The game of JumbleG

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Abstract

JumbleG is a Maker-Breaker game. Maker and Breaker take turns in choosing edges from the complete graph K_n. Maker's aim is to choose what we call an ϵ*-regular* graph (that is, the minimum degree is at least $(\frac{1}{2} - \epsilon)n$ and, for every pair of disjoint subsets $S, T \subset V$ of cardinalities at least ϵn, the number of edges $e(S, T)$ between S and T satisfies: $\left|\frac{e(S,T)}{|S||T|} - \frac{1}{2}\right| \leq \epsilon$.) In this paper we show that Maker can create an ϵ-regular graph, for $\epsilon \geq 2(\log n/n)^{1/3}$. We consider also a similar game, JumbleG2, where Maker's aim is to create a graph with minimum degree at least $\left(\frac{1}{2} - \epsilon\right) n$ and maximum co-degree at most $\left(\frac{1}{4} + \epsilon\right) n$, and show that Maker has a winning strategy for $\epsilon > 3(\log n/n)^{1/2}$. Thus, in both games Maker can create a pseudo-random graph of density $\frac{1}{2}$. This guarantees Maker's win in several other positional games, also discussed here.

1 Introduction

JumbleG is a Maker-Breaker game. Maker and Breaker take turns in choosing edges from the complete graph K_n on n vertices. Maker's aim is to choose a graph which is ϵ-regular (the definition follows).

Let $G = (V, E)$ be a graph of order n. We usually assume that the vertex set is $[n] =$

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$\{1,\ldots,n\}$. We call a pair S, T of non-empty disjoint subsets of $[n]$ *ϵ-unbiased* if

$$\left|\frac{e_G(S,T)}{|S|\,|T|} - \frac{1}{2}\right| \leq \epsilon, \tag{1}$$

where $e_G(S,T)$ is the number of $S-T$ edges in G. The graph G is *ϵ-regular* if

P1: $\delta(G) \geq (\frac{1}{2}-\epsilon)n$.

P2: Any pair S, T of disjoint subsets of $[n]$ with $|S|, |T| \geq \epsilon n$ is ϵ-unbiased.

Theorem 1 *Maker has a winning strategy in JumbleG provided $\epsilon \geq 2(\log n/n)^{1/3}$ and n is sufficiently large.*

We consider also a similar game, which we denote by JumbleG2. In this game Maker's aim is to create a graph with Properties P1 and P3, where

P3: Maximum co-degree is at most $(\frac{1}{4}+\epsilon)n$.

(The co-degree of vertices $u, v \in V(G)$ is the number of common neighbours of u and v in G.)

Here, too, Maker can win provided ϵ is not too small:

Theorem 2 *Maker has a winning strategy in JumbleG2 for all $\epsilon \geq 3(\log n/n)^{1/2}$ if n is sufficiently large.*

Theorems 1 and 2 are proved in Section 2. As shown in Section 3, our restrictions on ϵ are best possible, up to a logarithmic factor.

Although the goals of the above two games appear to be quite different, they are in fact very similar to each other — in both Maker tries to create a *pseudo-random graph* of density around $\frac{1}{2}$. Informally speaking, a pseudo-random graph $G=(V,E)$ is a graph on n vertices whose edge distribution resembles that of a truly random graph $G(n,p)$ of the same edge density $p = e(G)\binom{n}{2}^{-1}$. The reader can consult [12] for a recent survey on pseudo-random graphs. The fact that an ϵ-regular graph is pseudo-random with density $\frac{1}{2}$ is apparent from the definition. To see that degrees and co-degrees can guarantee pseudo-randomness we need to recall some notions and results due to Thomason. He introduced the notion of *jumbled* graphs [17]. A graph G with vertex set $[n]$ is (α,β)-jumbled if for every $S \subseteq [n]$ we have

$$\left|e_G(S) - \alpha\binom{|S|}{2}\right| \leq \beta|S|$$

where $e_G(S)$ is the number of edges of G contained in S.

