Turán's theorem in the hypercube

Noga Alon * Anja Krech † Tibor Szabó ‡

February 13, 2008

Abstract

We are motivated by the analogue of Turán's theorem in the hypercube Q_n: how many edges can a Q_d-free subgraph of Q_n have? We study this question through its Ramsey-type variant and obtain asymptotic results. We show that for every odd d it is possible to color the edges of Q_n with $\frac{(d+1)^2}{4}$ colors, such that each subcube Q_d is polychromatic, that is, contains an edge of each color. The number of colors is tight up to a constant factor, as it turns out that a similar coloring with $\binom{d+1}{2}+1$ colors is not possible. The corresponding question for vertices is also considered. It is not possible to color the vertices of Q_n with $d+2$ colors, such that any Q_d is polychromatic, but there is a simple $d+1$ coloring with this property. A relationship to anti-Ramsey colorings is also discussed.

We discover much less about the Turán-type question which motivated our investigations. Numerous problems and conjectures are raised.

1 Introduction

For graphs G and H, let $ex(G, H)$ denote the maximum number of edges in a subgraph of G which does not contain a copy of H. The quantity $ex(G, H)$ was first investigated in case G is a clique. Turán's Theorem resolves the problem precisely, when H is a clique as well.

In this paper, we study these Turán-type problems, when the base graph G is the n-dimensional hypercube Q_n. This setting was initiated by Erdős [8] who asked how many edges can a C_4-free subgraph of the hypercube contain. He conjectured the answer is

*Schools of Mathematics and Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel and IAS, Princeton, NJ 08540, USA. Email: nogaa@tau.ac.il.
Research supported in part by a USA-Israel BSF grant, by the Israel Science Foundation, by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University and by the Von Neumann Fund.

†Department of Mathematics and Theoretical Computer Science, Free University Berlin, D-14195 Berlin, Germany. E-mail: krech@math.fu-berlin.de

‡Institute of Theoretical Computer Science, ETH Zentrum, CH-8092 Zürich, Switzerland. E-mail: szabo@inf.ethz.ch.

$(\frac{1}{2}+o(1))e(Q_n)$ and offered \$100 for a solution. The current best upper bound, due to Chung [6], stands at $\approx .623e(Q_n)$. The best known lower bound is $\frac{1}{2}\left(n+\sqrt{n}\right)2^{n-1}$ (for $n=4^r$) due to Brass, Harborth and Nienborg [5].

Erdős [8] also raised the extremal question for even cycles. Chung [6] obtained that $\frac{ex(Q_n,C_{4k})}{e(Q_n))} \to 0$ for every $k \geq 2$, i.e. cycles with length divisible by 4, starting from 8 are harder to avoid than the four-cycle. She also showed that

$$\frac{1}{4}e(Q_n) \leq ex(Q_n, C_6) \leq (\sqrt{2}-1+o(1))e(Q_n).$$

Later Conder [7] improved the lower bound to $\frac{1}{3}e(Q_n)$ by defining a 3-coloring of the edges of the n-cube such that every color class is C_6-free. On the other hand it is shown in [1] that for any fixed k, in any k-coloring of the edges of a sufficiently large cube there are monochromatic cycles of every even length greater than 6. Note, however, that the Turán problem for cycles of length $4k+2$ is still wide open. For $k \geq 2$, it is not even known whether $ex(Q_n, C_{4k+2}) = o(e(Q_n))$.

In the present paper we consider a generalization of the C_4-free subgraph problem in a different direction, which we feel is the true analogue of Turán's Theorem in the hypercube. For arbitrary d we give bounds on $ex(Q_n, Q_d)$. For convenience we will talk about the complementary problem: i.e., let $f(n,d)$ denote the minimum number of edges one must delete from the n-cube to make it d-cube-free. Obviously $f(n,d) = e(Q_n) - ex(Q_n, Q_d)$. By a simple averaging argument one can see that for any fixed d the function $f(n,d)/e(Q_n)$ is non-decreasing in n, so a limit c_d exists. (In fact this limit exists for an arbitrary forbidden subgraph H, instead of Q_d). Erdős' conjecture then could be stated as $c_2 = \frac{1}{2}$.

Trivially $f(d,d) = 1$, so by the above $c_d \geq \frac{1}{d2^{d-1}}$. On the other hand, if one deletes edges of the hypercube on every d^{th} level, one obtains a Q_d-free subgraph. For this, observe that every d-dimensional subcube must span $d+1$ levels. Thus $c_d \leq \frac{1}{d}$.

