Dense graphs with cycle neighborhoods

ÁKOS SERESS AND TIBOR SZABÓ

ABSTRACT. For all $\varepsilon > 0$, we construct graphs with n vertices and $> n^{2-\varepsilon}$ edges, for arbitrarily large n, such that the neighborhood of each vertex is a cycle. This result is asymptotically best possible.

1. Introduction

For a graph $G(V, E)$ and $v \in V$, we denote by N_v the subgraph of G spanned by the neighborhood of v. Let $\mathcal{P}$ be a family of graphs. $G(V, E)$ is said to satisfy *local property* $\mathcal{P}$ if for all $v \in V$, $N_v \in \mathcal{P}$.

Early investigations dealt mostly with the case $|\mathcal{P}| = 1$, i.e., when all neighborhoods are isomorphic. The major question is the existence of an appropriate G. Good summaries of results of this type can be found in the survey papers Hell [6] and Sedlacek [13].

More recently, the cases when $\mathcal{P}$ consists of all cycles, all paths, or all matchings were investigated. At these instances, the question of existence is easy; so the main focus is on the following extremal problem. What is the maximum number of edges, $e = e(n)$, in graphs on n vertices satisfying one of these local properties?

For 3-connected planar graphs, Zelinka [15, 16] proved that $e(n) = 2n + 3\lfloor n/4 \rfloor - 6$ and $e(n) = (12/5)(n-2)$ in the cases of paths and matchings, respectively. Obviously, $e(n) = 3n - 6$ in the cycle case.

For general graphs, Clark, Entringer, McCanna, and Székely [4] showed, using a result of Ruzsa and Szemerédi [12], that $e(n) = o(n^2)$ in all three instances. Ruzsa and Szemerédi also gave graphs satisfying the local property 'matching' and having $> cn^{2-\frac{c'}{\sqrt{\log n}}} > n^{2-\varepsilon}$ edges (for a simple construction, see [4]). For the cases of paths and cycles, it is shown in [4] that $e(n) > cn \log n$, by adding diagonals to certain two-dimensional faces of hypercubes.

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The purpose of the present paper is to close the gap between the lower and upper bound in the cycle and path case. Our main result is

THEOREM 1.1. *There exists an infinite sequence S of integers and an absolute constant $c > 0$ such that for each $n \in S$, there is a graph on n vertices with $> n^{2-\frac{c}{\sqrt{\log\log n}}}$ edges and satisfying the local property 'cycle'.*

Graphs with cycle neighborhoods look locally as plane triangulations, so the minimum genus embeddings of the graphs constructed in Theorem 1.1 give a triangulation of the surface. These triangulations are automatically *clean triangulations*, as defined by Hartsfield and Ringel [5]: every face is a triangle, and every triangle of the graph is a face. Hartsfield and Ringel gave clean triangulations of genus g surfaces with $(4+o(1))g$ faces. (Later, Archdeacon [1] gave clean triangulations with $(4+o(1))g$ faces and with minimal essential cycle length $> k$, for all fixed k.) The condition that the number of faces is $(4+o(1))g$ is equivalent to $|V| = o(|E|)$ for the embedded graph $G(V,E)$; so, their graphs are dense in the sense that the number of edges is nonlinear. Our graphs provide clean triangulations essentially as dense as possible.

Topological properties of graphs with cycle neighborhoods were investigated e.g. by Malnic and Mohar [9]. That paper also describes the relation with the notion *representativity of graphs*, studied, among others, in [10],[11],[14].

A modification of the construction for Theorem 1.1 gives the following.

THEOREM 1.2. *There exists an infinite sequence S of integers and an absolute constant $c > 0$ such that for each $n \in S$, there is a graph on n vertices with $> n^{2-\frac{c}{\sqrt{\log\log n}}}$ edges and satisfying the local property 'path'.*

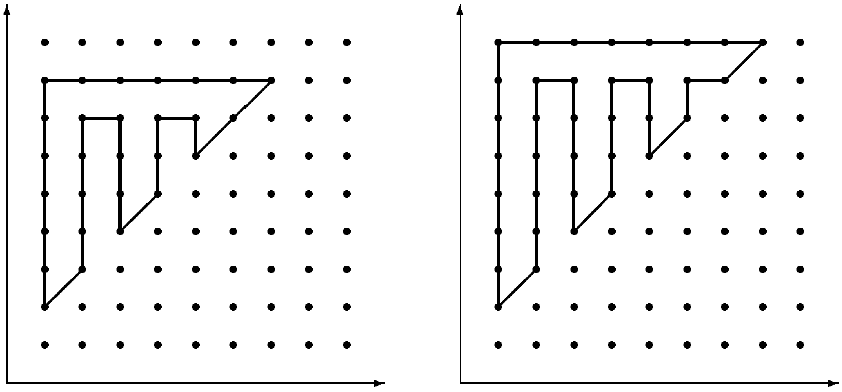
2. A hamiltonian graph

In this section, which serves as a preparation, we define an auxiliary graph $H(X,k)$ and prove that it is hamiltonian. Let X be a finite set, $|X| \geq 2$, and $k \geq 2$ an integer. The vertex set $V(X,k)$ of $H(X,k)$ is defined as the set of sequences of length $k|X|$ containing each element of X exactly k times: $V(X,k) = \{(v_1, v_2, \ldots, v_{k|X|}) : \forall x \in X(|\{i : v_i = x\}| = k)\}$. $\boldsymbol{u}, \boldsymbol{v} \in V(X,k)$ are adjacent if and only if they agree in exactly $k|X|-2$ positions (and, consequently, for the two exceptional coordinates i, j, $\boldsymbol{u}_i = \boldsymbol{v}_j$ and $\boldsymbol{u}_j = \boldsymbol{v}_i$).

