graph with vertex partition $V_1 \cup \ldots \cup V_r$, where $|V_i| = n$ for each i, that does not have an IT. Denote the maximum degree of G by Δ. Let $I = \{v_i, w_i : 1 \leq i \leq r-1\}$ be an IMC in G, where v_i and w_i are adjacent. Let $A_v = A_v(I)$ for each $v \in I$.

Lemma 3.1 *Let G be as in the Setup. Then*

(i) for each i we have $w_i \in A_{v_i}$ and $v_i \in A_{w_i}$, and $G[A_{v_i}, A_{w_i}]$ is a complete bipartite graph,

(ii) for any $a \in A_{v_i}$, $b \in A_{w_i}$ the set $I' = I \setminus \{v_i, w_i\} \cup \{a, b\}$ is an IMC,

(iii) the number $|V(G) \setminus \cup_{v \in I} A_v|$ of vertices that are dominated more than once by I satisfies $|V(G) \setminus \cup_{v \in I} A_v| \leq 2(r-1)\Delta - rn$,

(iv) for any subset $\mathcal{S}$ of $\{A_v : v \in I\}$ we have $|\bigcup_{C \in \mathcal{S}} C| \geq (|\mathcal{S}| - 4r + 4)\Delta + 2rn$.

(To give an idea of the size of these quantities, when we use this lemma to prove our main structural result Theorem 3.7, we have $\Delta < \frac{r-1}{2r-4}n$. Then the upper bound in (iii) becomes $\frac{n}{r-2}$ and the lower bound in (iv) is $|\mathcal{S}|\Delta - \frac{2n}{r-2}$.)

Proof. The first assertion of part (i) holds since v_i and w_i are adjacent by definition, and I is an *induced* matching.

Now we show that $G[A_{v_i}, A_{w_i}]$ is a complete bipartite graph. We fix i, and for convenience write $v = v_i$, $w = w_i$. The deletion of the edge vw disconnects the tree $\mathcal{G}_I$ into two components $\mathcal{G}_v$ and $\mathcal{G}_w$, where $\mathcal{G}_u$ is the tree containing the class of u for $u = v, w$. Then $I - \{v, w\}$ induces two IMCs I_v and I_w on the two sets of classes corresponding to the vertices of $\mathcal{G}_v$ and $\mathcal{G}_w$, respectively. Note that it is possible that say $\mathcal{G}_w$ consists of only one class, in which case I_w is empty.

Let $a \in A_v$, so its only neighbour in I is v. We claim that if the class of a were in $\mathcal{G}_v$, then G would have an independent transversal. To see this, note that a is not dominated by I_v and w is not dominated by I_w, so we can apply Lemma 2.1. Then the union of transversals $T_a^{I_v}$ and $T_w^{I_w}$ is an independent transversal in G, because a and w are not adjacent. This contradiction establishes our claim. Therefore the class of a is in $\mathcal{G}_w$. Similarly, if $b \in A_w$ then the class of b is in $\mathcal{G}_v$.

Now for any $a \in A_v$ and $b \in A_w$, if a and b were not adjacent, then the union of the transversals $T_b^{I_v}$ and $T_a^{I_w}$ would be an independent transversal in G. Thus a and b are adjacent and $G[A_v, A_w]$ is a complete bipartite graph.

For (ii), to prove that I' is an IMC we first note that by definition the only neighbour of a in I' is b and vice versa, so I' is an induced matching. Also, by the previous discussion, the class of a is in $\mathcal{G}_w$ and the class of b is in $\mathcal{G}_v$ (otherwise there is an IT), so the induced graph $\mathcal{G}_{I'}$ of I' is a reconnection of the two subtrees $\mathcal{G}_v$ and $\mathcal{G}_w$, thus a tree itself.

To establish (iii), we note that each element of I is adjacent to at most Δ vertices, but by Lemma 2.1 together they dominate $V(G)$. Hence at most $|I|\Delta - rn = 2(r-1)\Delta - rn$ vertices can be joined to more than one vertex from I.

For (iv), since each A_v has size at most Δ, we see that $\mathcal{S}$ must contain in its union at least $rn - (2(r-1)\Delta - rn) - (2(r-1) - |\mathcal{S}|)\Delta = (|\mathcal{S}| - 4r + 4)\Delta + 2rn$ vertices. $\square$

Next we prove a couple of technical lemmas. Let H be the spanning subgraph of G obtained by erasing all edges joining A_{v_i} to A_{w_i} for every i, and also all edges inside each A_v, $v \in I$. For a subset $\mathcal{S}$ of $\{A_v : v \in I\}$, we denote by $H_{\mathcal{S}}$ the induced subgraph of H on the vertex set $\cup_{C \in \mathcal{S}} C$. We will consider $H_{\mathcal{S}}$ as a vertex partitioned graph, where the partition classes are the *partite sets* $C \in \mathcal{S}$ (and *not* related to the usual V_i).

Lemma 3.2 *Let G be as in the Setup, let $\mathcal{S}$ be a subset of $\{A_v : v \in I\}$, and let H be as above. Suppose we have a set Q of $2|\mathcal{S}|-2$ vertices in $H_{\mathcal{S}}$ such that each member of $\mathcal{S}$ contains at least one of them. Then the number of vertices dominated by Q in $H_{\mathcal{S}}$ is at most $(|\mathcal{S}|-1)((4r-4)\Delta - 2rn)$.*

(When $\Delta < \frac{r-1}{2r-4}n$ this upper bound becomes $(|\mathcal{S}|-1)\frac{2n}{r-2}$.)

