Bounded transversals in multipartite graphs

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Abstract

Transversals in r-partite graphs with various properties are known to have many applications in graph theory and theoretical computer science. We investigate *f-bounded transversals* (or *f-BT*), that is, transversals whose connected components have order at most f. In some sense we search for the the sparsest f-BT-free graphs. We obtain estimates on the smallest maximum degree that 3-partite and 4-partite graphs without 2-BT can have and provide a greatly simplified proof of the best known general lower bound on the smallest maximum degree in f-BT-free graphs.

1 Introduction

Let G be a graph with a fixed partition of its vertex set. A *transversal* of G with respect to the given partition is a subset of vertices containing exactly one vertex from each partite set of the partition. In this paper we are interested in the subgraphs that are induced by transversals. Since no edge that joins two vertices in the same partite set is in any subgraph induced by a transversal, we will assume for convenience that the partite sets are independent, in other words G is a *multipartite graph*.

An *f-bounded transversal*, or *f-BT*, of a multipartite graph is a transversal T in which each component of the subgraph induced by T has at most f vertices. Let $\Delta_f(r,n)$ denote the smallest integer Δ such that there exists an r-partite graph with partite sets of size n, maximum degree Δ, and with no f-BT. Correspondingly we define $\Delta_f(n)$ when we do not want to restrict the number of parts in the graphs under consideration, that is, $\Delta_f(n) = \min \Delta_f(r,n)$. Finally, let Δ_f denote $\lim_{n\to\infty} \Delta_f(n)/n$ (this limit is easily seen to exist).

The study of the parameter $\Delta_f(r,n)$ was initiated by Bollobás, Erdős, and Szemerédi in 1975 [BES75], and in particular they asked for the determination of $\Delta_1(r,n)$ and $\Delta_1(n)$. This problem

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was addressed by many authors, and as a result of a series of works [BES75, Jin92, Yus97, Jin98, HST03, ST06, HS06], the precise value of $\Delta_1(r,n)$ is now known.

$$\Delta_1(r,n) = \begin{cases} \lceil \frac{rn}{2(r-1)} \rceil, & \text{if } r \text{ is even, and} \\ \lceil \frac{(r-1)n}{2(r-2)} \rceil, & \text{if } r \text{ is odd.} \end{cases} \tag{1}$$

This implies that $\Delta_1(n) = \lceil (n+1)/2 \rceil$. In this paper we are concerned with $\Delta_f(r,n)$ for more general values of f. The following theorem was first proved by Haxell, Szabó and Tardos in [HST03], and a different proof, based on topological methods, appears in [ST06] by Szabó and Tardos.

Theorem 1. *Let $f \geq 1$ and $n \geq 1$ be fixed integers, then $\Delta_f(n) > \frac{fn}{f+1}$.*

Both the arguments in [HST03] and [ST06] are quite technical. The aim of Section 2 is to give a much simpler proof of Theorem 1.

To complement Theorem 1, in [ST06] multipartite graphs are constructed for every f, that have small maximum degree and no f-BT.

Proposition 2 ([ST06])**.** *For all integers $f \geq 1$, $n \geq 2$, and $r \geq \sum_{i=0}^{f} n^i$, we have $\Delta_f(r,n) \leq n+1$.*

Hence one can conclude $\Delta_f(n) \leq n+1$ for every f.

The expression (1) for $\Delta_1(r,n)$ shows that Theorem 1 is asymptotically best possible for 1-bounded transversals (also called *independent transversals*). Already for 2-bounded transversals, subsequently called *matching transversals*, the asymptotic determination of $\Delta_2(n)$ remains open. The best known general lower bound to date is the $\frac{2}{3}n$ provided by Theorem 1. Trivially, for $n = 1$ it holds that $\Delta_2(1) = 2$, and for $n = 2$ it has been shown in [HST03] that $\Delta_2(2) = 3$. We conjecture that Proposition 2 is sharp for every $f \geq 2$.

Conjecture 3. $\Delta_2(n) = n + 1$.

Unfortunately, we don't even know whether $\Delta_2(n) = (1 + o(1))n$.

The statement of Theorem 1 for $f = 1$, i.e. $\Delta_1(n) > \frac{n}{2}$, is a fundamental result on independent transversals, with wide applications in combinatorics. Should the conjecture be true, this would be an easy consequence: Take a graph G of maximum degree Δ that is partitioned into vertex sets $V_1, \ldots, V_r$ of size at least 2Δ each. To find an independent transversal, refine arbitrarily the partition into classes $V_i = V_i' \cup V_i''$, such that $|V_i'|, |V_i''| \geq \Delta$ for each i. Then by the conjecture there is a matching transversal $v_i' \in V_i', v_i'' \in V_i''$. The auxiliary graph created by taking $G[\cup_{i=1}^{r} \{v_i', v_i''\}]$ and adding the auxiliary edges $v_i'v_i''$ is the union of two disjoint matchings, and hence two-colorable. Any of the two color classes forms an independent transversal of G. Observe also that via this argument for $\Delta = 2$, when the conjecture in fact is known to hold, one proves the existence of *two* disjoint independent transversals in any two-regular graph with a partition into sets of size four. It is widely believed that the selection of four pairwise disjoint independent transversals should also be possible [JT95, Problem 4.14].