THEOREM 2.1. *For all X, k, the graph $H(X,k)$ is hamiltonian.*

There are numerous algorithms for generating all permutations of a multiset, and some of them (cf. [3],[7],[8]) provide a sequence of permutations such that consecutive ones are obtained from each other by a transposition. In other words, these algorithms show that the graph $H(X,k)$ has a hamiltonian path. We have to prove slightly more; namely, the existence of a hamiltonian cycle.

For the proof, we need the following lemma. Let $V_k^n \subset \{0,1\}^n$ denote the set of 0-1 vectors containing exactly k 1's. We say that $\boldsymbol{u} \in V_k^n$ can be obtained



Case of even n Case of odd n

FIGURE 1

from $\boldsymbol{v} \in V_k^n$ by *shifting a block of* 1*'s* if $\boldsymbol{u}, \boldsymbol{v}$ agree in all but two coordinates $i < j$ and $\boldsymbol{u}_l = \boldsymbol{v}_l = 1$ for all $i < l < j$.

LEMMA 2.2. *For every $k \geq 2$ and $n \geq k$ there exists a cyclic ordering $\boldsymbol{v}_k^n[0]$, $\boldsymbol{v}_k^n[1], \ldots, \boldsymbol{v}_k^n[\binom{n}{k} - 1]$ of V_k^n such that*

$$\begin{aligned}
\boldsymbol{v}_k^n[\tbinom{n}{k} - 1] &= (\underbrace{1, 1, \ldots, 1}_{k-1}, 0, 1, 0 \ldots, 0), \\
\boldsymbol{v}_k^n[0] &= (\underbrace{1, 1, \ldots, 1, 1}_{k}, 0, 0, \ldots, 0), \\
\boldsymbol{v}_k^n[1] &= (0, \underbrace{1, 1, \ldots, 1, 1}_{k}, 0, \ldots, 0),
\end{aligned}$$

and for each $0 \leq i \leq \binom{n}{k} - 1$, $\boldsymbol{v}_k^n[i+1]$ can be obtained from $\boldsymbol{v}_k^n[i]$ by shifting a block of 1's. (*We use the convention $\boldsymbol{v}_k^n[\binom{n}{k}] = \boldsymbol{v}_k^n[0]$.*)

PROOF. We will prove the Lemma by induction on k. In the case $k = 2$, there is a natural bijection between V_2^n and the grid points (i, j) with $1 \leq i < j \leq n$, the grid point (i, j) corresponding to $\boldsymbol{v} \in V_2^n$ with $\boldsymbol{v}_i = \boldsymbol{v}_j = 1$ (and $\boldsymbol{v}_l = 0$ for all other coordinates). In this representation on grid points, shifting a block of 1's corresponds to one step parallel to a coordinate axis (for a block of length 1) or one step on the line $y = x + 1$ (for a block of length 2). A required cyclic ordering can be obtained from Figure 1.

Let us now assume that $k \geq 3$ and we proved the statement for $k - 1$. First, we notice that concatenating the vector $(1, 0, 0, ..., 0)$ of length $n - m$ to the elements of V_{k-1}^m, we obtain the set $V_k^{n,m+1}$ of those $\boldsymbol{v} \in V_k^n$ in which the last nonzero coordinate is in the $m{+}1^{\text{st}}$ position. Also, using the induction hypothesis for V_{k-1}^m, we obtain a cyclic ordering $\boldsymbol{v}_k^{n,m+1}[0], \boldsymbol{v}_k^{n,m+1}[1], \ldots, \boldsymbol{v}_k^{n,m+1}[\binom{m}{k-1} - 1]$ of $V_k^{n,m+1}$. Finally, we observe that $\boldsymbol{v}_k^{n,m+2}[1]$ and $\boldsymbol{v}_k^{n,m+2}[\binom{m+1}{k-1} - 1]$ can be obtained from $\boldsymbol{v}_k^{n,m+1}[1]$ and $\boldsymbol{v}_k^{n,m+1}[\binom{m}{k-1}-1]$, respectively, by shifting a block of 1's of length 1.

Using the above observation, we define the cyclic ordering of V_k^n as follows. For $1 \le i \le \binom{k}{k-1}-1$, $\boldsymbol{v}_k^n[i]$ is defined as the i^{th} element of $V_k^{n,k+1}$. The next $\binom{k+1}{k-1}-1$ elements are from $V_k^{n,k+2} \setminus \{\boldsymbol{v}_k^{n,k+2}[0]\}$, in the reverse ordering from $\boldsymbol{v}_k^{n,k+2}[\binom{k+1}{k-1}-1]$ to $\boldsymbol{v}_k^{n,k+2}[1]$. The next $\binom{k+2}{k-1}-1$ elements are from $V_k^{n,k+3} \setminus \{\boldsymbol{v}_k^{n,k+3}[0]\}$ in the forward ordering, etc. Finally, we use the elements $\boldsymbol{v}_k^{n,n}[0], \boldsymbol{v}_k^{n,n-1}[0], ..., \boldsymbol{v}_k^{n,k+1}[0]$ to get back to $\boldsymbol{v}_k^n[0] = \boldsymbol{v}_k^{n,k}[0] = (\underbrace{1,1,...,1}_{k},0,0,...0)$.