Proof. Fix an $|\mathcal{S}|$-element subset Q' of Q containing exactly one vertex from each member of $\mathcal{S}$. The partite sets opposite these $|\mathcal{S}|$ vertices are distinct and total at least $(|\mathcal{S}|-4r+4)\Delta + 2rn$ vertices by Lemma 3.1(iv). These represent neighbours of Q' in G, which are lost in H, thus the number of vertices dominated by Q' in $H_{\mathcal{S}}$ is at most $|\mathcal{S}|\Delta - ((|\mathcal{S}|-4r+4)\Delta + 2rn) = (4r-4)\Delta - 2rn$. The remaining $|\mathcal{S}|-2$ vertices lie in partite sets opposite partite sets whose sizes are at least $(5-4r)\Delta + 2rn$ (since this is a lower bound on the size of any partite set by Lemma 3.1(iv)). Hence each of these dominates at most $\Delta - ((5-4r)\Delta + 2rn) = (4r-4)\Delta - 2rn$, for a total of at most $(|\mathcal{S}|-1)((4r-4)\Delta - 2rn)$. $\square$

Lemma 3.3 *Let G be as in the Setup, let $S \subseteq [r-1]$ and let $\mathcal{S} = \{A_{v_i}, A_{w_i} : i \in S\}$. Suppose we have vertices $a_i \in A_{v_i}$ and $b_i \in A_{w_i}$ for each $i \in S$, with the property that $\{a_i, b_i : i \in S\}$ forms an IT in $H_{\mathcal{S}}$. Then $\{v_i, w_i : i \notin S\} \cup \{a_i, b_i : i \in S\}$ is an IMC in G.*

Proof. Suppose $\emptyset \subseteq R \subset S$ and we know that $\{v_i, w_i : i \notin R\} \cup \{a_i, b_i : i \in R\}$ is the vertex set of an IMC I_R in G. Let j be such that $j \in S \setminus R$. Then by definition of $a_j \in A_{v_j}$, $b_j \in A_{w_j}$, the only neighbour of a_j in $\{v_i, w_i : i \notin R\}$ is v_j, and the only neighbour of b_j in $\{v_i, w_i : i \notin R\}$ is w_j. Moreover, neither a_j nor b_j has a neighbour in $\{a_i, b_i : i \in R\}$ because of the IT condition on $H_{\mathcal{S}}$. Therefore $a_j \in A_{v_j}(I_R)$ and $b_j \in A_{w_j}(I_R)$, so by Lemma 3.1(ii) applied to I_R we have that $\{v_i, w_i : i \notin R \cup \{j\}\} \cup \{a_i, b_i : i \in R \cup \{j\}\}$ is an IMC in G. Repeating this argument we obtain the statement of the lemma. $\square$

We will prove the first structural result stated at the beginning of this section in two steps. We are now ready to make the first step, in which we show that for our fixed IMC I, each vertex that is not in any A_v, $v \in I$ is completely joined to some A_v, $v \in I$.

Lemma 3.4 *Let G be as in the Setup, and suppose $r \geq 7$. Suppose the maximum degree Δ of G satisfies $\Delta < \frac{r-1}{2r-4}n$. Let x be a vertex lying outside $\bigcup_{v \in I} A_v$. Then x is joined to every vertex in A_v for some $v \in I$.*

Proof. For convenience we set $\mathcal{C} = \{A_v : v \in I\}$. For $v \in I$ let $A'_v \subseteq A_v$ be those vertices in A_v not adjacent to x. Suppose on the contrary that x is not joined completely to any A_v, i.e. each A'_v is nonempty. We claim that there exist $a_i \in A'_{v_i}$ and $b_i \in A'_{w_i}$ forming an IT in the subgraph $H'_{\mathcal{C}}$ of $H_{\mathcal{C}}$ induced by $V(H'_{\mathcal{C}}) = \bigcup_{v \in I} A'_v$, with vertex classes $\{A'_v : v \in I\}$. This would then be an IT in $H_{\mathcal{C}}$, and would by Lemma 3.3 be an IMC in G, which, by definition, would not dominate x. Lemma 2.1 then implies that there is an independent transversal in G, a contradiction.

Suppose there is no IT in $H'_{\mathcal{C}}$. Let $I_0 = \emptyset$ and T_0 be a partial transversal of $H'_{\mathcal{C}}$ of maximum size. Then by Theorem 2.2(ii) applied to $H'_{\mathcal{C}}$ with (I_0, T_0) and vertex classes $\{A'_v : v \in I\}$, we know that there exists a subset $\mathcal{S}$ of $\mathcal{C}$ and $2|\mathcal{S}| - 2$ vertices in $\bigcup_{C\in\mathcal{S}} C'$ (each member of $\mathcal{S}$ containing at least one of them), that dominate all vertices in $\bigcup_{C\in\mathcal{S}} C'$ in $H_{\mathcal{S}}$. Therefore these vertices together with x dominate all of $H_{\mathcal{S}}$. But x dominates at most Δ, and the rest dominate at most $(|\mathcal{S}| - 1)((4r - 4)\Delta - 2rn)$ by Lemma 3.2. Hence $H_{\mathcal{S}}$, which contains at least $(|\mathcal{S}| - 4r + 4)\Delta + 2rn$ vertices by Lemma 3.1(iv), has size at most $(|\mathcal{S}|-1)((4r-4)\Delta-2rn)+\Delta$. This implies $(|\mathcal{S}|-4r+4)\Delta+2rn \leq (|\mathcal{S}| - 1)((4r - 4)\Delta - 2rn) + \Delta$, from which we conclude $|\mathcal{S}|2rn \leq \Delta((4r - 5)|\mathcal{S}| + 1)$. But then $\frac{|\mathcal{S}|2rn}{(4r-5)|\mathcal{S}|+1} \leq \Delta < \frac{r-1}{2r-4}n$ giving $(r - 5)(|\mathcal{S}| - 1) < 4$.