Thomason showed that one can check for pseudo-randomness via jumbledness by checking degrees and co-degrees. Suppose that $G = (V, E)$ has minimum degree at least αn and no two vertices have more than $\alpha^2 n + \mu$ common neighbours. Then, (see Theorem 1.1 of [17] and its proof) for every $s \leq n$, every set $S \subseteq V$ of size $|S| = s$ satisfies:

$$\left| e(S) - \alpha \binom{s}{2} \right| \leq \frac{((s-1)\mu + \alpha n)^{1/2} + \alpha}{2} s\,, \tag{2}$$

and therefore G is (α, β)-jumbled with $\beta = ((\alpha n + (n-1)\mu)^{1/2} + \alpha)/2$.

Now suppose that for some $\epsilon = \Omega(1/n)$ a graph G on n vertices has minimum degree at least $\alpha n = \left(\frac{1}{2} - \epsilon\right) n$ and maximum co-degree at most $\left(\frac{1}{4} + \epsilon\right) n = \alpha^2 n + (2\epsilon - \epsilon^2)n$. Then a routine calculation, based on (2), shows that G is ϵ'-regular for $\epsilon' = \Omega(\epsilon^{1/4})$. Thus Theorem 2 can be used to show that Maker can create an ϵ-regular graph with $\epsilon = n^{-1/8+o(1)}$ — a weaker result than the one provided by the direct application of Theorem 1. Indeed, let $|S| = s$, $|T| = t \geq \epsilon' n$, $\mu = (2\epsilon - \epsilon^2)n$, and $\epsilon' \geq \Omega(\epsilon^{1/4}) \geq \Omega(n^{-1/4})$. Then

$$\begin{aligned}
&\left| \frac{e_G(S,T)}{st} - \frac{1}{2} \right| = \frac{1}{st} \left| e_G(S \cup T) - e_G(S) - e_G(T) - (\alpha + \epsilon)st \right| \\
&\leq \frac{1}{st} \left(\left| e_G(S \cup T) - \alpha \binom{s+t}{2} \right| + \left| e_G(S) - \alpha \binom{s}{2} \right| + \left| e_G(T) - \alpha \binom{t}{2} \right| \right) + \epsilon \\
&\leq \frac{((s+t)\mu + \alpha n)^{1/2} + \alpha}{2st}(s+t) + \frac{(s\mu + \alpha n)^{1/2} + \alpha}{2t} + \frac{(t\mu + \alpha n)^{1/2} + \alpha}{2s} + \epsilon \\
&\leq \frac{(s+t)^{3/2}\mu^{1/2}}{2st} + \frac{(s\mu)^{1/2}}{2t} + \frac{(t\mu)^{1/2}}{2s} + \frac{4((\alpha n)^{1/2} + \alpha)}{2\min\{s,t\}} + \epsilon \\
&\leq \frac{((1+\epsilon')n)^{1/2}(2\epsilon n)^{1/2}}{\epsilon' n} + \frac{(2n\epsilon n)^{1/2}}{2\epsilon' n} + \frac{(2n\epsilon n)^{1/2}}{2\epsilon' n} + \frac{2n^{1/2}}{\epsilon' n} + \epsilon \\
&\leq c\frac{\epsilon^{1/2}}{\epsilon'} \\
&\leq \epsilon'\,.
\end{aligned}$$

Pseudo-random graphs are known to have many nice properties. Hence, Maker's ability to create a pseudo-random graph guarantees his win in several other positional games. For example, using a result of [11], one can guarantee Maker's success in creating $\frac{n}{4} - O(n^{5/6}\log^{1/6} n)$ pairwise edge-disjoint Hamiltonian cycles. This is trivially best possible up to the error-term and confirms a conjecture of Lu [13] in a strong form. We will discuss this and other games in Section 4.