Formally, let

$$\boldsymbol{v}_k^n[j] = \begin{cases} v_k^{n,m+1}[i], & \text{if } j = \sum\limits_{l=k-1}^{m-1}(\binom{l}{k-1}-1)+i \text{ for some } k \le m \le n-1 \\ & \text{and } 1 \le i \le \binom{m}{k-1}-1 \text{ with } m-k \text{ even,} \\ v_k^{n,m+1}[\binom{n}{k}-i], & \text{if } j = \sum\limits_{l=k-1}^{m-1}(\binom{l}{k-1}-1)+i \text{ for some } k \le m \le n-1 \\ & \text{and } 1 \le i \le \binom{m}{k-1}-1 \text{ with } m-k \text{ odd,} \\ v_k^{n,k+\binom{n}{k}-j}[0], & \text{if } \binom{n}{k}-n+k \le j \le \binom{n}{k}. \end{cases}$$

□

Now we are ready to prove Theorem 2.1 by induction on $|X|$. For the case $|X| = 2$, Lemma 2.2 gives an even stronger result than needed about $V_k^{2k} \cong V(X,k)$.

Next, suppose that $|X| \ge 3$, and write X in the form $X = \{x\} \cup X'$, $|X'| = |X|-1$. By the induction hypothesis, there is a hamiltonian cycle $\boldsymbol{u}[0], \boldsymbol{u}[1], ..., \boldsymbol{u}[m-1]$, $m = (k|X'|)!/((k!)^{|X'|})$ in $H(X',k)$ and, by Lemma 2.2, a cyclic ordering $\boldsymbol{v}[0], \boldsymbol{v}[1], ..., \boldsymbol{v}[l-1]$, $l = \binom{k|X|}{k}$ in $V_k^{k|X|}$.

There is a natural bijection between $V(X,k)$ and $V(X',k) \times V_k^{k|X|}$ defined as follows. Given $\boldsymbol{z} \in V(X,k)$, the positions of the k occurences of x in $\boldsymbol{z}$ determine an element $\boldsymbol{v} \in V_k^{k|X|}$ and deleting these k coordinates from $\boldsymbol{z}$ we obtain an element $\boldsymbol{u} \in V(X',k)$. We shall use the notation $\boldsymbol{z} = (\boldsymbol{u}, \boldsymbol{v})$ to denote this correspondence.

It is clear from the definition of the graph $H(X,k)$ that for all $0 \le i < m$ and $\boldsymbol{v} \in V_k^{k|X|}$, $\boldsymbol{z} = (\boldsymbol{u}[i], \boldsymbol{v})$ and $\boldsymbol{z'} = (\boldsymbol{u}[i+1], \boldsymbol{v})$ are connected in $H(X,k)$. Also, for all $0 \le i < l$ and $\boldsymbol{u} \in V(X',k)$, $\boldsymbol{z} = (\boldsymbol{u}, \boldsymbol{v}[i])$ and $\boldsymbol{z'} = (\boldsymbol{u}, \boldsymbol{v}[i+1])$ are connected in $H(X,k)$ since $\boldsymbol{z'}$ can be obtained from $\boldsymbol{z}$ by shifting a block of x's and so $\boldsymbol{z}, \boldsymbol{z'}$ differ in exactly two coordinates. Finally, we note that $m = |V(X',k)|$ is even.

Using the observations from the previous paragraph, it is clear that

$(\boldsymbol{u}[0], \boldsymbol{v}[0]), (\boldsymbol{u}[0], \boldsymbol{v}[1]), \dots, (\boldsymbol{u}[0], \boldsymbol{v}[l-1])$,
$(\boldsymbol{u}[1], \boldsymbol{v}[l-1]), (\boldsymbol{u}[1], \boldsymbol{v}[l-2]), \dots, (\boldsymbol{u}[1], \boldsymbol{v}[0])$,
$(\boldsymbol{u}[2], \boldsymbol{v}[0]), (\boldsymbol{u}[2], \boldsymbol{v}[1]), \dots, (\boldsymbol{u}[2], \boldsymbol{v}[l-1])$,
$\vdots$

$(\boldsymbol{u}[m-1], \boldsymbol{v}[l-1]), (\boldsymbol{u}[m-1], \boldsymbol{v}[l-2]), \ldots, (\boldsymbol{u}[m-1], \boldsymbol{v}[0])$
is a hamiltonian cycle in $H(X,k)$. $\Box$

3. Equations in $V(X,k)$

In this section we specify the choice of the set X in the construction of the graph $H(X,k)$ so that $V(X,k)$ becomes a subset of a vector space. Hence we can talk about adding the elements of $V(X,k)$. The main result of the section is that certain equations have only few solutions in $V(X,k)$.

Let d be a positive integer such that $p = 8d+1$ is prime. Also, let $F \subset (d, 2d] = \{d+1, d+2, ..., 2d\}$ be a set such that F contains no arithmetic progression of length 3. Finally, we define $X \subset GF(p)$ as $X = F \cup (-F)$. In the following proposition, all additions are defined in $GF(p)$.

Proposition 3.1. *Let $z \in X$ be fixed. Then the equation $x + y = 2z$ has a unique solution $(x,y) \in X \times X$, namely $x = y = z$.*

Proof.

$$\begin{aligned} \text{If } x, y \in F \text{ then} \quad & x + y \in (2d, 4d]; \\ \text{if } x, y \in -F \text{ then} \quad & x + y \in (4d, 6d); \\ \text{otherwise} \quad & x + y \in (-d, d). \end{aligned}$$

So, if $x + y = 2z$ for some $z \in F$ then $2z \in (2d, 4d]$ and both of x, y must be in F. But F is free of arithmetic progressions of length 3, so the only possibility is $x = y = z$.