In Section 3 we determine $\Delta_2(3,n)$, and find bounds for $\Delta_2(4,n)$. Our study of a fixed number of partite sets is motivated by the work of Jin [Jin92] on independent transversals; his precise results on $\Delta_1(4,n)$ and $\Delta_1(5,n)$ were fundamental in making the correct conjecture [ST06] about $\Delta_1(r,n)$ that eventually led to its determination [HS06].

Proposition 4. *For all integers $n \geq 1$, we have $\Delta_2(3,n) = \lceil 3n/2 \rceil$.*

A generalization of the upper bound construction in Proposition 4 leads to a slight improvement on Proposition 2 when $f = 2$.

Proposition 5. *For all integers $n \geq 2$, and $r > n$, we have $\Delta_2(r,n) \leq n+1$.*

Our results on $\Delta_2(4,n)$ are summarized as follows.

Theorem 6. *Let $n \geq 1$ be a fixed integer, then $\Delta_2(4,n) \geq 4n/3$. On the other hand for every n divisible by 7 there are 4-partite graphs G with parts of size n, $\Delta(G) = \frac{10n}{7}$, and no matching transversal. For general n it follows that $4n/3 \leq \Delta_2(4,n) \leq 10n/7+3$.*

Independent transversals are widely applicable in many combinatorial problems including problems involving the strong chromatic number [Alo92], the linear arboricity [Alo88, Ber08], list coloring [Hax01], or SAT [Hax09]. Matching transversals were used in relation to relaxed colorings of bounded degree graphs [HST03, BS, BS07].

In Section 4 we show how our study of $\Delta_2(r,n)$ can be viewed as a Turán-type problem for r-partite graphs, a topic that has been investigated by many authors over the years. In particular we highlight an apparent common theme of "finiteness" amongst these problems.

1.1 Notation

As usual we denote by $\Delta(G)$ the maximum degree of the graph G. We define the degree of a vertex v into a certain vertex subset W by $d_W(v)$. The *cross-degree* $d^\times(v_i,v_j)$ defined for two vertices v_i, v_j from distinct partite sets V_i and V_j is $d_{V_i}(v_j) + d_{V_j}(v_i)$. For a graph G and two sets of vertices $U, W \subseteq V(G)$, we say U *dominates* W in G if for every $v \in W$ there exists $u \in U$ such that $uv \in E(G)$. If U consists of only two vertices u' and u'', then we often say that the pair u', u'' dominates W in G instead of saying that U dominates W in G. Let $\mathcal{G}_r(n_1,\ldots,n_r)$ denote the family of r-partite graphs G where each of the partite sets V_i, with $i \in \{1,\ldots,r\}$, contains exactly n_i vertices. When $n_1 = \ldots = n_r = n$ we use the abbreviation $\mathcal{G}_r(n)$. For any subset W of vertices, we use the notation $G[W]$ to denote the subgraph of G induced by W, and $\Gamma(W) = \{y : wy \in E(G) \text{ for some } w \in W\}$ to denote the neighbourhood of W. A *component* C of a graph G is a maximal set of vertices of G (w.r.t. containment) such that $G[C]$ is connected.

2 Bounded transversals

We will derive Theorem 1 from the following more general result.

Theorem 7. *Let f and r be positive integers. Let G be an r-partite graph with a partition into vertex classes $V_1, \ldots, V_r$. Suppose G has no f-BT, but the graph $G_1 = G[V_2 \cup \ldots \cup V_r]$ has an f-BT. Then there exists a subset $S \subseteq \{1, \ldots, r\}$ of indices and a set $Z \subseteq \bigcup_{i \in S} V_i$ such that*

(i) $1 \in S$,

(ii) *Z dominates $\cup_{i \in S} V_i$ in G,*

(iii) $|Z| \leq \left(\frac{f+1}{f}\right)(|S| - 1)$,

(iv) *all components of $G[Z]$ contain at least $f+1$ vertices.*

Proof. We prove Theorem 7 by induction on r. The theorem is trivially true when $r \leq f$, so assume $r \geq f+1$ and that the statement is true for smaller values of r.

Let G be a graph as in the statement of the theorem. Then for every f-BT T of G_1 and every vertex v of V_1 the component $C_{v,T}$ of $G[T \cup \{v\}]$ containing v has at least $f+1$ vertices. Let $v \in V_1$ be an arbitrary vertex. We select an f-BT T of G_1 which minimizes the order of the component $C_{v,T} = C$. We form a new multi-partite graph H from G by

- removing the vertex set $W = \Gamma_G(C)$ from G (note that $C \subset W$),
- unifying the remaining vertices in $\cup_{V_j \cap C \neq \emptyset} V_j$ into one new vertex class Y_1 (and removing any edges inside Y_1),
- creating classes $Y_i = V_i \setminus W$ in H, for each index i with $V_i \cap C = \emptyset$.

Note that each class, apart from possibly Y_1, is nonempty because it still contains an element of T, and the remainder of T in these classes forms an f-BT of all classes of H except Y_1.

Case 1: $Y_1 = \emptyset$.

In this case set $S = \{j : V_j \cap C \neq \emptyset\}$ and $Z = C$. Then property (i) is valid, since $v \in C \cap V_1$. For (ii), observe that Z dominates all of $\cup_{j \in S} V_j$ in G, because $Y_1 = \emptyset$. Moreover, since C contains exactly one vertex from each class in S and C has at least $f+1$ vertices, we have $|Z| = |S| \leq (\frac{f+1}{f})(|S| - 1)$, verifying (iii). Finally, property (iv) holds because $Z = C$ consists of exactly one component of order at least $f+1$.