Note that by Theorem 2.2(i) we have $|\mathcal{S}| \geq 2$. For $|\mathcal{S}| \geq 3$ we have a contradiction because $r \geq 7$. Suppose that $|\mathcal{S}| = 2$. Observe that $\mathcal{S} \neq \{A_{v_i}, A_{w_i}\}$ for any index i, since otherwise $H_{\mathcal{S}}$ would have no edges, so there would be no dominating set either. Assume without loss of generality that $\mathcal{S} = \{A_{v_i}, A_{v_j}\}$ for some $i \neq j$. Let $Q = \{q_i, q_j\}$ be a set dominating A'_{v_i} and A'_{v_j} in $H_{\mathcal{S}}$. By Theorem 2.2(ii) they must be in different partite sets, say $q_i \in A'_{v_i}$ and $q_j \in A'_{v_j}$. Then q_i, q_j and x dominate the four partite sets $A_{v_i}, A_{v_j}, A_{w_i}, A_{w_j}$. Thus the size of these four partite sets, which is at least $2rn - (4r - 8)\Delta$ by Lemma 3.1(iv), must be at most 3Δ. We conclude $2rn \leq (4r - 5)\Delta$, a contradiction for $r \geq 5$.

Our contradiction implies the existence of the IT in $H_{\mathcal{C}}$, which is an IMC in G, not dominating x, a contradiction. □

Finally we can complete the proof by showing that extra vertices such as x in the previous lemma that are completely joined to A_{v_i} are also adjacent to all extra vertices y joined completely to A_{w_i}. For convenience we state the lemma below for $i = 1$, but the same argument gives the result for each i.

Lemma 3.5 *Let G be as in the Setup. Suppose $r \geq 7$ and $\Delta < \frac{r-1}{2r-4}n$. Suppose a is adjacent to all of A_{w_1}, and b to all of A_{v_1}. Then a is adjacent to b.*

Proof. Suppose on the contrary that a and b are not joined. Now for $v \in I \setminus \{v_1, w_1\}$ let $A'_v \subseteq A_v$ be those vertices not adjacent to a nor to b. Certainly these sets are all nonempty because otherwise a and b together would dominate three members of $\{A_v : v \in I\}$, which by Lemma 3.1(iv) have total size at least $(7 - 4r)\Delta + 2rn$, which is impossible since $\Delta < \frac{r-1}{2r-4}n$ implies $(7 - 4r)\Delta + 2rn < 2\Delta$.

We claim that there exist $a_i \in A'_{v_i}$ and $b_i \in A'_{w_i}$ forming an IT in $H_{\tilde{\mathcal{C}}}$, where $\tilde{\mathcal{C}} = \{A_v : v \in I \setminus \{v_1, w_1\}\}$. If not then by Theorem 2.2(ii) we know there exists a nonempty subset $\mathcal{S}$ of $\tilde{\mathcal{C}}$ and $2|\mathcal{S}| - 2$ vertices in $\bigcup_{C\in\mathcal{S}} C'$ that dominate $\bigcup_{C\in\mathcal{S}} C'$, and each member of $\mathcal{S}$ contains at least one of them. But then these vertices together with a and b dominate all of $\bigcup_{C\in\mathcal{S}} C \cup A_{v_1} \cup A_{w_1}$. Here a and b can each dominate at most Δ (and they dominate A_{v_1} and A_{w_1}), and the rest dominate at most $(|\mathcal{S}|-1)((4r-4)\Delta-2rn)$ in $\bigcup_{C\in\mathcal{S}} C$ by Lemma 3.2. But $|\bigcup_{C\in\mathcal{S}} C \cup A_{v_1} \cup A_{w_1}| \geq (|\mathcal{S}|-4r+6)\Delta+2rn$ by Lemma 3.1(iv). Thus we conclude $(|\mathcal{S}| - 4r + 6)\Delta + 2rn \leq (|\mathcal{S}| - 1)((4r - 4)\Delta - 2rn) + 2\Delta$ which implies $|\mathcal{S}|2rn \le (4r-5)|\mathcal{S}|\Delta$. But this contradicts our assumption on Δ for $r \ge 5$. Therefore the a_i and b_i exist as claimed.

Now by Lemma 3.3 applied with the subset $\tilde{\mathcal{C}}$, we find that $I' = \{v_1, w_1\} \cup \{a_i, b_i : 2 \le i \le r-1\}$ is an IMC of G, with the property that all neighbours of a and b in I' lie in $\{v_1, w_1\}$. First we note that neither a nor b is completely joined to both $A_{v_1}(I')$ and $A_{w_1}(I')$. Indeed, if this were true then its degree would be at least $2rn - (4r-6)\Delta$ by Lemma 3.1(iv). But then this would imply $\frac{2r}{4r-5}n \le \Delta$, contradicting our assumption on Δ.

Now if we can find $v_1' \in A_{v_1}(I')$ and $w_1' \in A_{w_1}(I')$, such that v_1' is not adjacent to a and w_1' is not adjacent to b, then by Lemma 3.1(ii) $I'' = \{v_1', w_1'\} \cup \{a_i, b_i : 2 \le i \le r-1\}$ is an IMC in which a is joined only to w_1' and b is joined only to v_1'. Thus $a \in A_{v_1}(I'')$ and $b \in A_{w_1}(I'')$, so by Lemma 3.1(i) they are adjacent.

Therefore to complete the proof, we just need to show that such v_1' and w_1' exist. If not, then (without loss of generality) each of a and b is completely joined to $A_{v_1}(I')$. Recall that A_{w_1} is defined to be the set of vertices joined only to w_1 in I. But $v_1 \in I$ and each vertex of $A_{v_1}(I')$ is joined to v_1 by definition, so we conclude $A_{w_1} \cap A_{v_1}(I') = \emptyset$. Then since by assumption a is joined to all of $A_{w_1} \cup A_{v_1}(I')$ we find $deg(a) \ge |A_{w_1}| + |A_{v_1}(I')|$.