Similarly, if $z \in -F$ then both of x, y must be in $-F$. $-F$ is free of arithmetic progressions as well, implying that $x = y = z$. $\Box$

By Theorem 2.1, the graph $H(X,k)$ is hamiltonian. Let us fix a hamiltonian circuit $\boldsymbol{h}[1], \boldsymbol{h}[2], \ldots, \boldsymbol{h}[m]$, $m = (k|X|)!/((k!)^{|X|})$. To simplify later notation, we also assume that $\boldsymbol{h}[1]$ can be obtained from $\boldsymbol{h}[m]$ by exchanging the last two coordinates.

$V(X,k)$ can be considered as a subset of the vector space $GF(p)^{k|X|}$. For $\boldsymbol{v} \in GF(p)^{k|X|}$, we denote by $\boldsymbol{v}^-$ the vector obtained from $\boldsymbol{v}$ by deleting the last two coordinates and by $\boldsymbol{v}^*$ the sum of the last two coordinates of $\boldsymbol{v}$.

In the following lemma, additions are defined in the vector space $GF(p)^{k|X|}$. Also, we use the convention $\boldsymbol{h}[m+1] = \boldsymbol{h}[1]$.

Lemma 3.2. (i) *For all $1 \le j \le m$, the equation $\boldsymbol{x} + \boldsymbol{y} = 2\boldsymbol{h}[j]$ has a unique solution $(\boldsymbol{x}, \boldsymbol{y}) \in V(X,k) \times V(X,k)$, namely $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{h}[j]$.*

(ii) *For all $1 \le j \le m$, the equation $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{h}[j] + \boldsymbol{h}[j+1]$ has two solutions $(\boldsymbol{x}, \boldsymbol{y}) \in V(X,k) \times V(X,k)$, namely $\boldsymbol{x} = \boldsymbol{h}[j], \boldsymbol{y} = \boldsymbol{h}[j+1]$ and $\boldsymbol{x} = \boldsymbol{h}[j+1], \boldsymbol{y} = \boldsymbol{h}[j]$.*

(iii) *The only solution of the equation $(\boldsymbol{h}[j] + \boldsymbol{h}[j+1])^- = 2\boldsymbol{h}[1]^-$ is $j = m$.*

PROOF. (i) For each $1 \le i \le k|X|$, Proposition 3.1 implies that the i^{th} coordinate of $\boldsymbol{x}$ and $\boldsymbol{y}$ must be the same as the i^{th} coordinate of $\boldsymbol{h}[j]$. Hence $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{h}[j]$.

(ii) $\boldsymbol{h}[j]$ and $\boldsymbol{h}[j+1]$ agree in $k|X|-2$ coordinates i and, by Proposition 3.1, the i^{th} coordinate of $\boldsymbol{x}$ and $\boldsymbol{y}$ must be the same as of $\boldsymbol{h}[j]$ for these i. Also, each element of X must occur exactly k times as coordinate of $\boldsymbol{x}$ (and $\boldsymbol{y}$). Therefore, it is uniquely determined which two elements of X occur in the remaining two coordinates of $\boldsymbol{x}$. The two possible orderings of these two elements give the solutions stated in the Lemma.

(iii) By Proposition 3.1, $\boldsymbol{h}[j], \boldsymbol{h}[j+1]$, and $\boldsymbol{h}[1]$ agree in the first $k|X|-2$ coordinates. Also, the two elements of X occuring in the last two positions in $\boldsymbol{h}[j], \boldsymbol{h}[j+1]$, and $\boldsymbol{h}[1]$ must be the same; therefore, one of $\boldsymbol{h}[j], \boldsymbol{h}[j+1]$ must be equal to $\boldsymbol{h}[1]$ and the other one can be obtained by exchanging the last two coordinates of $\boldsymbol{h}[1]$. □

4. Cycle neighborhoods

In this section, we define graphs $G(p,k)$ such that the neighborhood of each vertex spans a cycle. Theorem 1.1 can be obtained by the appropriate choice of the parameters p, k.

We start with the definition of $G(p,k)$. Let $p = 8d+1$ and $X \subset GF(p)$ as defined in Section 3. The vertex set of $G(p,k)$ is defined as $V(p,k) = A \cup B \cup C \cup D$, where $A \cong B \cong C \cong GF(p)^{k|X|}$ and $D \cong GF(p)^{k|X|-1}$.

Before defining the edge set $E(p,k)$ of $G(p,k)$, we introduce some further notation. Since we work with three copies of $GF(p)^{k|X|}$, we write $\boldsymbol{v}_A, \boldsymbol{v}_B, \boldsymbol{v}_C$ if we consider $\boldsymbol{v} \in GF(p)^{k|X|}$ as an element of A, B, or C, respectively. The elements of D will be always written in the form $(\boldsymbol{d}, t)$ for some $\boldsymbol{d} \in GF(p)^{k|X|-2}$ and $t \in GF(p)$. Finally, recall that we denote by $\boldsymbol{v}^-$ the vector obtained from $\boldsymbol{v} \in GF(p)^{k|X|}$ by deleting the last two coordinates and by $\boldsymbol{v}^*$ the sum of the last two coordinates of $\boldsymbol{v}$. Also, $\boldsymbol{h}[1], \boldsymbol{h}[2], \ldots, \boldsymbol{h}[m]$, $m = (k|X|)!/((k!)^{|X|})$ is a hamiltonian circuit in $H(X,k)$ and $\boldsymbol{h}[1], \boldsymbol{h}[m]$ differ in the last two coordinates.