Case 2: $Y_1 \neq \emptyset$.

We plan to use induction, so first recall that H has an f-BT of all of its classes except Y_1. Now we verify that H does not have an f-BT of *all* of its classes. Suppose on the contrary that T' is an f-BT for H. Let z be the vertex of T' in Y_1. Then we have that in G, z is contained in some V_j with $V_j \cap C \neq \emptyset$. By the definition of H, there is no edge joining any vertex of T' (including

z) to any vertex of C. Thus if $z \in V_1$ we find that $T' \cup (C \setminus \{v\})$ is an f-BT of G, which is a contradiction. If $z \in V_j$ for some $j \neq 1$, let w be the vertex of C in V_j. Then $T^* = T' \cup (C \setminus \{v, w\})$ is an f-BT of G_1 with the property that the component of $T^* \cup \{v\}$ containing v is smaller than C. This contradicts our choice of T. We conclude that H has no f-BT.

Let t denote the number of vertex classes that intersect C, then $t \geq f+1$. Since H has $r-t+1 < r$ vertex classes, by the induction hypothesis applied to H and the class Y_1, there exist a set S' of indices containing 1 together with a vertex set $Z' \subseteq \cup_{j \in S'} Y_j$ that satisfies the conclusions (i)–(iv). We set $S = S' \cup \{j : V_j \cap C \neq \emptyset\}$ and $Z = Z' \cup C$. Then (i) holds. To check (ii), note that every vertex of $\cup_{i \in S} V_i$ that is also contained in $V(H)$ is dominated by a vertex of Z', and all the remaining vertices of $\cup_{i \in S} V_i$ are dominated by C. For (iii) we have $|S| = |S'| - 1 + t$ and $|Z| = |Z'| + t \leq (\frac{f+1}{f})(|S'| - 1) + t \leq (\frac{f+1}{f})(|S'| - 2 + t)$ since $t \leq (\frac{f+1}{f})(t-1)$. Therefore $|Z| \leq (\frac{f+1}{f})(|S| - 1)$ as required. Finally for (iv) note that since C has at least $f+1$ vertices, each component of $G[Z]$ has at least $f+1$ vertices. □

To see that Theorem 7 immediately implies Theorem 1, we assume on the contrary that there is a graph G with a vertex partition into classes of size n and maximum degree $\Delta \leq \frac{fn}{f+1}$, that has no f-BT. Suppose first that G_1 does contain an f-BT. Therefore according to Theorem 7 there exists the subset S and the subset Z such that conclusions (i)-(iv) hold. Observe here that the number of vertices in a graph of maximum degree Δ that can be dominated by a set of size at most $\left(\frac{f+1}{f}\right)(|S| - 1)$, is at most $\left(\frac{f+1}{f}\right)(|S| - 1)\Delta$. Since $\cup_{i \in S} V_i$ contains $n|S| \geq \left(\frac{f+1}{f}\right)\Delta|S| > \left(\frac{f+1}{f}\right)(|S| - 1)\Delta$ vertices, Z cannot dominate all of them, contradicting conclusion (ii) of Theorem 7. On the other hand, if G_1 does not contain an f-BT, then we repeat the above argument with G replaced by G_1. □

Remark. In fact in [HST03, ST06] Theorem 1 is proven with the lower bound $\Delta_f(n) \geq \frac{f(n+1)}{f+1}$, which for $f \geq 2$ is slightly larger (by one) for all but two residue classes (mod $f+1$). This stronger statement is a straightforward consequence of the following.

Theorem 8. *Let $f, n \geq 1$ and $r \geq 2$ be fixed integers, then $\Delta_f(r, n) \geq \frac{f}{f+1}\left(\frac{nr}{r-1} + 1 - \frac{1}{f}\right)$.*

Theorem 8 can be proved with a minor modification of the proof of Theorem 1, which takes into account the fact that for $f \geq 2$, some vertices of each component of Z with more than two vertices are dominated more than once.

3 Matching transversals

In this section we investigate 3- and 4-partite graphs. We start by establishing a couple of general facts.

Lemma 9. *A graph $G \in \mathcal{G}_r(n)$ with $r \in \{3, 4\}$ contains* no *matching transversal if and only if every pair of vertices $v_i \in V_i, v_j \in V_j$ from distinct partite sets of G dominates at least one of the other partite sets V_k with $k \in \{1, \ldots, r\} \setminus \{i, j\}$.*

Proof. G having a matching transversal is equivalent to the existence of r vertices $v_{i_1} \in V_{i_1}, \dots, v_{i_r} \in V_{i_r}$ from distinct partite sets, such that the edge set of $G[\{v_{i_1}, \dots, v_{i_r}\}]$ is a subset of $\{v_{i_1}v_{i_2}\}$ if $r = 3$, and of $\{v_{i_1}v_{i_2}, v_{i_3}v_{i_4}\}$ if $r = 4$. This in turn is equivalent to the existence of two vertices $v_{i_1} \in V_{i_1}$ and $v_{i_2} \in V_{i_2}$ in two distinct partite sets that have a common non-neighbor in each of the remaining partite set(s). This is just saying that v_{i_1} and v_{i_2} do not dominate any of the remaining partite sets. □

Corollary 10. *Let $G \in \mathcal{G}_3(n) \cup \mathcal{G}_4(n)$. Suppose there are vertices v_i and v_j from distinct partite sets of G such that*

$$d^{\times}(v_i, v_j) > 2\Delta(G) - n.$$

Then G has a matching transversal.
In particular, G has a matching transversal if $\Delta(G) < 3n/2$ and $d^{\times}(v_i, v_j) = 2n$.