To estimate $|A_{v_1}(I')|$ we observe that each vertex of $A_{v_1} \setminus A_{v_1}(I')$ is joined to v_1, and hence by definition also joined to another vertex of I'. Therefore by Lemma 3.1(iii) we know $|A_{v_1} \setminus A_{v_1}(I')| \le 2(r-1)\Delta - rn$, and so $|A_{v_1}(I')| \ge |A_{v_1}| - |A_{v_1} \setminus A_{v_1}(I')| \ge |A_{v_1}| - 2(r-1)\Delta + rn$. Therefore $deg(a) \ge |A_{w_1}| + |A_{v_1}| - 2(r-1)\Delta + rn \ge (6-4r)\Delta + 2rn - 2(r-1)\Delta + rn$ by Lemma 3.1(iv), which tells us that $\Delta \ge (8-6r)\Delta + 3rn$. But then $\Delta \ge \frac{3r}{6r-7}n$, which contradicts our assumption on Δ for $r \ge 7$. Therefore v_1' and w_1' exist as required. $\square$

Now we are ready to prove that every minimal counterexample to Theorem 1.1 has to be the vertex-disjoint union of complete bipartite graphs. The following lemma is an easy consequence of Theorem 2.2.

Lemma 3.6 *Let G be an r-partite graph with vertex partition $V_1 \cup \ldots \cup V_r$ that does not have an IT, and suppose $|V_i| = n$ for each i and $\Delta < \frac{r-1}{2r-4}n$. Let e be an edge of G and suppose e prevents an IT (i.e. $G - e$ has an IT). Then e lies in an IMC.*

Proof. Let $U = \{v_1, \ldots, v_r\}$, $v_i \in V_i$, be an almost independent transversal inducing the lone edge $e = v_1v_2$. Then $T_0 = \{v_2, \ldots, v_r\}$ is a maximum size partial independent transversal. Let $I_0 := \{v_1, v_2\}$. The pair (I_0, T_0) is easily seen to be feasible, because the failure of condition (e) would immediately imply the existence of an IT in G.

We now apply Theorem 2.2 with (I_0, T_0), to obtain a feasible pair (I, T). Then by Theorem 2.2(iii), I is an IMC. Moreover Theorem 2.2(i) implies that I contains e. $\square$

Putting the above results together gives the structural theorem of minimal counterexamples.

Theorem 3.7 *Let G be an r-partite graph with vertex partition $V_1 \cup \ldots \cup V_r$, where $r \geq 7$ and $|V_i| = n$ for each i. Suppose G has no independent transversal, but the deletion of any edge creates one. If $\Delta < \frac{r-1}{2r-4}n$, then G is a union of $r-1$ vertex-disjoint complete bipartite graphs.*

Proof. By Theorem 2.2(iii), there is an IMC I_0 in G. Fix one, say $I_0 = \{v_i, w_i : 1 \leq i \leq r-1\}$ where v_i is adjacent to w_i for each i. Then by Lemmas 3.4 and 3.5 we have that the vertex set of G is partitioned into partite sets $A_1^*, \ldots, A_{r-1}^*, B_1^*, \ldots, B_{r-1}^*$, where each $G[A_i^*, B_i^*]$ is a complete bipartite graph, and $A_{v_i}(I_0) \subseteq A_i^*$ and $A_{w_i}(I_0) \subseteq B_i^*$ for each i.

Suppose G has an edge xy that does not lie in any of the bipartite subgraphs $G[A_i^*, B_i^*]$. By Lemma 3.6 we know that xy lies in some IMC I. Let us assume without loss of generality that $x \in A_1^*$, then $y \notin B_1^*$.

Suppose first that $y \in A_1^*$. Then the whole class B_1^* is dominated more than once by the IMC I. Therefore by Lemma 3.1(iii) the number of vertices in B_1^* is at most $2(r-1)\Delta - rn$. On the other hand $|B_1^*| \geq |A_{w_1}(I_0)| \geq (5-4r)\Delta + 2rn$ by Lemma 3.1(iv), giving us $(2r-2)\Delta - rn \geq (5-4r)\Delta + 2rn$. This implies $\Delta \geq \frac{3r}{6r-7}n$, a contradiction for $r \geq 7$.

Suppose now that $y \in C$ for some $C \in \{A_2^*, \ldots, A_r^*, B_2^*, \ldots, B_r^*\}$, say without loss of generality $y \in A_2^*$. By Lemma 3.1(i) and (iv), the edge xy lies in a complete bipartite graph J with a total of at least $(6-4r)\Delta + 2rn$ vertices, and with at least $(5-4r)\Delta + 2rn$ vertices in each class J_x and J_y. Here J_x and J_y denote the partite sets of J containing x and y respectively, and J_x contains all vertices whose only neighbour in I is y, and J_y contains all those whose only neighbour in I is x.

Suppose first that both $J_y \cap B_1^*$ and $J_x \cap B_2^*$ are non-empty and let $u \in J_y \cap B_1^*$ and $w \in J_x \cap B_2^*$ be arbitrary vertices. Then $deg(w) + deg(u) \geq (|A_2^*| + |J_y \cap B_1^*|) + (|A_1^*| + |J_x \cap B_2^*|) \geq |A_2^*| + |J_y| + |B_1^*| - |J_y \cup B_1^*| + |A_1^*| + |J_x| + |B_2^*| - |J_x \cup B_2^*|$. But since $|J_y \cup B_1^*| \leq deg(x) \leq \Delta$ and $|J_x \cup B_2^*| \leq deg(y) \leq \Delta$ we find by Lemma 3.1(iv)

$$\begin{aligned} 2\Delta &\geq deg(w) + deg(u) \geq (|J_y| + |J_x|) + (|A_1^*| + |A_2^*| + |B_1^*| + |B_2^*|) - 2\Delta \\ &\geq (6-4r)\Delta + 2rn + (8-4r)\Delta + 2rn - 2\Delta = (12-8r)\Delta + 4rn. \end{aligned}$$

This implies $\Delta \geq \frac{2r}{4r-5}$, a contradiction for $r \geq 5$.