The edge set $E(p,k)$ of $G(p,k)$ is defined as follows. The sets $A \cup D$, B, C are independent. Let $\boldsymbol{a} \in A, \boldsymbol{b} \in B, \boldsymbol{c} \in C, (\boldsymbol{d}, t) \in D$. Then

$$\begin{aligned}
\{\boldsymbol{a},\boldsymbol{b}\} &\in E(p,k) && \text{iff } \boldsymbol{a}-\boldsymbol{b} \in V(X,k) \ (\leftrightarrow\ \boldsymbol{b}-\boldsymbol{a} \in V(X,k)) \\
\{\boldsymbol{a},\boldsymbol{c}\} &\in E(p,k) && \text{iff } \boldsymbol{a}-\boldsymbol{c} \in V(X,k) \ (\leftrightarrow\ \boldsymbol{c}-\boldsymbol{a} \in V(X,k)) \\
\{\boldsymbol{b},\boldsymbol{c}\} &\in E(p,k) && \text{iff (i) } \boldsymbol{c}-\boldsymbol{b} = \boldsymbol{h}[j]+\boldsymbol{h}[j+1] \ \text{ for some } 1 \le j \le m-1 \\
& && \text{or (ii) } \boldsymbol{c}-\boldsymbol{b} = 2\boldsymbol{h}[1] \ \text{ or } \boldsymbol{c}-\boldsymbol{b} = 2\boldsymbol{h}[m] \\
\{\boldsymbol{b},(\boldsymbol{d},t)\} &\in E(p,k) && \text{iff } \boldsymbol{b}^- = \boldsymbol{d} \text{ and } \boldsymbol{b}^* = t \\
\{\boldsymbol{c},(\boldsymbol{d},t)\} &\in E(p,k) && \text{iff } (\boldsymbol{c}-2\boldsymbol{h}[1])^- = \boldsymbol{d} \text{ and } (\boldsymbol{c}-2\boldsymbol{h}[1])^* = t
\end{aligned}$$

If $Y \subset V(p,k)$ and $\boldsymbol{z} \in V(p,k)$ then we denote the set of neighbors of $\boldsymbol{z}$ in Y by $Y_{\boldsymbol{z}}$. Recall that $N_{\boldsymbol{z}}$ denotes the subgraph of $G(p,k)$ spanned by the neighbors of $\boldsymbol{z}$.

LEMMA 4.1. *For each $\boldsymbol{a} \in A$, $N_{\boldsymbol{a}}$ is a cycle of length $2m$.*

PROOF. By the definition of $E(p,k)$, $B_{\boldsymbol{a}} = \{(\boldsymbol{a} - \boldsymbol{h}[i])_B \ : \ 1 \le i \le m\}$ and $C_{\boldsymbol{a}} = \{(\boldsymbol{a} + \boldsymbol{h}[l])_C \ : \ 1 \le l \le m\}$.

$(\boldsymbol{a} - \boldsymbol{h}[i])_B$ is adjacent to $(\boldsymbol{a} + \boldsymbol{h}[l])_C$ if and only if (i) there exists $1 \le j \le m-1$ such that $(\boldsymbol{a}+\boldsymbol{h}[l]) - (\boldsymbol{a} - \boldsymbol{h}[i]) = \boldsymbol{h}[j] + \boldsymbol{h}[j+1]$ or (ii) $(\boldsymbol{a}+\boldsymbol{h}[l]) - (\boldsymbol{a} - \boldsymbol{h}[i]) = 2\boldsymbol{h}[j]$ for some $j \in \{1, m\}$.

In case (i), we obtain that $\boldsymbol{h}[l] + \boldsymbol{h}[i] = \boldsymbol{h}[j] + \boldsymbol{h}[j+1]$. By Lemma 3.2(ii), the only solutions are $l = j,\ i = j+1$ and $l = j+1,\ i = j,\ 1 \le j \le m-1$. This means that

$$\{(\boldsymbol{a} - \boldsymbol{h}[i])_B, (\boldsymbol{a} + \boldsymbol{h}[i-1])_C\} \in E(p,k) \text{ for } 2 \le i \le m$$

and

$$\{(\boldsymbol{a} - \boldsymbol{h}[i])_B, (\boldsymbol{a} + \boldsymbol{h}[i+1])_C\} \in E(p,k) \text{ for } 1 \le i \le m-1.$$

In case (ii), we obtain that $\boldsymbol{h}[l] + \boldsymbol{h}[i] = 2\boldsymbol{h}[j]$. By Lemma 3.2(i), the only solutions are $l = i = j = 1$ and $l = i = j = m$. These solutions define two more edges in $N_{\boldsymbol{a}}$:

$$\{(\boldsymbol{a} - \boldsymbol{h}[m])_B, (\boldsymbol{a} + \boldsymbol{h}[m])_C\} \text{ and } \{(\boldsymbol{a} - \boldsymbol{h}[1])_B, (\boldsymbol{a} + \boldsymbol{h}[1])_C\}.$$

Thus, altogether there are $2m$ edges in $N_{\boldsymbol{a}}$ and they form a cycle as indicated on Figure 2.

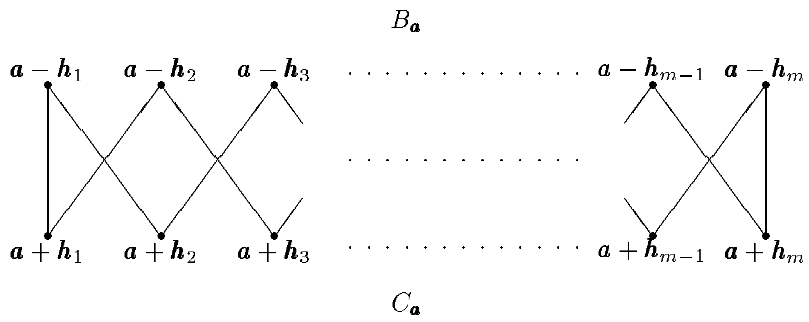


FIGURE 2

□

LEMMA 4.2. *For each $\boldsymbol{b} \in B$, $N_{\boldsymbol{b}}$ is a cycle of length $2m+2$.*

PROOF. By the definition of $E(p,k)$, $A_{\boldsymbol{b}} = \{(\boldsymbol{b} + \boldsymbol{h}[l])_A \ : \ 1 \le l \le m\}$ and $C_{\boldsymbol{b}} = \{(\boldsymbol{b} + \boldsymbol{h}[j] + \boldsymbol{h}[j+1])_C \ : \ 1 \le j \le m-1\} \cup \{(\boldsymbol{b} + 2\boldsymbol{h}[1])_C, (\boldsymbol{b} + 2\boldsymbol{h}[m])_C\}$. $D_{\boldsymbol{b}}$ contains just one element: $(\boldsymbol{b}^-, \boldsymbol{b}^*)$.