In the present paper we improve on these trivial bounds.

Theorem 1.

$$\Omega\left(\frac{\log d}{d2^d}\right) = c_d \leq \begin{cases} \frac{4}{(d+1)^2} & \textit{if } d \textit{ is odd} \\ \frac{4}{d(d+2)} & \textit{if } d \textit{ is even.} \end{cases}$$

We conjecture that our construction is essentially optimal for $d=3$.

Conjecture 2.

$$c_3 = \frac{1}{4}.$$

The best known lower bound on c_3 is $1-\left(\frac{5}{8}\right)^{1/4} \approx 0.11$ and follows from some property of the 4-dimensional cube. (A Q_3-free subgraph of Q_4 cannot contain more than 10 vertices of degree 4; see the paper of Graham, Harary, Livingston and Stout [10]).

For arbitrary d we are less confident; it would certainly be very interesting to determine how fast c_d tends to 0, when d tends to infinity.

Problem 3. *Determine the order of magnitude of c_d.*

We tend to think that c_d is larger than inverse exponential, but feel that we are very far from understanding the truth. In fact all our arguments are set in the related Ramsey-type framework, rather than the original Turán-type. A coloring of the edges of Q_n is called *d-polychromatic* if every subcube of dimension d is *polychromatic* (i.e. it has all the colors represented on its edges). Let $pc(n,d)$ be the largest integer p such that there exists a d-polychromatic coloring of the edges of Q_n in p colors. Clearly, $pc(n,d) \leq d2^{d-1}$ and $f(n,d) \leq e(Q_n)/pc(n,d)$. Since $pc(n,d)$ is a non-increasing function in n, it stabilizes for large n. Let p_d be this limit, then we have $c_d \leq 1/p_d$. We can determine p_d up to a factor of 2.

Theorem 4.

$$\binom{d+1}{2} \geq p_d \geq \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even.} \end{cases}$$

The lower bound implies the upper bound in Theorem 1. It would be interesting to resolve the following problem.

Problem 5. *Determine the asymptotic behaviour of p_d.*

The lower bound in Theorem 1 is a consequence of some known results on the analogous problem for vertices of the cube. Let $g(n,d)$ be the minimum number of vertices one must delete from the n-cube to make it d-cube-free. Clearly $g(n,d) \leq f(n,d)$. Again, simple averaging shows that for any fixed d the function $g(n,d)/2^n$ is non-decreasing in n, so a limit c_d^0 exists.

The problem of determining $g(n,d)$ was investigated early and widely by several research communities mostly in a dual formulation under the different names of t-independent sets [12], qualitatively t-independent 2-partitions [14] and (n,t)-universal vector sets [16], where $t = n - d$. These investigations mostly deal with the case when d is *large*, i.e. very close to n. The lone result we are aware of about $g(n,d)$ for d small compared to n is due to E. A. Kostochka [13], who proves that $c_2^0 = 1/3$ (the same result has been obtained later and independently by Johnson and Entringer [11]). In both papers it is also shown that the unique smallest set breaking all copies of Q_2 is in the form of every third level of the cube. In general we know very little.

Proposition 6.

$$\frac{1}{d+1} \geq c_d^0 \geq \frac{\log d}{2^{d+2}}.$$

Again, the Ramsey analogue of the problem is more clear. In fact we have here a precise result. A coloring of the vertices of Q_n is called *d-polychromatic* if every subcube of dimension d has all the colors represented on its vertices. Let $pc^0(n,d)$ be the largest integer p such that there exists a d-polychromatic coloring of the vertices of Q_n in p colors. Clearly, $pc^0(n,d) \leq 2^d$ and $g(n,d) \leq 2^n/pc^0(n,d)$. Since $pc^0(n,d)$ is a non-increasing function of n, it stabilizes for large n. Let p_d^0 be this limit, then we have $c_d^0 \leq 1/p_d^0$. We can determine p_d^0 for every d.

Theorem 7.

$$p_d^0 = d + 1.$$

1.1 Relation to rainbow colorings

In this subsection we point out a relation between the established notion of anti-Ramsey coloring and the one of polychromatic coloring introduced in this paper. We also note how Theorem 4 could be applied to improve a result of [2].