Proof. The two vertices v_i and v_j together have less than n adjacent vertices outside of $V_i \cup V_j$. Hence they do not dominate any other partite set of G. According to Lemma 9, G contains a matching transversal.

□

3.1 Tripartite graphs

Here we give the proof of Proposition 4.

Proof of Proposition 4. The proof that $\Delta_2(3, n) \leq \lceil 3n/2 \rceil$ is postponed to Proposition 20 (with $r = 3$).

For the lower bound let $G \in \mathcal{G}_3(n)$ with partite sets V_1, V_2, V_3, that contains no matching transversal. In the following the three indices i, j and k denote three distinct integers from the set $\{1, 2, 3\}$. We first observe that for every vertex $v \in V_i$ there is a $j \in \{1, 2, 3\} \setminus \{i\}$, such that $d_{V_j}(v) = n$. Indeed, otherwise there is a vertex $v \in V_i$ that has a non-neighbor v_j in V_j and a non-neighbor v_k in V_k. Hence v_j and v_k do not dominate V_i, a contradiction according to Lemma 9.

Hence we can define the following refined partition $V_{1,2}, V_{1,3}, V_{2,1}, V_{2,3}, V_{3,1}$ and $V_{3,2}$ of the partition V_1, V_2, V_3 of G, see also Figure 1:

$$V_{i,j} = \{v \in V_i \mid d_{V_j}(v) = n\}.$$

We can assume the partition to be well-defined, as otherwise we have a vertex adjacent to every vertex of both other classes, making the maximum degree $2n$.

According to Corollary 10, $d^{\times}(v_i, v_j) < 2n$ for any pair of vertices v_i and v_j from distinct partite sets. Hence $V_{i,j}$ or $V_{j,i}$ is empty for every pair of indices i, j. Suppose $V_{1,2} = \emptyset$, then $V_{1,3} = V_1$. That means every vertex v of V_1 has degree n into V_3, which in turn implies that every vertex $v_3 \in V_3$ has degree n into V_1. Therefore $d^{\times}(v, v_3) = 2n$, a contradiction. □

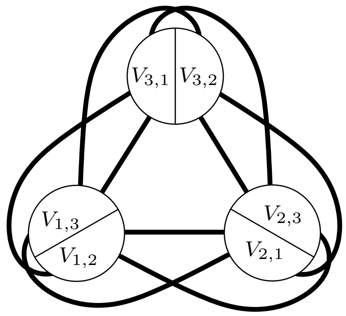

Figure 1: Graph G with refined partite sets $V_{i,j} \subseteq V_i$

3.2 Four-partite graphs

Here we aim to estimate $\Delta_2(4, n)$.

Proof of Theorem 6. Let $G \in \mathcal{G}_4(n)$. We define a directed graph $K(G)$ on the vertex set $V(K(G)) = \{V_1, V_2, V_3, V_4\}$ by putting an arc (V_i, V_j) in $K(G)$ if there exists a vertex $v_i \in V_i$ with $d_{V_j}(v_i) = n$. From Corollary 10 it immediately follows that if G has no matching transversal and $\Delta(G) < 3n/2$ then $K(G)$ contains no cycles of length two.

Lemma 11. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal. Suppose V_i is a vertex of out-degree at most 1 in $K(G)$. Let V_j be the out-neighbour of V_i if it exists, otherwise let V_j be an arbitrary class different from V_i. Let k and ℓ denote the indices in $\{1, \dots, 4\} \setminus \{i, j\}$ in arbitrary order. Then one of the following holds.*

(i) (V_ℓ, V_k) *is an arc of* $K(G)$*,*

(ii) (V_k, V_j) *is an arc of* $K(G)$*,*

(iii) (V_j, V_i) *is an arc of* $K(G)$*,*

(iv) There exists a vertex $w \in V_k$, such that $3\Delta(G) \geq 4n + d_{V_i}(w)$. In particular, $\Delta(G) \geq \frac{4}{3}n$.

Proof. Let $v \in V_j$ be a vertex with largest degree into V_i. If $d_{V_i}(v) = n$, then case *(iii)* holds. Hence assume that v has a non-neighbour x in V_i. Since there is no arc in $K(G)$ from V_i into $\{V_k, V_\ell\}$, we can find vertices $w \in V_k$ and $z \in V_\ell$ that are not adjacent to x. Then by Lemma 9, we see that v and w dominate V_ℓ, w and z dominate V_j, and v and z dominate V_k. Thus

$$d(v) + d(w) + d(z) \geq 3n + d_{V_i}(v) + d_{V_i}(w) + d_{V_i}(z).$$

Let us look at the vertex z. If $d_{V_k}(z) = n$ then *(i)* holds. Thus we may assume z has a non-neighbour $u \in V_k$. If $d_{V_j}(u) = n$ then *(ii)* holds, so we may assume u has a non-neighbour $t \in V_j$. Then z and t have a common non-neighbour in V_k, so by Lemma 9 we must have that z

and t dominate V_i. By choice of v we conclude that $d_{V_i}(z) \geq n - d_{V_i}(t) \geq n - d_{V_i}(v)$. Thus in particular if none of $(i) - (iii)$ hold then