$(\boldsymbol{b}+\boldsymbol{h}[l])_A$ is adjacent to $(\boldsymbol{b}+\boldsymbol{h}[j]+\boldsymbol{h}[j+1])_C$ if and only if there exists $1 \le i \le m$ such that $(\boldsymbol{b} + \boldsymbol{h}[j] + \boldsymbol{h}[j+1]) - (\boldsymbol{b} + \boldsymbol{h}[l]) = \boldsymbol{h}[i]$, i.e. $\boldsymbol{h}[j] + \boldsymbol{h}[j+1] = \boldsymbol{h}[l] + \boldsymbol{h}[i]$.

By Lemma 3.2(ii), the only solutions are $l = j,\ i = j+1$ and $l = j+1,\ i = j$, $1 \le j \le m-1$. This means that

$$\{(\boldsymbol{b} + \boldsymbol{h}[j] + \boldsymbol{h}[j+1])_C, (\boldsymbol{b} + \boldsymbol{h}[j])_A\} \in E(p,k) \text{ for } 1 \le j \le m-1$$

and

$$\{(\boldsymbol{b} + \boldsymbol{h}[j] + \boldsymbol{h}[j+1])_C, (\boldsymbol{b} + \boldsymbol{h}[j+1])_A\} \in E(p,k) \text{ for } 1 \le j \le m-1.$$

$(\boldsymbol{b} + 2\boldsymbol{h}[j])_C$ $(j = 1, m)$ is adjacent to $(\boldsymbol{b} + \boldsymbol{h}[l])_A$ if and only if there exists $1 \le i \le m$ such that $(\boldsymbol{b} + 2\boldsymbol{h}[j]) - (\boldsymbol{b} + \boldsymbol{h}[l]) = \boldsymbol{h}[i]$ i.e. $2\boldsymbol{h}[j] = \boldsymbol{h}[l] + \boldsymbol{h}[i]$. By Lemma 3.2(i), the only solutions are $l = i = j = 1$ and $l = i = j = m$. These solutions define two more edges in $N_{\boldsymbol{b}}$:

$$\{(\boldsymbol{b} + 2\boldsymbol{h}[1])_C, (\boldsymbol{b} + \boldsymbol{h}[1])_A\} \text{ and } \{(\boldsymbol{b} + 2\boldsymbol{h}[m])_C, (\boldsymbol{b} + \boldsymbol{h}[m])_A\}.$$

Finally, we claim that there are exactly two edges in $N_{\boldsymbol{b}}$ which are adjacent to $(\boldsymbol{b}^-, \boldsymbol{b}^*)$. Since $(2\boldsymbol{h}[1])^- = (2\boldsymbol{h}[m])^-$ and $(2\boldsymbol{h}[1])^* = (2\boldsymbol{h}[m])^*$,

$$\{(\boldsymbol{b} + 2\boldsymbol{h}[1])_C, (\boldsymbol{b}^-, \boldsymbol{b}^*)\} \text{ and } \{(\boldsymbol{b} + 2\boldsymbol{h}[m])_C, (\boldsymbol{b}^-, \boldsymbol{b}^*)\}$$

are in $E(p,k)$. Moreover, by Lemma 3.2(iii), $(\boldsymbol{b} + \boldsymbol{h}[j] + \boldsymbol{h}[j+1])_C$ and $(\boldsymbol{b}^-, \boldsymbol{b}^*)$ are not adjacent for $1 \le j \le m-1$.

Hence there are $2m+2$ edges in $N_{\boldsymbol{b}}$ and they form a cycle as indicated on Figure 3.

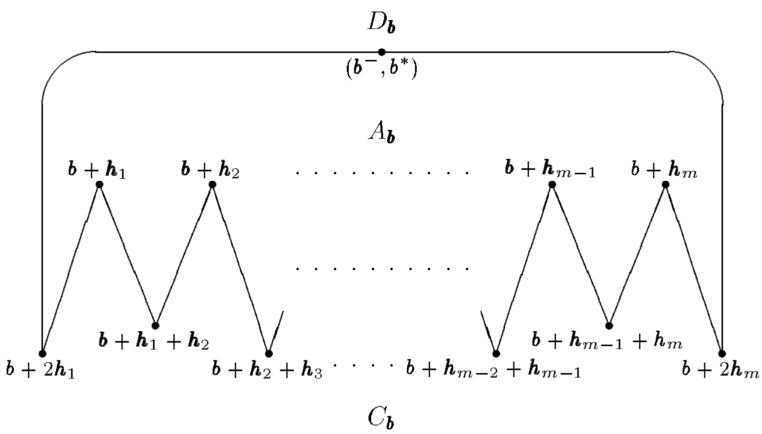


FIGURE 3

□

LEMMA 4.3. *For each $\boldsymbol{c} \in C$, $N_{\boldsymbol{c}}$ is a cycle of length $2m+2$.*

PROOF. This Lemma can be proven analogously to Lemma 4.2. $N_{\boldsymbol{c}}$ is indicated on Figure 4.