Suppose now that $J_x \cap B_2^*$ is empty. (The case when $J_y \cap B_1^* = \emptyset$ is similar.) This means that all vertices in B_2^* are dominated more than once by I, since they are dominated by y and J_x contains all those dominated only by y. The number of such vertices is at most $2(r-1)\Delta - rn$ by Lemma 3.1(iii), while the size of one partite set is at least $(5-4r)\Delta + 2rn$ by Lemma 3.1(iv). This implies $(2r-2)\Delta - rn \geq (5-4r)\Delta + 2rn$, giving $\Delta \geq \frac{3r}{6r-7}n$, a contradiction for $r \geq 7$.

Therefore no such edge xy can exist, and G must be the union of vertex-disjoint complete bipartite graphs. □

4 Proof of Theorem 1.1

In this section we focus on the case of odd number of parts. From the previous section we know that any minimal counterexample to our main theorem has to be the vertex-disjoint union of complete bipartite graphs. By proving the following theorem we finish the proof of Theorem 1.1.

Theorem 4.1 *Let $r = 2t + 1$ be an odd integer. Let G be the union of $2t$ vertex disjoint complete bipartite graphs, with a vertex r-partition $V(G) = V_1 \cup \cdots \cup V_r$ into classes of size n. If the maximum degree $\Delta(G) < \frac{t}{2t-1}n$ then G has an independent transversal of the classes $V_1, \ldots, V_r$.*

Proof. Suppose on the contrary that G has no independent transversal. By Theorem 2.2(iii) there is an IMC I that induces $r - 1$ edges, which defines a tree structure $\mathcal{G}_I$ on the classes of G. Let us choose a root-class, for which all subtrees not containing the root have order at most t (for any tree of order $2t + 1$ one can find such a vertex; this is basically the only time we use the fact that the number of parts is odd). We color the vertices of I according to the tree structure: for an edge induced by I we color the vertex in the parent class white and the vertex in the child class black. We call these $2t$ black and $2t$ white vertices *distinguished* and denote them by $b_1, \ldots b_{2t}$ and $w_1, \ldots, w_{2t}$, respectively, where b_i is adjacent to w_i. The root contains only white vertices and all other classes contain exactly one black vertex. We call the white neighbour of the unique black vertex in class C, the *parent vertex* of C. By Lemma 2.1 the set T of black vertices is an almost complete independent transversal, which is only missing a vertex from the root class. For a white vertex w, we define the subtree $\mathcal{G}_w$ of w to be the subtree of the class-graph $\mathcal{G}_I$ that contains all the classes that are descendants of the class of the black neighbour of w (including the class of the black neighbour).

Our plan is to change the black vertices in some classes such that we still have a partial independent transversal, but now we are able to make a black/white switch on a path to the root and create a complete independent transversal.

Note that, since G is the vertex-disjoint union of $r - 1$ complete bipartite graphs, each edge induced by I lies in a distinct complete bipartite graph. Therefore every vertex of G has exactly one distinguished neighbour, and every component of G contains exactly one black vertex. This immediately implies the following.

Fact 1. Let Z and X be disjoint sets of black vertices. Then $N_G(Z) \cup X$ is an independent set.

We also note here two more technical facts that we will need in the proof.

Fact 2. The unique distinguished neighbour of a vertex v is either a white vertex on the path from the class of v to the root class, or a black vertex in a class that is not on the path from the class of v to the root.

Proof. For distinguished vertices the statement is true by definition. Let $v \in V_i$ be non-distinguished and suppose the statement is false. Although v is dominated by I, so Lemma 2.1 cannot be applied directly, by our assumption the lone distinguished neighbour of v is *not* in T_i^I (as it is defined in the proof of Lemma 2.1). So the transversal $T_i^I \cup \{v\}$ is independent, a contradiction. $\square$

Fact 3. No set $\mathcal{L}$ of at most t classes is dominated by *fewer* than $2|\mathcal{L}|$ vertices. In particular, for any white vertex w_i there is a non-distinguished vertex $u \in \bigcup\{V_i : V_i \in V(\mathcal{G}_{w_i})\}$ in the classes of the subtree $\mathcal{G}_{w_i}$ of w_i, which has a distinguished neighbour $w \neq w_i$ in a class *outside* the classes of $\mathcal{G}_{w_i}$.
Proof. A set of $2|\mathcal{L}| - 1$ vertices can dominate at most $(2|\mathcal{L}| - 1)\Delta$ vertices, and $(2|\mathcal{L}| - 1)\Delta < (2|\mathcal{L}| - 1)\frac{t}{2t-1}n \leq (2|\mathcal{L}| - 1)\frac{|\mathcal{L}|}{2|\mathcal{L}|-1}n = |\mathcal{L}|n$.

For the second part, by Fact 2 the black vertex b_i cannot dominate any vertex in any class of $\mathcal{G}_{w_i}$. Thus the $|\mathcal{G}_{w_i}| \leq t$ classes in $\mathcal{G}_{w_i}$ cannot be completely dominated by the $2|\mathcal{G}_{w_i}| - 1$ distinguished vertices in $I \cap (\bigcup\{V_i : V_i \in V(\mathcal{G}_{w_i})\} \cup \{w_i\} - \{b_i\})$. □

In our proof we define a sequence of vertices $z_1, u_1, z_2, u_2, \ldots, z_q, u_q$ with the following properties.