$$3\Delta(G) \geq d(v) + d(w) + d(z) \geq 3n + n + d_{V_i}(w).$$

This implies $3\Delta(G) \geq 4n$ and therefore $\Delta(G) \geq 4n/3$. □

Corollary 12. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal, and let α be a constant such that $\Delta(G) < (1+\alpha)n$. Suppose V_i is a vertex of out-degree at most 1 in $K(G)$. Let V_j be the out-neighbour of V_i if it exists, otherwise let V_j be an arbitrary class different from V_i. Let k and ℓ denote the indices in $\{1, \ldots, 4\} \setminus \{i, j\}$. Suppose that none of (V_j, V_i), (V_ℓ, V_k), (V_k, V_j), (V_k, V_ℓ), and (V_ℓ, V_j) is an arc of $K(G)$. Then V_j contains a vertex y with $d_{V_i}(y) > (2 - 3\alpha)n$.*

Proof. We continue from conclusion (iv) of the previous lemma. We apply a similar argument to vertex w that was used there for vertex z. Since $(V_k, V_\ell) \notin E(K(G))$, we have a vertex $u' \in V_\ell$ that is not adjacent to w. Since $(V_\ell, V_j) \notin E(K(G))$, u' has a non-neighbour $y \in V_j$. Then w and y have a common non-neighbour in V_ℓ, so by Lemma 9 they must dominate V_i, so $d_{V_i}(w) \geq n - d_{V_i}(y)$. Hence

$$3(1+\alpha)n > 3\Delta(G) \geq 4n + d_{V_i}(w) \geq 5n - d_{V_i}(y).$$

This implies $d_{V_i}(y) > (2 - 3\alpha)n$, as required. □

The following proposition proves a strengthening of Theorem 6 for graphs $G \in G_4(n)$ when $K(G)$ contains at most one arc.

Proposition 13. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal.*
(a) If $K(G)$ has no arc then $\Delta(G) \geq \frac{11}{8}n$.
(b) If $K(G)$ has exactly one arc then $\Delta(G) \geq \frac{7}{5}n$.

Proof. First suppose $K(G)$ has no arc and $\Delta(G) < \frac{11}{8}n$. By Corollary 12, applied with $V_i = V_2$ and $V_j = V_1$, we find that V_1 contains a vertex v_1 with $d_{V_2}(v_1) \geq (2 - 3 \cdot \frac{3}{8})n$. Applying Corollary 12 with $V_i = V_1$ and $V_j = V_2$, we find that V_2 contains a vertex v_2 with $d_{V_1}(v_2) \geq (2 - 3 \cdot \frac{3}{8})n$. Thus we have that the cross-degree $d^\times(v_1, v_2) \geq (4 - 6 \cdot \frac{3}{8})n = \frac{14}{8}n > 2\Delta(G) - n$. Thus by Corollary 10 we have a matching transversal, a contradiction.

Let now (V_i, V_j) be the unique arc of $K(G)$ and assume that $\Delta(G) < \frac{7}{5}n$. Then Lemma 11 gives us a vertex $y \in V_j$ with $d_{V_i}(y) \geq (2 - 3 \cdot \frac{2}{5})n$. Together with the vertex $z \in V_i$ that has a full degree into V_2, we find a pair with cross-degree $d^\times(y, z) \geq (3 - 3 \cdot \frac{2}{5})n = \frac{9}{5}n > 2\Delta(G) - n$. Thus Corollary 10 establishes that there is a matching transversal in G, a contradiction. □

Now we deal with the cases when $K(G)$ has more than one arc.

Lemma 14. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal, and suppose $\Delta(G) < 4n/3$. Then it holds true that* *(i)* $\Delta^+(K(G)) = 3$, *or*

(ii) $\Delta^-(K(G)) \geq 2$.

Proof. Suppose on the contrary that there is a graph $G \in \mathcal{G}_4$, $\Delta(G) < 4n/3$, with no matching transversal, and $\Delta^+(K(G)) \leq 2$ and $\Delta^-(K(G)) \leq 1$. Without loss of generality let V_1 denote a vertex of $K(G)$ with maximum out-degree.

If $d^+(V_1) = 1$, then denote by V_2 the out-neighbor of V_1. Applying Lemma 11 with $V_i = V_1, V_j = V_2$, and $V_k = V_3$ we observe that case (i) must hold, i.e., the arc (V_4, V_3) is in $K(G)$. Indeed conclusion (ii) does not hold since vertex V_2 would have in-degree two, (iii) does not hold since then V_1 and V_2 would contain vertices with cross-degree $2n$, guaranteeing a matching transversal by Corollary 10, and (iv) contradicts the assumption $\Delta(G) < \frac{4}{3}n$. Using now Lemma 11 with $V_k = V_4$ instead, we conclude similarly that the arc (V_3, V_4) is also present in $K(G)$. Hence, again by Corollary 10 there is a matching transversal in G, a contradiction.