2 Playing JumbleG

In this section we prove Theorems 1 and 2. The proofs are quite similar and are based on the approach of Erdős and Selfridge [9] via potential functions.

Lemma 3 *If the edges of a hypergraph $\mathcal{F}$ satisfy $\sum_{X\in\mathcal{F}} 2^{-|X|} < 1/4$ then Maker can force a 2-colouring of $\mathcal{F}$.*

Proof. Let a round consist of a move of Maker followed by a move of Breaker. At the start of a round, let C_M, C_B denote the set of edges chosen so far by Maker and Breaker, R denote the unchosen edges and for $X \in \mathcal{F}$ let $\delta_{X,M}, \delta_{X,B}$ be the indicators of $X \cap C_M \neq \emptyset$, $X \cap C_B \neq \emptyset$ respectively. Let $\delta_X = \delta_{X,M} + \delta_{X,B}$. We use the potential function

$$\Phi = \sum_{\substack{X\in\mathcal{F}\\ \delta_X \leq 1}} 2^{-|X\cap R|+1-\delta_X}.$$

This represents the expected number of monochromatic sets if the unchosen edges are coloured at random. Our assumption is that $\Phi < \frac{1}{2}$ at the start and we will see that it can be kept this way until the end of the last complete round. In case n is odd, Maker with his last choice can at most double the value of Φ. In any case at the end of the play $\Phi < 1$. Also, at the end $R = \emptyset$; thus $\delta_X \geq 2$ for all $X \in \mathcal{F}$, showing that Maker has achieved his objective.

It remains to show that Maker can ensure that the value of Φ never increases after one complete round is played. Suppose that in some round, Maker chooses an edge a and Breaker chooses an edge b. Let Φ' be the new value of Φ. Then

$$\begin{aligned}
&\Phi' - \Phi \\
&= -\sum_{\substack{a,b\in X\\ \delta_X=0}} 2^{1-|X\cap R|} - \sum_{\substack{a\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{b\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} + \sum_{\substack{a\in X, b\notin X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} + \sum_{\substack{a\notin X, b\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} \\
&\leq -\left(\sum_{\substack{a\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{a\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} \right) + \left(\sum_{\substack{b\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{b\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|} \right)
\end{aligned}$$

which is non-positive if Maker chooses a to maximise $\sum_{\substack{a\in X\\ \delta_{X,B}=1}} 2^{-|X\cap R|} - \sum_{\substack{a\in X\\ \delta_{X,M}=1}} 2^{-|X\cap R|}$. $\square$

Lemma 4 *Let $\epsilon = \epsilon(n)$ tend to zero with n. Let $\delta > 1$ be fixed. Let $t = \lceil \delta\epsilon^{-2}\log n \rceil$. Then for all sufficiently large n Maker can ensure that any pair of disjoint subsets of V, both of size at least t, is ϵ-unbiased.*

Proof. Assume that $t \le n/2$ for otherwise there is nothing to prove. This means that $\epsilon > \left(\frac{2\log n}{n}\right)^{1/2}$.

Let $k = \lceil (\frac{1}{2}+\epsilon)\, t^2 \rceil$. Let $\mathcal{T}$ consist of pairs (S,T) of disjoint subsets of V, both of size *exactly* t. Recall that $e_M(S,T)$ counts the number of Maker's edges connecting S to T. A simple averaging argument shows that it is enough to show that Maker can guarantee that

$$t^2 - k < e_M(S,T) < k, \quad \text{for all } (S,T) \in \mathcal{T}. \tag{3}$$

(Indeed, let S', T' have size at least t each. The expectation of $\frac{e_M(S,T)}{t^2}$, where S,T are random t-subsets of S',T', is $\frac{e_M(S',T')}{|S'||T'|}$. By (3) this cannot differ from $\frac{1}{2}$ by more than ϵ, as required.)