- The z_i are distinct black vertices; the u_i are not black.
- z_i and u_i are in the same class. Note that this implies the u_i are all distinct.
- For $i = 1, \ldots, q-1$, u_i is adjacent to z_{i+1}.

The construction of the sequence goes as follows (see Figure 3). Let w_1 be an arbitrary white vertex in the root-class. (Here for convenience we may re-number the vertices b_i and w_i.) We define $z_1 = b_1$, the black neighbour of w_1. By Fact 3 there is a non-distinguished vertex g_1 in the classes of the subtree $\mathcal{G}_{w_1}$ that has a distinguished neighbour $w \neq w_1$ *outside* the subtree $\mathcal{G}_{w_1}$. By Fact 2, w must be black, say $w = b_2$. The initial segment $z_1, u_1, \ldots, z_{i_1}, u_{i_1}$ of our sequence is then defined by $u_{i_1} = g_1$, the z_k's for $k \leq i_1$ are the black vertices in the classes on the path from the class of b_1 to the class of g_1 in $\mathcal{G}_I$ in the same order, while u_k for $1 \leq k \leq i_1 - 1$ is the white neighbour of z_{k+1}.

In general, if b_j is defined, we define g_j to be the (existing) non-distinguished vertex in a class in the subtree $\mathcal{G}_{w_j}$ of w_j, that has a distinguished neighbour $w \neq w_j$ outside of $\mathcal{G}_{w_j}$. (If more than one such vertex exists we just choose one arbitrarily.) We then define the next segment $z_{i_{j-1}+1}, u_{i_{j-1}+1}, \ldots, z_{i_j}, u_{i_j}$ of our sequence by $z_{i_{j-1}+1} = b_j$, $u_{i_j} = g_j$, the z_k's for $i_{j-1} < k \leq i_j$ are the black vertices in the classes that are on the path from the class of b_j to the class of g_j in $\mathcal{G}$, while u_k for $i_{j-1} + 1 \leq k \leq i_j - 1$ is the white neighbour of z_{k+1}.

If any of these new black vertices participated already in our sequence, we stop the sequence right before the repetition, so the last vertex is u_q, and the candidate for z_{q+1} is already some z_i in the sequence.

If the distinguished neighbour w of g_j is black, say $w = b_{j+1}$, then we go on and construct the next segment of our sequence.

We can build our sequence as long as we don't repeat a black vertex z_i and the distinguished neighbour of g_j outside the subtree $\mathcal{G}_{w_j}$ is not white. Since our graph is finite, so will be our sequence.

Case 1. Our sequence ends, because z_{q+1} would be equal to some z_i, $i \leq q$. We improve on the almost complete independent transversal T of black vertices by making switches (see Figure 3). Let z_t be the black vertex whose class V_t is closest to the root among all z_j, $i \leq j \leq q$, i.e. its