If $d^+(V_1) = 2$, then denote by V_2 the vertex in $K(G)$ that is not an out-neighbor of V_1. We apply Lemma 11 with $i = 2$, $j = 1$, $k =$ and $\ell = 4$. This is possible since V_2 can not have an arc going towards V_3 and V_4, otherwise these classes would have in-degree at least two in $K(G)$. Note (V_ℓ, V_k) is not an arc of $K(G)$, since then V_3 would have in-degree at least two. Also, (V_k, V_j) is not an arc, since then we would find a pair of vertices in V_1 and V_3 with cross-degree $2n$, contradicting that there is no matching transversal in G. Finally (V_j, V_i) is not an arc of $K(G)$ by the definition of V_2. Then by Lemma 11 we conclude the existence of a vertex in V_1 with degree larger than $\frac{4}{3}n$, contradicting our assumption. ☐

Let us next show that Conclusion (i) of Lemma 14 in fact implies Conclusion (ii) of the same lemma.

Lemma 15. *Let $G \in \mathcal{G}_4(n)$ with $\Delta(G) < 4n/3$ be a graph with no matching transversal such that there is a vertex V in $K(G)$ with $d^+_{K(G)}(V) = 3$. Then $\Delta^-(K(G)) \geq 2$.*

Proof. Let us assume without loss of generality that $V = V_4$ and choose three vertices $v_i \in V_4$ with $d_{V_i}(v_i) = n$, $i \in \{1, 2, 3\}$. Assume on the contrary that $\Delta^-(K(G)) = 1$. First we observe that $K(G)$ contains no other arcs than (V_4, V_i), $i \in \{1, 2, 3\}$. Arcs (V_i, V_4) are not present because that would create vertex pairs of cross-degree $2n$, other arcs are not possible because they would make the in-degree of V_1, V_2 or V_3 at least two.

Claim 16. *For every $v \in V_i$, $i \in \{1, 2, 3\}$, there is a $j \in \{1, 2, 3\} \setminus \{i\}$ such that $d_{V_j}(v) > 2n/3$.*

Proof. According to Lemma 9 the pair v, v_i dominates another part $V_j, j \in \{1, 2, 3\} \setminus \{i\}$. Due to the fact that $d_{V_i}(v_i) = n$, and thus $d_{V_j}(v_i) < 4n/3 - n = n/3$, it holds for v that $d_{V_j}(v) > 2n/3$. ☐

Since we assume $\Delta(G) < 4n/3$, we can classify every vertex $v \in V_i$ according to whether its degree to V_j or to V_k is larger than $2n/3$, for $\{i,j,k\} = \{1,2,3\}$. Hence we obtain a partition of V_i into classes $V_{i,m}$ with $m \in \{1,2,3\} \setminus \{i\}$ as follows,

$$V_{i,m} = \{v \in V_i \mid d_{V_m}(v) > 2n/3\}.$$

Claim 17. *For every vertex $v \in V_1 \cup V_2 \cup V_3$, $n/3 < d_{V_4}(v) < 2n/3$.*

Proof. Let $v \in V_i$. The inequality $d_{V_4}(v) < 2n/3$ immediately follows from Claim 16 and that $\Delta(G) < \frac{4}{3}n$. For the other direction, first we find a vertex $w \in V_j$ that is not adjacent to v, and then a vertex $z \in V_k$ that is not adjacent to w. Both of these vertices exist because there is no arc in $K(G)$ other than the arcs leaving V_4. Thus v and z must dominate V_4. According to Claim 16, $d_{V_\ell}(z) > 2n/3$, for some $\ell \in \{1,2,3\} \setminus \{j\}$ and hence $d_{V_4}(z) < 2n/3$. This implies $d_{V_4}(v) > n/3$. □

We are now ready to prove the lemma. We prove that $|V_{i,k}| + |V_{j,k}| < n$, for any i,j,k, $\{i,j,k\} = \{1,2,3\}$. To see this let w be a vertex in the common non-neighborhood of v_i and v_j in V_k. Such a vertex exists because each of v_i and v_j has at most $n/3$ neighbors in V_k. We claim that w is adjacent to every vertex of $V_{i,k}$ and of $V_{j,k}$. Indeed, any vertex $u \in V_{i,k}$ has degree less than $2n/3$ in V_j, since it has degree more than $2n/3$ in V_k. The vertex v_i has degree less than $n/3$ in V_j, since it has degree n in V_i. So v_i and u do not dominate V_j, so they must dominate V_k. Since w is not adjacent to v_i, the vertex u must be. An analogous argument shows that any vertex of $V_{j,k}$ is adjacent to w.

By Claim 17 we have $d_{V_4}(w) > n/3$, so the neighborhood of w in $V_i \cup V_j$, containing $V_{i,k} \cup V_{j,k}$, must have fewer than n vertices.

This provides a contradiction, since then the union of the sets $V_{1,2} \cup V_{3,2}$, $V_{2,1} \cup V_{3,1}$, and $V_{1,3} \cup V_{2,3}$, that is $V_1 \cup V_2 \cup V_3$ has fewer than $3n$ elements.

□

To complete the proof of the lower bound in Theorem 6 we show that Conclusion (*ii*) of Lemma 14 implies $\Delta(G) \geq 4n/3$.