□

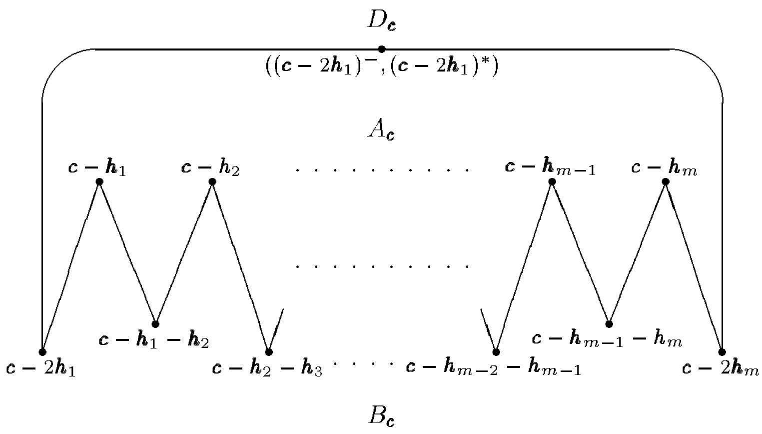


FIGURE 4

LEMMA 4.4. *For each* $(\boldsymbol{d}, t) \in D$, $N_{\boldsymbol{d}}$ *is a cycle of length* $16d+2$.

PROOF. By the definition of $E(p,k)$, $B_{(\boldsymbol{d},t)} = \{(\boldsymbol{d}, s, t-s)_B \ : \ s \in GF(p)\}$ and $C_{(\boldsymbol{d},t)} = \{((\boldsymbol{d}, s', t-s') + 2\boldsymbol{h}[1])_C \ : \ s' \in GF(p)\}$. Moreover, $(\boldsymbol{d}, s, t-s)_B$ and $((\boldsymbol{d}, s', t-s') + 2\boldsymbol{h}[1])_C$ are connected if and only if (i) $((\boldsymbol{d}, s', t-s') + 2\boldsymbol{h}[1]) - (\boldsymbol{d}, s, t-s) = \boldsymbol{h}[j] + \boldsymbol{h}[j+1]$ for some $1 \le j \le m-1$ or (ii) $((\boldsymbol{d}, s', t-s') + 2\boldsymbol{h}[1]) - (\boldsymbol{d}, s, t-s) = 2\boldsymbol{h}[j]$ for some $j \in \{1, m\}$.

By Lemma 3.2(iii), there are no edges in $N_{\boldsymbol{d}}$ defined in case (i). It is clear that case (ii) defines the edges

$$\{(\boldsymbol{d}, s, t-s)_B, ((\boldsymbol{d}, s, t-s) + 2\boldsymbol{h}[1])_C \ : \ s \in GF(p)\}$$

in $N_{\boldsymbol{d}}$. Also,

$$\{(\boldsymbol{d}, s, t-s)_B, ((\boldsymbol{d}, s, t-s) + 2\boldsymbol{h}[m])_C \ : \ s \in GF(p)\}$$

are edges in $N_{\boldsymbol{d}}$ since $(2\boldsymbol{h}[1])^- = (2\boldsymbol{h}[m])^-$ and $(2\boldsymbol{h}[1])^* = (2\boldsymbol{h}[m])^*$ and so $((\boldsymbol{d}, s, t-s) + 2\boldsymbol{h}[m])_C \in C_{(\boldsymbol{d},t)}$.

Thus there are $16d+2$ edges in $N_{\boldsymbol{d}}$ and, since p is a prime, they form a cycle as indicated on Figure 5.

□

To finish the proof of Theorem 1.1, let the integer $k \ge 2$ be fixed. By a result of Behrend [2], X can be chosen such that

$$|X| > d^{1 - \frac{c'}{\sqrt{\log d}}} \tag{4.1}$$

for some absolute constant $c' > 0$. Let $n = |V(p,k)|$. Then

$$\log\log n = \log\log(3 + \frac{1}{8d+1})(8d+1)^{k|X|} = (1+o(1))\log d, \tag{4.2}$$

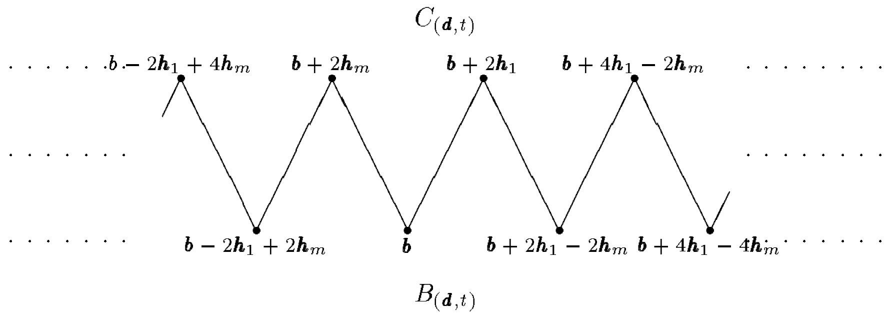


FIGURE 5

as $d \to \infty$. As before, let $m = (k|X|)!/((k!)^{|X|})$. Since most vertices of $G(p,k)$ have degree at least $2m$, it is enough to prove that

$$m > n^{1-\frac{c}{\sqrt{\log\log n}}} \tag{4.3}$$

for some constant $c > 0$. By Stirling's formula,

$$m > \Big(\frac{|X|}{(2\pi k)^{\frac{1}{2k}}}\Big)^{k|X|}.$$

This, combined with 4.1 and 4.2, proves 4.3. □

5. Path neighborhoods

In this section, we construct graphs $J(p,k)$ such that the neighborhood of each vertex is a path.