If Maker is able to ensure that all k-element subsets of the edge-set $S:T = \{\{x,y\} \mid x \in S,\, y \in T\}$ are properly 2-colored (i.e. not monochromatic) for every $(S,T) \in \mathcal{T}$, then he has achieved his goal. A direct application of Lemma 3 is not possible however: there are simply too many of these k-sets and the criterion does not hold. We need to cut down on the number of sets.

Define $\ell = \lceil 2t^2\epsilon \rceil$ and $\lambda = \lceil 2^{\ell} n^{-2t} \rceil$. For $(S,T) \in \mathcal{T}$ we prove the existence of a collection $\mathcal{X}_{S,T}$, of ℓ-subsets of the edge-set $S:T = \{\{x,y\} \mid x \in S,\, y \in T\}$ such that (i) $|\mathcal{X}_{S,T}| = \lambda$ and (ii) each k-set $B \subseteq S:T$ contains at least one member of $\mathcal{X}_{S,T}$. Let us show that if the elements of $\mathcal{X}_{S,T}$ are chosen at random, independently with replacement, then this property is almost surely satisfied. In estimating this probability we will use the following auxiliary inequalities: $\ell = o(t^2)$ and

$$\begin{aligned} \frac{\binom{k}{\ell}}{\binom{t^2}{\ell}} &= \prod_{i=0}^{\ell-1} \frac{k-i}{t^2-i} = \left(\frac{k}{t^2}\right)^{\ell} \prod_{i=0}^{\ell-1} \left(1 - \frac{i(t^2-k)}{t^2k - ki}\right) \\ &\ge \left(\frac{1}{2}+\epsilon\right)^{\ell} \exp\left\{-\frac{\ell^2}{2t^2} + O(\epsilon^2 \ell)\right\}. \end{aligned}$$

The probability that there is a k-subset of $S:T$ which does not contain a member of $\mathcal{X}_{S,T}$ is at most

$$\begin{aligned} &\binom{t^2}{k}\left(1 - \frac{\binom{k}{\ell}}{\binom{t^2}{\ell}}\right)^{\lambda} \le 2^{t^2} \exp\left\{-\lambda \left(\frac{1}{2}+\epsilon\right)^{\ell} e^{-\frac{\ell^2}{2t^2} + O(\epsilon^2\ell)}\right\} \\ &= 2^{t^2} \exp\left\{-n^{-2t} e^{2\epsilon\ell - \frac{\ell^2}{2t^2} + O(\epsilon^2\ell)}\right\} = 2^{t^2} \exp\left\{-e^{-2t\log n + (2+o(1))\epsilon^2 t^2}\right\} = o(1), \end{aligned}$$

so a family $\mathcal{X}_{S,T}$ with the required property does exist.

Let $\mathcal{F} = (\binom{[n]}{2}, \mathcal{E})$ be the hypergraph with hyper-edges $\mathcal{E} = \bigcup_{(S,T)\in\mathcal{T}} \mathcal{X}_{S,T}$. (We will use the term *hyper-edges* to distinguish them from the edges of K_n). To complete the proof it is enough to show that Maker can ensure that the choices $E_M, E_B \subset \binom{[n]}{2}$ of Maker, Breaker respectively are a 2-colouring of $\mathcal{F}$. This follows from Lemma 3 in view of the inequality

$$|\mathcal{E}|\, 2^{-\ell} \leq \binom{n}{t}^2 \lambda 2^{-\ell} = o(1). \tag{4}$$