Lemma 18. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal such that $\Delta^-_{K(G)} \geq 2$. Then $\Delta(G) \geq 4n/3$.*

Proof. Assume that $\Delta(G) < 4n/3$ and let V_3 be a class with in-degree at least two, with V_1 and V_2 its in-neighbors. Further denote by $v_i \in V_i$ a vertex with $d_{V_3}(v_i) = n$, $i \in \{1,2\}$. We first observe the following:

Claim 19. *No pair v_1, w with $w \in V_3$ dominates V_2.*

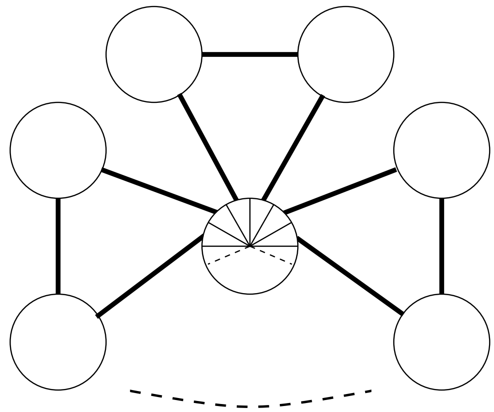

Figure 2: An r-partite graph with no matching transversal.

Proof. Suppose v_1, w dominate V_2. Since $d_{V_3}(v_1) = n$, and thus $d_{V_2}(v_1) < n/3$ we have that $d_{V_2}(w) > 2n/3$. Hence $d^{\times}(w, v_2) > 5n/3$, a contradiction to Corollary 10 and the fact $\Delta(G) < 4n/3$. $\square$

By the Claim v_1 and w should dominate V_4 for every $w \in V_3$. That means that all the non-neighbors of v_1 in V_4, and there are more than $2n/3$ of them, should be adjacent to *every* vertex in V_3. That means the cross-degree of any $w \in V_3$ and any non-neighbor of v_1 in V_4 is larger than $5n/3$, again a contradiction to Corollary 10. $\square$

This concludes the proof of Lemma 18 and the lower bound in Theorem 6. $\square$

3.3 Upper bounds

First we give a simple construction to show that the following holds.

Proposition 20. *For any two integers $r \geq 3, n \geq 1$, we have*

$$\Delta_2(r,n) \leq \begin{cases} n + \lceil \frac{n}{r-1} \rceil, & \textit{if } r \textit{ is odd,} \\ n + \lceil \frac{n}{r-2} \rceil, & \textit{if } r \textit{ is even.} \end{cases}$$

Proof. Assume first that r is odd. Let G be an r-partite graph such that $G[V_{2i} \cup V_{2i-1}] \cong K_{n,n}$, for $1 \leq i \leq (r-1)/2$. Partition the part V_r into $r-1$ almost equally sized parts $V_{r,i}$, with $\lfloor n/(r-1) \rfloor \leq |V_{r,i}| \leq \lceil n/(r-1) \rceil$, for $i \in \{1, \ldots, r-1\}$ and connect every vertex in V_j with every vertex in $V_{r,j}$, see Figure 2. Hence $\Delta(G) = n + \lceil n/(r-1) \rceil$. In case r is even, then simply add another part to G with n isolated vertices.

Obviously in every transversal T, the two vertices in $T \cap V_{2i}$ and $T \cap V_{2i-1}$ form an edge in G. Hence every vertex of V_r is adjacent to an endpoint of an edge of $G[T]$ and thus G does not contain a matching transversal. $\square$

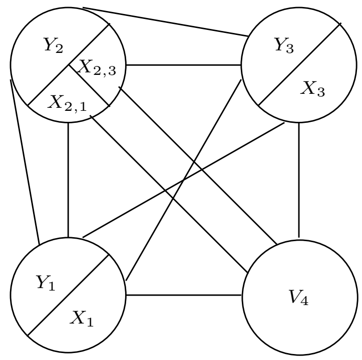

Figure 3: The construction

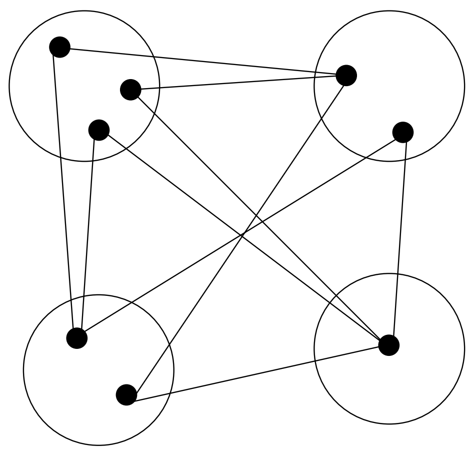

Figure 4: The graph J_0

Note here that Proposition 20 improves Proposition 2 for matching transversals in the following sense. For integers $n > 0$ and $r > n$, Proposition 20 shows the existence of r-partite graphs with parts of size n and maximum degree $\Delta(G) = n + 1$ without a matching transversal, as opposed to Proposition 2 which requires $r \geq \sum_{i=0}^{2} n^i$. This constitutes Proposition 5.

Now we construct a graph $G \in G_4(n)$ with no matching transversal and $\Delta(G) = 10n/7$, provided that n is divisible by 7. Consider the graph G in Figure 3. Here an edge indicates that the corresponding vertex classes are joined by a complete bipartite graph, in other words G is a blow-up of the graph J_0 shown in Figure 4. It is easy to check that J_0 has no matching transversal, thus neither does G.

For an optimal choice of the sizes of the vertex classes, namely $|X_1| = 4n/7, |Y_1| = 3n/7, |X_{2,1}| = n/7, |X_{2,3}| = n/7, |Y_2| = 5n/7, |X_3| = 4n/7, |Y_3| = 3n/7$, we observe that no vertex in the graph G has degree larger than $\Delta(G) = 10n/7$.