$J(p,k)$ is obtained from $G(p,k)$ by deleting the vertices in D and the edges

$$\{\{\boldsymbol{v}_B, (\boldsymbol{v}+2\boldsymbol{h}[1])_C\} \;:\; \boldsymbol{v} \in GF(p)^{k|X|}\}.$$

The following lemma proves the correctness of this construction.

LEMMA 5.1. *The neighborhood of each vertex in $J(p,k)$ is a path of length $2m-1$.*

PROOF. Each $\boldsymbol{a} \in A$ is connected to the same vertices in $G(p,k)$ and $J(p,k)$. Also, by Lemma 3.2(ii), the only edge deleted in the neighborhood of $\boldsymbol{a}$ is $\{(\boldsymbol{a}-\boldsymbol{h}[1])_B, (\boldsymbol{a}+\boldsymbol{h}[1])_C\}$.

For each $\boldsymbol{b} \in B$, two vertices were deleted from the neighborhood of $\boldsymbol{b}$: $\boldsymbol{b}+2\boldsymbol{h}[1] \in C$ and $(\boldsymbol{b}^-, \boldsymbol{b}^*) \in D$. These two vertices were connected in $G(p,k)$, so the $J(p,k)$ neighborhood of $\boldsymbol{b}$ is obtained by deleting two consecutive vertices in a cycle.

Similarly, the $J(p,k)$ neighborhood of $\boldsymbol{c} \in C$ is obtained by deleting the consecutive vertices $\boldsymbol{c}-2\boldsymbol{h}[1] \in B$ and $((\boldsymbol{c}-2\boldsymbol{h}[1])^-, (\boldsymbol{c}-2\boldsymbol{h}[1])^*)$ in $N_{\boldsymbol{c}}$. □

6. A related problem

In this section, we present another construction of a graph $L(V, E)$ with local property 'cycle', and state an extremal problem related to the ones investigated in this paper. Although the graph $L(V, E)$ will have only $(3/4)|V|^{3/2}$ edges, the construction is much simpler than the one described in Section 4.

Let $A = (a_{i,j})$, $B = (b_{i,j})$, $X = (x_{i,j})$, and $Y = (y_{i,j})$, $1 \le i, j \le k$, be $k \times k$ matrices. We identify each matrix with the set of its entries and we suppose that the sets A, B, X, Y are pairwise disjoint. Let $n = 4k^2$ and define the vertex set of $L(V, E)$ as the set consisting of the $4k^2$ elements of A, B, X, Y. The edges of $L(V, E)$ are defined as follows.

The sets A, B, and $X \cup Y$ are independent. $x_{i,j}$ is connected to $a_{i,t}$ for $1 \le t \le k$ (the i^{th} row of A) and to $b_{j,t}$ for $1 \le t \le k$ (the j^{th} row of B). $y_{i,j}$ is connected to $a_{t,i}$ for $1 \le t \le k$ (the i^{th} column of A) and to $b_{t,j}$ for $1 \le t \le k$ (the j^{th} column of B). Finally, $a_{i,j}$ is connected to $b_{i+t,j+t}$ and $b_{i+t,j+t+1}$ for $1 \le t \le k$ (two diagonals of B; the indices are mod k).

It is easy to see that the neighborhood of each vertex spans a cycle: for example, $N_{x_{1,1}}$ is the cycle

$$(b_{1,1}, a_{1,1}, b_{1,2}, a_{1,2}, b_{1,3}, \ldots, b_{1,k}, a_{1,k})$$

and $N_{a_{1,1}}$ is the cycle

$$(b_{1,1}, x_{1,1}, b_{1,2}, y_{1,2}, b_{2,2}, x_{1,2}, b_{2,3}, y_{1,3}, \ldots, y_{1,k}, b_{k,k}, x_{1,k}, b_{k,1}, y_{1,1}).$$

The number of edges is $6k^3 = (3/4)n^{3/2}$.

As mentioned in the introduction, Clark, Entringer, McCanna, and Székely [4] proved that $e(n) = o(n^2)$ for the local properties cycle, path, and matching. In fact, they showed that for fixed k, if every edge of a graph $G(n, E)$ is contained in at least one and at most k triangles then $|E| = o(n^2)$. This result motivates the following question. Let $\mathcal{G}(n, c)$ be the family of graphs with n vertices and $\ge cn^2$ edges satisfying the property that each edge occurs in a triangle. For $G \in \mathcal{G}(n, c)$, let $f(G)$ be the minimal number k such that each edge of G occurs in $\le k$ triangles, and let $f(\mathcal{G}(n, c)) = \min\{f(G) : G \in \mathcal{G}(n, c)\}$.

Adding a complete bipartite graph between A and B to the graph $L(V, E)$ constructed above, we obtain a graph $G \in \mathcal{G}(n, 1/16)$ with $f(G) = \sqrt{n}/2$. Similar constructions were given by N. Alon and L. Székely (personal communications). In all three examples, G contains a large complete bipartite graph, which implies $f(G) > c'\sqrt{n}$. We believe that this cannot be avoided:

CONJECTURE 6.1. *$f(\mathcal{G}(n, c)) = c'\sqrt{n}$, where the constant c' depends only on c.*

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Department of Mathematics, The Ohio State University, Columbus, OH 43210
E-mail address: akos@math.ohio-state.edu

Department of Mathematics, The Ohio State University, Columbus, OH 43210, Department of Computer Science, Eötvös University, Budapest, Hungary H-1088
E-mail address: szabote@math.ohio-state.edu