An edge-coloring $r : E(H) \to \{1, 2, \ldots\}$ of a graph H is called *rainbow* if no two edges of H receive the same color. A coloring c of the edges of graph G is called *H-anti-Ramsey* if the restriction of c to any subgraph $H_0 \subseteq G$, $H_0 \cong H$, is *not* rainbow. Let $ar(G, H)$ be the largest number of colors used in an H-anti-Ramsey coloring of G. The function $ar(G, H)$ was introduced by Erdős, Simonovits and T. Sós [9]. It is well-known that $ar(G, H) \leq ex(G, H)$ since taking one arbitrary edge from each color class of an H-anti-Ramsey coloring one must obtain an H-free subgraph of G.

For any graph G and H, we call a p-coloring $c : E(G) \to \{1, \ldots p\}$ of the edges of G *H-polychromatic* if every subgraph $H_0 \subseteq G$, $H_0 \cong H$, has *all* the p colors represented on its edges. Let $pc(G, H)$ be the largest number p such that there is an H-polychromatic coloring of the edges of G. The following proposition establishes a relationship between H-anti-Ramsey and H-polychromatic colorings.

Proposition 8.

$$ar(G, H) \geq \left(1 - \frac{2}{pc(G, H)}\right) e(G).$$

Proof. Given an H-polychromatic coloring c of G with $p = pc(G, H)$-colors, we define an H-anti-Ramsey coloring r of G with at least $(1 - 2/p)e(G)$ colors. Let F be the set of edges formed by the union of the two smallest color classes of c. The coloring r will be chosen constant on F, say all edges in F receive color 1. All other edges of G will receive distinct colors. Then we used at least $\left(1 - \frac{2}{p}\right) e(G) + 1$ colors. Also, the coloring r defined this way is H-anti-Ramsey since each copy of H in G contains at least two edges of F, and thus at least two edges receive the color 1 in every copy of H. □

In a recent paper [2], Axenovich, Harborth, Kemnitz, Möller, and Schiermeyer investigated Q_d-anti-Ramsey colorings of Q_n. Lower and upper bounds for $ar(Q_n, Q_d)$ are found. In particular for fixed d, the leading terms of their bounds amount to

$$\left(1 - \frac{4}{d2^d}\right) e(Q_n) \geq ar(Q_n, Q_d) \geq \left(1 - \frac{1}{d}\right) e(Q_n).$$

One can improve the upper bound applying Theorem 1, and the lower bound using the polychromatic coloring of Theorem 4 .

Corollary 9.

$$\left(1 - \Omega\left(\frac{\log d}{d2^d}\right)\right) e(Q_n) \geq ar(Q_n, Q_d) \geq \left(1 - \frac{8}{d^2} - O\left(\frac{1}{d^3}\right)\right) e(Q_n).$$

Notation. We consider the cube as a set of n-dimensional $0-1$-vectors, where the coordinates are labeled by the first n positive integers, $[n] = \{1, \ldots, n\}$. A d-dimensional subcube of the n-dimensional cube is denoted by a vector from $\{0, 1, \star\}^n$ which contains d $\star$-entries; the stars represent the non-constant coordinates of the subcube. For a subcube D of the n-dimensional cube we denote by $ONE(D)$, $ZERO(D)$, and $STAR(D)$ the set of labels of those coordinates which are 1, 0, and $\star$, respectively.

2 Q_d-free subgraphs of Q_n

In this section we give a proof of the lower bound in Theorem 4.

Proof. First assume that d is odd. We define a $\frac{(d+1)^2}{4}$-coloring of the edges of Q_n, which is d-polychromatic.

We color the edges of Q_n with elements of $\mathbb{Z}_{\frac{d+1}{2}} \times \mathbb{Z}_{\frac{d+1}{2}}$ in the following way. The edge e with a star at coordinate a is colored with the vector whose first coordinate is $|\{x \in ONE(e) : x < a\}| \pmod{\frac{d+1}{2}}$ and whose second coordinate is $|\{x \in ONE(e) : x > a\}| \pmod{\frac{d+1}{2}}$.
Now consider a d-dimensional subcube C of Q_n with $STAR(C) = \{a_1, \ldots, a_d\}$, where $a_1 < a_2 < \cdots < a_d$. Let s be the vertex of C with the least number of ones. So for each vertex x of C we have that $ONE(s) \subseteq ONE(x) \subseteq ONE(s) \cup \{a_1, \ldots, a_d\}$.