If 7 does not divide n, then let $n' > n$ denote the smallest integer divisible by 7. First suppose

$n' \geq 14$. Let G be the graph as in Figure 3 with partite sets containing n' vertices each and set $r = n' - n$. Then $r \leq 6$. Delete r vertices arbitrarily from each of V_4, Y_1, Y_2 and X_3. The resulting graph G' has no matching transversal, has n vertices in each class, and has maximum degree at most $\Delta(G) - r \leq 10(n+r)/7 - r = 10n/7 + 3r/7 < 10n/7 + 3$.

Otherwise $n \leq 6$. For $n \leq 5$ observe that $\lfloor 10n/7 + 3 \rfloor \geq 2n$, so the complete tripartite graph $K_{n,n,n}$ together with n isolated vertices is an example of a graph in $\mathcal{G}_4(n)$ with $\Delta \leq 10n/7 + 3$ and no matching transversal. For $n = 6$ we can construct a graph in $\mathcal{G}_4(6)$ with maximum degree 9 by blowing up the vertices of J_0 (see Figure 4) by the amounts 4, 1, 1 (for the class with 3 vertices), 3, 3 (classes with 2 vertices) and 6 (class with 1 vertex).

This concludes the proof of Theorem 6. □

4 Multipartite Turán-problems

The first study of transversal problems, by Bollobás, Erdős, and Szemerédi [BES75], considered the r-partite complement formulation, as follows. Let H be a fixed graph with r vertices and consider $G \in \mathcal{G}_r(n)$. We say that a (not necessarily induced) subgraph T of G is an *H-transversal* of G if $T \cong H$ and $V(T)$ has one vertex in each class of G. Then the parameter of interest is $\delta(n, H)$, the largest integer such that there exists an H-transversal-free graph from $\mathcal{G}_r(n)$ with minimum degree $\delta(n, H)$. Thus for example $\delta(n, K_r) = (r-1)n - \Delta_1(r, n)$ for each r. In [BES75] the first bounds on $\delta(n, K_r)$ were obtained, and the first exact value found was $\delta(n, K_3) = n$, by Graver (cf [BES75]). Now that $\Delta_1(r, n)$ is known for every r, the quantities $\delta(n, K_r)$ are also known, but for other graphs H not much is known about $\delta(n, H)$. For cycles, of course K_3 is also the cycle C_3 so $\delta(n, C_3) = n$. Our problem of finding $\Delta_2(4, n)$ is equivalent to finding $\delta(n, C_4)$, in particular we show that $\frac{11n}{7} - 3 \leq \delta(n, C_4) \leq \frac{5n}{3}$. It would be interesting to investigate the problem for $H = C_l$, and determine the limit of $\lim_{n\to\infty} \delta(n, C_l)/n$, for general l.

The problem of finding $\delta(n, H)$ could be viewed as a Turán-type problem for r-partite graphs (where $r = |V(H)|$), in the sense that one seeks the densest graph in $\mathcal{G}_r(n)$ containing no subgraph isomorphic to H. Here density is measured in terms of the minimum degree. One might ask the same question using the usual edge density $\frac{|E(G)|}{\binom{r}{2}n^2}$ as a measure of the density of $G \in \mathcal{G}_r(n)$, but this question turns out to be equivalent to finding the usual Turán number $ex(r, H)$, the largest integer m such that there exists an H-free graph J on r vertices with m edges. (This follows, for example, from the work of Bollobás, Erdős, and Straus [BES74]). Extremal examples are given by blow-ups of any J with $ex(r, H)$ edges, by blowing up each vertex of J to a set of n vertices.

Another notion of density for the multi-partite Turán problem was considered by Bondy, Shen, Thomassé and Thomassen [BSTT06]. Instead of the minimum degree, they considered the *minimum edge density* of the bipartite subgraphs induced by the r-partition. They proved in particular that if all three edge densities in a tripartite graph are at least the golden ratio then the graph contains a triangle. The extremal examples for this problem (K_3-transversal-free graphs that attain the maximum number of edges in the limit) are also blow-ups of a single graph, in this case the tripartite 5-cycle.

When *minimum degree* measures the density of K_r-transversal-free graphs (or in the complement formulation independent transversal-free graphs) the critical minimum degree $\delta(n, K_r)$ is again established by a graph that is a blow-up of a single graph refining the given partition [ST06, HS06]. However, we do not know whether C_4 exhibits a similar behaviour. Our constructions in the previous section are all blow-ups as well.

Conjecture 21. *For n large enough, the critical minimum degree $\delta(n, C_4)$ is attained by a blowup of the 4-partite complement of J_0 (see Figure 4).*

As a consequence we would obtain $\Delta_2(4, n) = \frac{10}{7}n + O(1)$. We were able to verify this conjecture only for a special sub-class of $\mathcal{G}_4(n)$, consisting of those graphs in $\mathcal{G}_4(n)$ for which $\Gamma(v) = \Gamma(u)$ for all $u, v \in V_4$ (see [Ber08]).

Finally, we believe that a "finiteness" phenomenon similar to that stated in Conjecture 21 should exist for arbitrary graphs H.

Conjecture 22. *For every graph H with l vertices, there is a constant $r(H)$ and integer $n_0(H)$ such that for every $n \geq n_0(H)$ there exists a graph G on n vertices with $\delta(G) = \delta(n, H)$ such that G is a blow-up of an H-transversal-free l-partite graph $J \in \mathcal{G}(r_1, \dots, r_l)$ with $r_1 + \cdots + r_l = r(H)$.*

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