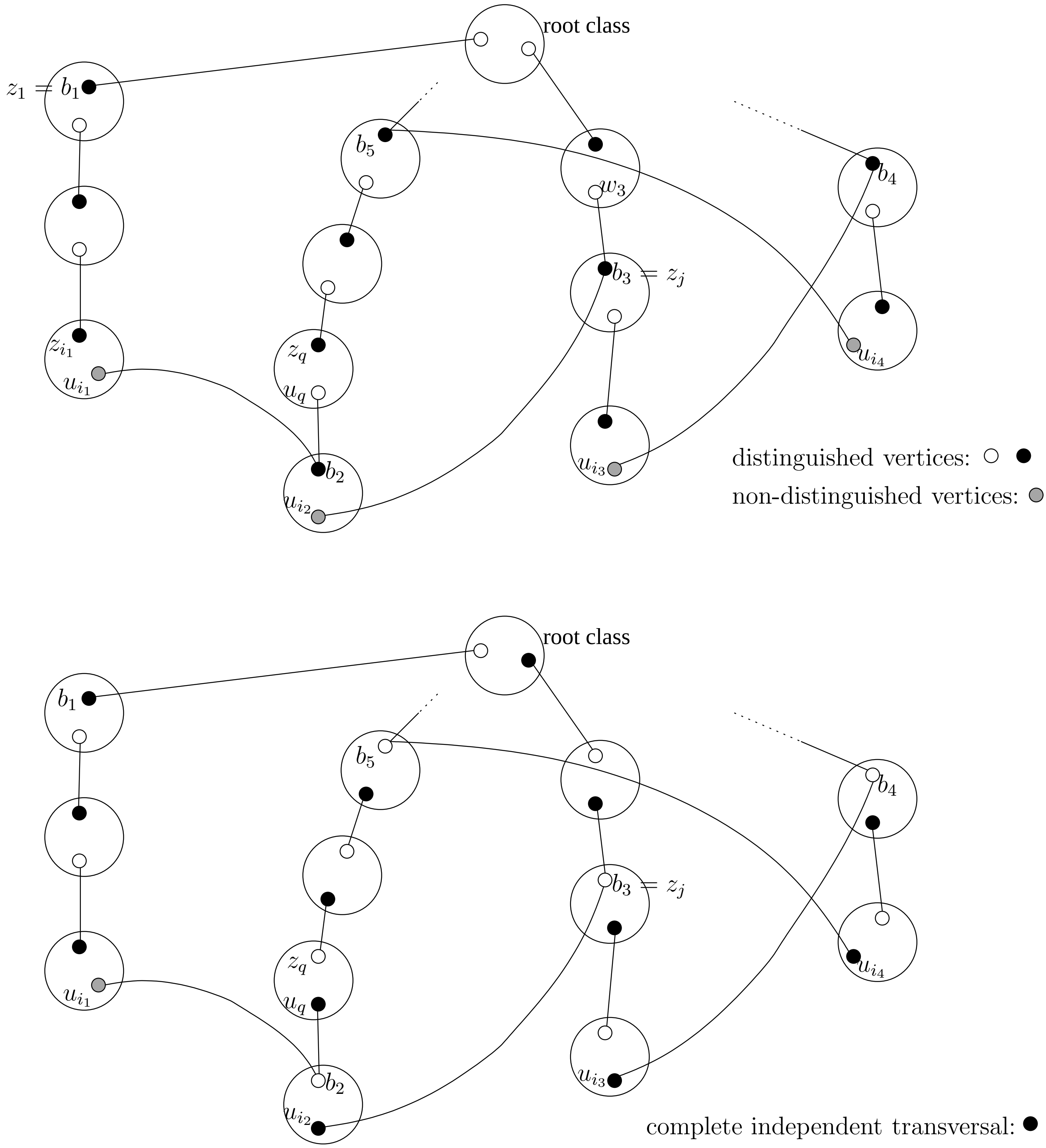

Figure 3: Case 1, before and after the switch

path P to the root is shortest. Let $b(P)$ denote the set of black vertices in the classes of P, and let $w(P)$ denote their white neighbours. (Note then that $b(P) \cap \{z_i \dots, z_q\} = z_t$ and $w(P)$ contains a vertex in the root class.) We form the set T' by removing $Z = \{z_i, \dots, z_q\} \cup b(P)$ from T and adding $U = \{u_i, \dots, u_q\} \cup w(P)$. We claim that T' is a (complete) independent transversal of G. To see that T' is independent, apply Fact 1 to the sets Z and $X = T \setminus Z$ of black vertices, and observe that $U \subseteq N(Z)$ because u_q is adjacent to z_i. To check that T' is a transversal, recall that u_j and z_j were in the same class for each j, and note that $w(P)$ contains a vertex of each class of P including the root class, except for V_t. But V_t contains the vertex $u_t \in T'$. Therefore T' is an independent transversal as claimed.

Case 2. Our sequence stops, because the distinguished neighbour $w \neq w_k$ of $g_k = u_{i_k}$ outside the subtree $\mathcal{G}_{w_k}$ is white. Note then that $k \geq 2$.

By Fact 2, the class of w is above the class of $u_{i_k} = u_q$ in the class-graph tree $\mathcal{G}_I$. Since it is outside the subtree of w_k, w is also above b_k (but $w \neq w_k$!).

We identify the *last time* our sequence entered the subtree $\mathcal{G}_w$. By the property of the sequence, the last vertex in our sequence *not* contained in this subtree is not black, say u_j (note this vertex exists since $k \geq 2$). Then we claim $w \neq u_j$. To see this, note that if $w = u_j$ then for some index $l \leq k$, the vertex $b_l = z_{i_{l-1}+1}$ would be in a class above (or equal to) the class of w, while $g_l = u_{i_l}$ would be in a class below w. So $l < k$, since b_k is below w. But then by definition z_{i_l+1}, the distinguished neighbour of g_l, is outside of the subtree of w_l, and thus outside of the subtree of w as well. This is a contradiction, since then the entry of the sequence into $\mathcal{G}_w$ from u_j was not the last one.

We create a complete independent transversal T' as follows (see Figure 4). Let P denote the path from the class V_j containing u_j to the root class, then since V_j is not in $\mathcal{G}_w$ we know that none of the classes containing $\{u_{j+1}, \dots, u_q\}$ are in P. Let $b(P)$ denote the set of black vertices in the classes of P, and let $w(P)$ denote their white neighbours (again $w(P)$ contains a vertex in the root class). We form the set T' by removing $Z = \{z_{j+1}, \dots, z_q\} \cup b(P)$ from T and adding $U = \{u_j, \dots, u_q\} \cup w(P)$. We claim that T' is a (complete) independent transversal of G. It is a transversal because u_i replaces z_i for each $j + 1 \leq i \leq q$, and each class of P including the root class gets a vertex of $w(P)$, except for V_j. But V_j contains the vertex $u_j \in T'$, so every class of G contains an element of T'.

To check that T' is independent, first apply Fact 1 to the sets Z and $X = T \setminus Z$ of black vertices, and observe that $U \setminus \{u_q\} \subseteq N(Z)$. Thus it remains only to show that u_q is not adjacent to any vertex of $T' \setminus \{u_q\}$. Since u_q is adjacent to w, it is certainly independent of $X = T \setminus Z$ because it has exactly one distinguished neighbour. Let b denote the black neighbour of w. Suppose on the contrary that u_q is adjacent to some $x \in U$. Let z denote the black neighbour of x, then $z \in Z$, and x, z, w and b are all in the same component of G. But then we must have $z = b$ because this component contains only one black vertex, say $z = z_s$ where $j + 1 \leq s \leq q$. Now $s = j + 1$ is not possible, since otherwise by construction the next non-distinguished vertex in the sequence will be u_v for some $v \leq q$, and the distinguished neighbour z_{v+1} of u_v will be outside $\mathcal{G}_w$ and *not* equal