We will show that all $\frac{(d+1)^2}{4}$ colors appear on edges of C whose star is at position $a_{\frac{d+1}{2}}$. Let (u, v) be an arbitrary element of $\mathbb{Z}_{\frac{d+1}{2}} \times \mathbb{Z}_{\frac{d+1}{2}}$.
Let $l := |\{x \in ONE(s) : x < a_{\frac{d+1}{2}}\}| \pmod{\frac{d+1}{2}}$ and
$r := |\{x \in ONE(s) : x > a_{\frac{d+1}{2}}\}| \pmod{\frac{d+1}{2}}$. Choose any $k \equiv u - l \pmod{\frac{d+1}{2}}$ elements K from $\{a_1, \ldots, a_{\frac{d+1}{2}-1}\}$ and any $p \equiv v - r \pmod{\frac{d+1}{2}}$ elements L from $\{a_{\frac{d+1}{2}+1}, \ldots, a_d\}$. Define s' by $ONE(s') = ONE(s) \cup K \cup L$. Then the edge incident to s' and having star at position $a_{\frac{d+1}{2}}$ has color (u, v).

For even d a similar construction works; the only difference is that we take the number of ones left of the label of the edge modulo $\frac{d}{2}$ and the number of ones to the right modulo $\frac{d+2}{2}$. Then one can prove that among the edges with label $\frac{d}{2}$ all colors appear. $\square$

3 Upper bound in the Ramsey problems.

First we prove the upper bound in Theorem 4.

Proof of Theorem 4 Suppose we have a d-polychromatic p-edge-coloring c of Q_n where n is huge. We will use Ramsey's theorem for d-uniform hypergraphs with $p^{d2^{d-1}}$ colors. We define a $p^{d2^{d-1}}$-coloring of the d-subsets of $[n]$. Fix an arbitrary ordering of the edges of Q_d. For an arbitrary subset S of the coordinates, define $cube(S)$ to be the subcube whose $\star$ coordinates are at the positions of S and all its other coordinates are 0, i.e. $STAR(cube(S)) = S$ and $ZERO(cube(S)) = [n] \setminus S$. Let S be a d-subset of $[n]$ and define the color of S to be the vector whose coordinates are the c-values of the edges of the d-dimensional subcube $cube(S)$ (according to the fixed ordering of the edges of Q_d). By Ramsey's theorem, if n is large enough, there is a set $T \subseteq [n]$ of $d^2 + d - 1$ coordinates such that the color-vector is the same for any d-subset of T. Let us now fix a set S of d particular coordinates from T: those ones which are the $(id)^{th}$ elements of T for some $i = 1, \ldots, d$. Hence any two elements of S have at least $d - 1$ elements of T in between.

Claim 10. *The c-value of an edge e of $cube(S)$ depends only on the number of 1s to the left of the $\star$ of e and the number of 1s to the right of this $\star$.*

Proof. Let e_1 and e_2 be two edges of $cube(S)$ such that they have the same number of 1s to the left of their respective star and the same number of 1s to the right as well. We can find d coordinates S' from T such that $STAR(e_2) \cup ONE(e_2) \subseteq S'$ (i.e., e_2 is an edge of $cube(S')$), and the vector e_2 restricted to S' is *equal* to the vector e_1 restricted to S. Indeed, there are enough unused 0-coordinates of e_2 in T between any two elements of S.
Now, since every d-subset of T has the same color-vector, the corresponding edges of the cubes $cube(S)$ and $cube(S')$ have the same c-value. In particular the colors of e_1 and e_2 are equal. The claim is proved. $\square$

To finish the proof of the upper bound in Theorem 4 we just note that there are exactly $1 + \ldots + d = \binom{d+1}{2}$ many ways to separate at most $d-1$ 1s by a $\star$. By the Claim a d-polychromatic edge-coloring is not possible with more colors. $\square$

With a very similar argument one can prove the matching upper bound in the analogous question for vertices.

Proof of Theorem 7 Assume we have a d-polychromatic coloring of the vertices of Q_n. Let us define a d^{2^d}-coloring of the d-tuples of $[n]$. For a d-subset S let the color be determined by the vector of the 2^d colors of the vertices of the subcube $cube(S)$ with $STAR(cube(S)) = S$ and $ZERO(cube(S)) = [n] \setminus S$ (according to some fixed ordering of the vertex set of Q_d). By Ramsey's theorem there is a set T of d^2+d-1 coordinates such that the color-vector is the same for any d-subset of T. Let us again fix d coordinates S in T such that any two elements of S have at least $d-1$ elements of T in between (in a way similar to the one in the edge-coloring case).