$\square$

Proof of Theorem 1. To ensure that all degrees of Maker's graph are appropriate we use a trick similar to the one in the proof of the previous lemma. Let $k = \lceil (1/2+\epsilon)n \rceil$. Maker again would like to use Lemma 3 and ensure that all k-subsets of the edges incident with vertex i are properly 2-colored. These are again too many; we define $\ell = \lceil 10\epsilon^{-1}\log n \rceil$, $M = \lceil 2^\ell/n^2 \rceil$, and $\mu = nM$. We want to find a collection $A_1, A_2, \ldots, A_\mu$ of ℓ-sets such that, for $1 \leq i \leq n$, every k-subset of the edges incident with i contains at least one of $A_{(i-1)M+j}$, $1 \leq j \leq M$. As before, we construct the sets A_i randomly. The probability that there is a bad k-subset (containing no chosen ℓ-set) is at most

$$\binom{n-1}{k}\left(1 - \frac{\binom{k}{\ell}}{\binom{n-1}{\ell}}\right)^M \leq 2^n \exp\left\{-\frac{2^\ell}{n^2}\frac{k^\ell}{n^\ell}e^{-\ell^2/n}\right\} \leq 2^n \exp\left\{-((1+\epsilon)e^{-\ell/n})^\ell\right\} < n^{-2}$$

for large n, and so the desired sets exist.

For Property **P2** let $t = \lceil 6\epsilon^{-2}\log n \rceil$. By our assumption on ϵ, we have $t < \epsilon n$. Define $\mathcal{X}_{S,T}$ as in the proof of Lemma 4. Namely, let $\ell' = \lceil 2t^2\epsilon \rceil$ and $\lambda = \lceil 2^{\ell'} n^{-2t} \rceil$. For $(S,T) \in \mathcal{T}$ (that is, S, T are disjoint t-sets) let $\mathcal{X}_{S,T}$ be a collection ℓ'-subsets of $S:T$ such that (i) $|\mathcal{X}_{S,T}| = \lambda$ and (ii) every $\lceil (\frac{1}{2}+\epsilon)t^2 \rceil$-set contains at least one member of $\mathcal{X}_{S,T}$.

Let $\mathcal{F}$ be the hypergraph with the edge set $\mathcal{E}_1 \cup \mathcal{E}_2 = \{A_1, A_2, \ldots, A_\mu\} \cup \bigcup_{(S,T)\in\mathcal{T}} \mathcal{X}_{S,T}$.

Lemma 4 (or rather its proof) implies that it suffices for Maker to force a 2-colouring of $\mathcal{F}$. Indeed, the definition of the sets A_i will imply Property P1. To see that P2 will also hold, observe that for any $S, T \in \mathcal{T}$, we will have

$$\left|\frac{e_M(S,T)}{t^2} - \frac{1}{2}\right| \leq \epsilon,$$

while the claim for general $|S|, |T| \geq t$ follows by averaging.

It remains to check that $\mathcal{F}$ satisfies the conditions of Lemma 3 for large n. The initial value Φ of the potential function satisfies

$$\Phi \leq Mn2^{-\ell} + \Phi(\mathcal{E}_2) = o(1). \tag{5}$$

(Here we have used (4).) This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. This time for Property P1 we define $\ell = \lfloor \epsilon n \rfloor$, as before $M = \lceil 2^\ell/n^2 \rceil$, $\mu = nM$, $k = \lceil (1/2+\epsilon)n \rceil$. The family $A_1, A_2, \ldots, A_\mu$ should satisfy: For $1 \leq i \leq$

n, every k-subset of the edges incident with i contains at least one of $A_{(i-1)M+j}$, $1 \le j \le M$. We construct the A_i randomly. Suppose that we randomly choose M ℓ-subsets of $[n-1]$ independently with replacement. The probability that there is a k-subset of $[n-1]$ which contains no chosen ℓ-set is at most