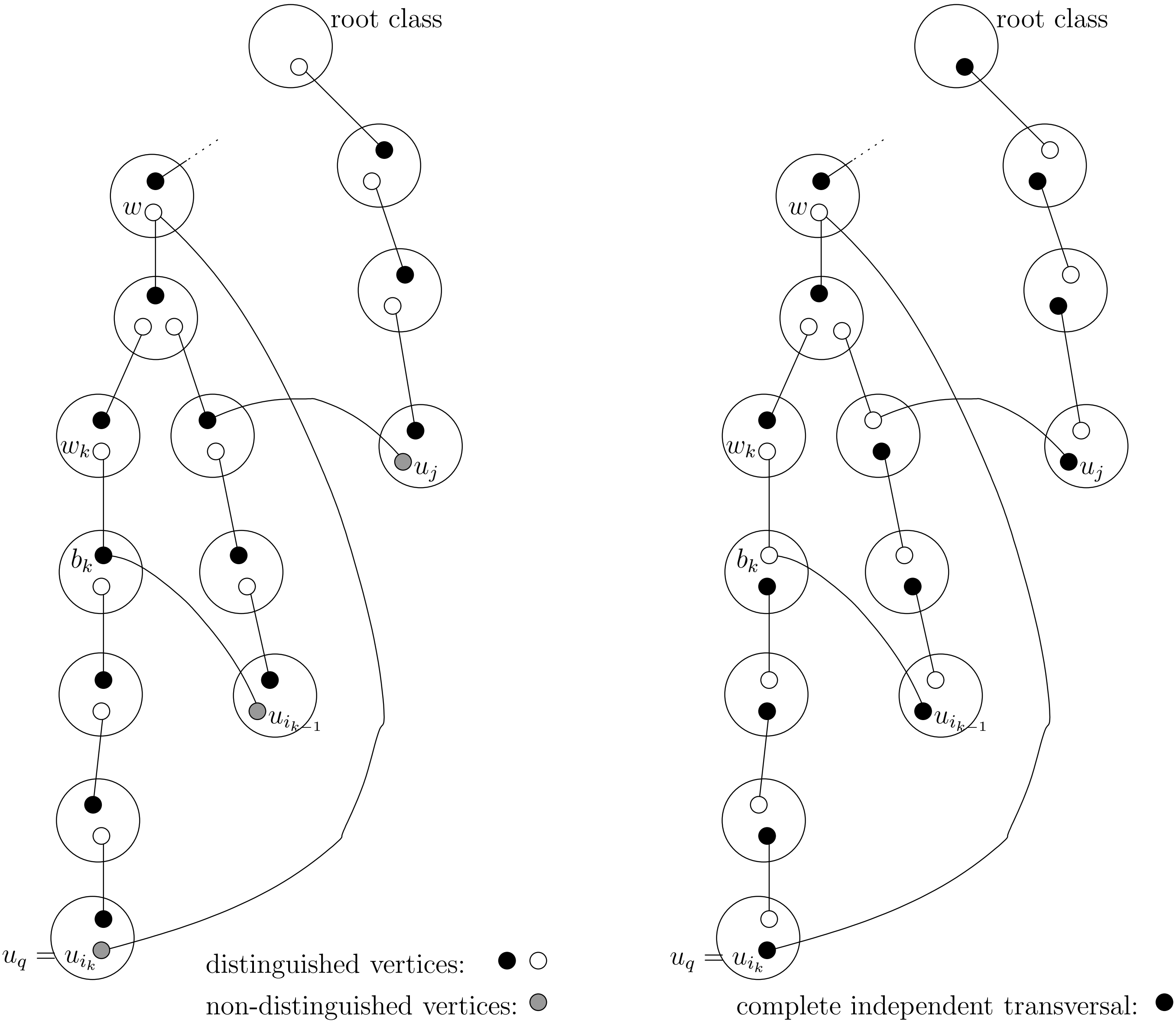

Figure 4: Case 2, before and after the switch

to w. This would contradict the fact that this is the last time the sequence enters $\mathcal{G}_w$. Therefore $j+2 \le s \le q$. Then $x = u_{s-1} \in T'$ must be a non-distinguished vertex in our sequence, since w is the only distinguished neighbour of $z_s = b$. But all non-distinguished vertices in $U \setminus \{u_j\}$ that are candidates for u_{s-1} are in classes below $b = z_s$, so by Fact 2 they cannot be adjacent to z_s (which is black). Thus u_q cannot have any such neighbour $x \in U$, giving that T' is an independent transversal, and thus contradicting our assumption on G. This completes the proof of the theorem. $\square$

5 Open problems

An intriguing problem remains unsolved regarding the number of independent transversals if the maximum degree is below the threshold $\Delta_r(n)$. In particular, Bollobás, Erdős and Szemerédi introduced the function $f_r(n)$ which is the largest number f such that every r-partite graph with parts of size n and maximum degree $\Delta_r(n) - 1$ has at least f independent transversals. Trivially $f_2(n) = n$. In [7] Bollobás, Erdős and Szemerédi determined $f_3(n)$ precisely and obtained that, quite surprisingly, $f_3(n) = 4$ for every $n \ge 4$. Jin [12] proved that $f_4(n) = \Theta(n^3)$, but the behaviour of the function $f_r(n)$ for $r \ge 5$ is a complete mystery. For r even we conjecture that $f_r(n) = \Theta(n^{r-1})$. For odd r the only thing we dare to predict is that $f_r(n) = O(n^{r-2})$; for the threshold $\Delta(r, n)$ is so "unnaturally" high for odd r. At this point even $f_r(n) = \Theta(1)$ is a possibility. It would certainly be very interesting to gain more information; maybe the structural theorems of the present paper could be of use.

Another, less precise goal is related to the alternative proof of $\Delta_r \ge \frac{r}{2(r-1)}$ through labeled triangulations and Sperner's Lemma, a method developed by Aharoni and others in e.g. [1] and [2]. It would be very desirable to understand the difference between the even and odd case by means of the topological properties of odd and even dimensional triangulated spheres.

Finally, we only stated the structural theorems of Section 3 for $r \ge 7$. It should certainly be possible to extend our methods and obtain results about the structure of slightly sub-optimal independent transversal-free examples for $r \le 6$, that is for $r = 4, 6$. Also, we did not investigate the stability of the optimal examples in the case of odd r. For $r = 3, 5$ the example is not unique; there are examples where the base graph is *not* the union of $r - 2$ bipartite graphs. We do not know what happens for odd $r \ge 7$.

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