Claim 11. *The color of a vertex in $cube(S)$ depends only on its number of 1s.*

Proof. Let v_1 and v_2 be two vectors from $cube(S)$ such that $|ONE(v_1)| = |ONE(v_2)|$. We can find d coordinates S' from T such that $ONE(v_2) \subseteq S'$ and the vector v_2 restricted to S' is *equal* to the vector v_1 restricted to S. Indeed, there are enough unused 0-coordinates in T between any two elements of S to do this. Now, since T is monochromatic according to our color-vectors, the color of v_1 and v_2 is the same as well. The claim is proved. $\square$

To finish the proof of the upper bound in Theorem 7 we just note that there are exactly $d+1$ possible values for the number of 1*s* on d coordinates. By the Claim a d-polychromatic coloring is not possible with more colors.

For the lower bound in Theorem 7 one can color each vertex of the cube by the number of its non-zero coordinates modulo $d+1$. This gives a d-polychromatic vertex coloring in $d+1$ colors. $\square$

4 A lower bound on c_d

The lower bound in Proposition 6 can be deduced from earlier results on the d-independent set problem and is essentially stated (implicitly) in [10]. For completeness we sketch the proof.

Let G be a set of g vertices which intersects all d-cubes of the n-cube. This happens if and only if, interpreting these vertices as subsets of an n-element base set X, G shatters all $(n-d)$-element subsets of X. (A family $\mathcal{F}$ of subsets *shatters* a given subset K, if all the $2^{|K|}$ subsets of K can be represented as $K \cap F$ for some $F \in \mathcal{F}$.) Now let M_G be the $g \times n$ $0-1$-matrix whose rows correspond to the elements of G. Then the columns of M_G can be interpreted as a family L of n subsets of a g-element base set Y, such that all the 2^{n-d} parts of the Venn diagram of any $n-d$ members of L are nonempty. (A family L satisfying this property is usually called $(n-d)$-*independent*.)

Thus determining $g(d+t,d)$ is the same problem as determining the largest size of a t-independent family. This was first done by Schönheim [15] and Brace and Daykin [4] for $t=2$ and later reproved and generalized by many others, e.g. Kleitman and Spencer [12].

It is known that $g(d+2,d) \geq \log d$ and thus the lower bound on c_d^0 follows by the monotonicity of $g(n,d)/2^n$. The lower bound in Theorem 1 also follows since $f(d+2,d) \geq g(d+2,d)$ and $f(n,d)/e(Q_n)$ is non-decreasing.

5 Remarks and More Open Problems

Remark. The following Claim shows that if c_d is indeed larger than inverse exponential, then one has to search for the evidence in very large, i.e. doubly exponential, dimensions.

For simplicity we write here the proof for c_d^0 (the vertex version); the argument for c_d follows along similar lines.

Claim. For any $p \leq \frac{2^d}{2d}$, there is a d-polychromatic p-coloring of the n-cube, with $n = \frac{1}{2}\exp\left\{\frac{2^d}{2dp}\right\}$. In particular, for any $\epsilon > 0$ and $n \leq \frac{1}{2}\exp\left\{2^{(1-\epsilon)d}\right\}$,

$$g(n,d) \leq \frac{2d}{2^{\epsilon d}} \cdot 2^n.$$

Proof. We randomly color the vertices of Q_n with p colors. For each vertex v select a color uniformly at random from $\{1,\ldots,p\}$, choices being independent from the choices on all other vertices. For a d-cube D, let A_D be the event that there is a color which does not appear on the vertices of D. The probability of A_D is at most $p\,(1-1/p)^{2^d}$. Each d-cube intersects less than $2^d\binom{n}{d}$ other d-cubes. Obviously A_D is independent from the set of all events $A_{D'}$ where D' is disjoint from D.

For $p \leq \frac{2^d}{2d}$ and $n = \frac{1}{2}\exp\left\{\frac{2^d}{2dp}\right\}$,

$$e \cdot p\left(1-\frac{1}{p}\right)^{2^d} 2^d \binom{n}{d} \leq e^{1+\log p - \frac{2^d}{p} + d\log 2n} = o_d(1).$$

Hence the Local Lemma implies that with nonzero probability all p colors are represented on all d-cubes.

For the second part of the Claim, choose $p = 2^{\epsilon d}/2d$ and leave out the vertices of the sparsest color class in a d-polychromatic p-coloring of the n-cube. $\square$

Open Problems. Since $f(n,2)$ is known to be strictly larger than one third of the number of edges in Q_n for large n [6], it is clear that $p_2 = 2$. Bialostocki [3] proved that in any 2-polychromatic edge-two-coloring of Q_n the color classes are asymptotically equal. The next natural question is the determination of p_3, which is either $4, 5$ or 6. Once p_3 is known, it would be interesting to generalize Bialostocki's theorem and decide whether in any 3-polychromatic p_3-edge-coloring of Q_n, each color class contains approximately $\frac{1}{p_3}e(Q_n)$ edges.

Everything above could be generalized, quite straightforwardly, but would not answer the following problems:

Turán-type: Let $f^{(l)}(n,d)$ be the smallest integer f such that there is a family of f l-faces of Q_n, such that every d-face contains at least one member of this family. Again, $f^{(l)}(n,d)/\binom{n}{l}2^{n-l}$ is non-decreasing, so there is a limit $c_d^{(l)}$. Determine it!

Ramsey-type: A coloring of the l-faces of Q_n is *d-polychromatic* if for every d-face S and color s there is an l-face of S with color s. Let $pc^{(l)}(n,d)$ be the largest number of colors with which there is a d-polychromatic coloring of the l-faces of Q_n. Again, the limit $p_d^{(l)}$ of $pc^{(l)}(n,d)$ exists. Determine it!

Acknowledgment. We would like to thank an anonymous referee for pointing out reference [2] to us.

References

[1] N. Alon, R. Radoičić, B. Sudakov, and J. Vondrák, A Ramsey-type result for the hypercube, *Journal of Graph Theory*, to appear.

[2] M. Axenovich, H. Harborth, A. Kemnitz, M. Möller, I. Schiermeyer, Rainbows in the hypercube, *Graphs and Combinatorics*, to appear.

[3] A. Bialostocki, Some Ramsey type results regarding the graph of the n-cube, *Ars Combinatoria*, **16-A** (1983) 39-48.

[4] A. Brace and D. E. Daykin, Sperner type theorems for finite sets, *Proceedings of the British Combinatorial Conference, Oxford*, (1972), 18-37.

[5] P. Brass, H. Harborth, and H. Nienborg, On the maximum number of edges in a C_4-free subgraph of Q_n, *Journal of Graph Theory*, **19** (1995), 17-23.

[6] F. Chung, Subgraphs of a hypercube containing no small even cycles, *Journal of Graph Theory*, **16** (1992), 273-286.

[7] M. Conder, Hexagon-free subgraphs of hypercubes, *Journal of Graph Theory*, **17** (1993), 477-479.

[8] P. Erdős, Some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics*, B. Bollobás, ed., Academic Press (1984), 1-17.

[9] P. Erdős, M. Simonovits, V. T. Sós, Anti-Ramsey theorems, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol II, 633-643. *Colloq. Math. Soc. János Bolyai*, **10**, North-Holland, Amsterdam, 1975.

[10] N. Graham, F. Harary, M. Livingston, and Q. Stout, Subcube fault-tolerance in hypercubes, *Information and Computation*, **102** (1993), 280-314.

[11] K.A. Johnson and R. Entringer, Largest induced subgraphs of the n-cube that contain no 4-cycles, *Journal of Combinatorial Theory, Series B*, **46** (1989), 346-355.

[12] D. Kleitman and J. Spencer, Families of k-independent sets, *Discrete Mathematics*, **6** (1973), 255-262.

[13] E. A. Kostochka, Piercing the edges of the n-dimensional unit cube. (Russian) *Diskret. Analiz Vyp.* **28** *Metody Diskretnogo Analiza v Teorii Grafov i Logiceskih Funkcii* (1976), 55–64.

[14] A. Rényi, **Foundations of Probability**, Wiley New York, 1971.

[15] J. Schönheim, A generalization of results of P. Erdős, G. Katona, and D. J. Kleitman concerning Sperner's theorem, *Journal of Combinatorial Theory, Series A*, **11** (1971), 111-117.

[16] G. Seroussi, N.H. Bshouty, Vector sets for exhaustive testing of logic circuits, *IEEE Transactions on Information Theory*, **34** (1988), 513-522.