$$
\begin{aligned}
&\binom{n-1}{k}\left(1-\frac{\binom{k}{\ell}}{\binom{n-1}{\ell}}\right)^M \\
&\le 2^n \exp\left\{-\frac{\binom{k}{\ell}M}{\binom{n}{\ell}}\right\} \\
&= 2^n \exp\left\{-M\,\frac{k\cdots(k-\lfloor \ell/2\rfloor+1)}{n\cdots(n-\lfloor \ell/2\rfloor+1)}\cdot\frac{(k-\lfloor \ell/2\rfloor)\cdots(k-\ell+1)}{(n-\lfloor \ell/2\rfloor\cdots(n-\ell+1)}\right\} \\
&\le 2^n \exp\left\{-M\left(\frac{k-\ell/2}{n}\right)^{\lfloor \ell/2\rfloor}\cdot\left(\frac{k-\ell}{n}\right)^{\lceil \ell/2\rceil}\right\} \\
&\le \exp\left\{n\log 2-\frac{2^\ell}{n^2}\left(\frac{1}{2}+\frac{\epsilon}{2}\right)^{\lfloor \ell/2\rfloor}\left(\frac{1}{2}\right)^{\lceil \ell/2\rceil}\right\} \\
&= \exp\left\{n\log 2-\frac{(1+\epsilon)^{\lfloor \ell/2\rfloor}}{n^2}\right\} \\
&< n^{-2}
\end{aligned}
$$

for $\epsilon \ge 3(\log n/n)^{1/2}$, so the required family exists.

For Property P3 we take a collection $B_1, B_2, \ldots, B_\rho$ of ℓ-sets where $\rho = \binom{n}{2}N$ and $N = \lceil 4^\ell/n^3\rceil$. For each pair $i, j \in [n]$ select N random ℓ-subsets of $[n] \setminus \{i, j\}$ so that each $\lceil(1/4+\epsilon)n\rceil$-set contains at least one of them. The hyper-edges are $\{(i, x) : x \in A\}$ $\cup\{(j, x) : x \in A\}$ for each random $A \subseteq [n] \setminus \{i, j\}$. $B_1, B_2, \ldots, B_\rho$ are chosen randomly and now with $k = \lceil(1/4+\epsilon)n\rceil$ the probability that there is a k-subset of $[n-2]$ which contains no chosen ℓ-set is at most

$$
\binom{n-2}{k}\left(1-\frac{\binom{k}{\ell}}{\binom{n-2}{\ell}}\right)^N \le \exp\left\{n\log 2-\frac{(1+2\epsilon)^{\lfloor \ell/2\rfloor}}{n^3}\right\} < n^{-3}
$$

for large n, and so the sets exist.

We will use Lemma 3 and so we need to check that the initial potential is less than 1/4. Now the initial value of the potential function is at most

$$
Mn2^{1-\ell} + Nn^2 2^{1-2\ell} = o(1)
$$

and this completes the proof of Theorem 2. □

3 Breaker's Strategies

In this section we show that up to a small power of $\log n$, our restrictions on ϵ are sharp in both Theorems 1 and 2 or, even more strongly, with respect to each of the Properties P1–P3.

Property P1

Theorem 2 gives immediately that Maker can guarantee a graph with minimum degree at least $n/2 - 3\sqrt{n \log n}$. A similar result has been previously obtained by Székely [16] by applying a lemma of Beck [2, Lemma 3] which in turn is based on the Erdős–Selfridge method. This comes quite close to a result of Beck [3] who proved that Breaker can force the minimum degree of Maker's graph to be $n/2 - \Omega(\sqrt{n})$.

Property P2

Let $c > 0$ be any constant which is less than $6^{-1/3}$, n be large, and $\epsilon = cn^{-1/3}\log^{1/3} n$.

Here we prove that *no* graph of order n can satisfy Property P2 for this ϵ, which shows that the restriction on ϵ in Theorem 1 is sharp up to a multiplicative constant. The proof is based on ideas of Erdős and Spencer [10].

Let G be an arbitrary graph of order n. Let $m = \lceil \epsilon n \rceil$. Let X be a random m-subset of $V(G)$ chosen uniformly. For $y \in V(G)$, let $\mathcal{E}_y$ be the event that $y \notin X$ and $|\,|\Gamma(y) \cap X| - m/2| > \epsilon m$, where $\Gamma(y)$ denoted the set of neighbours of y in G.

Let us show that for every y,

$$\mathbf{Pr}(\mathcal{E}_y) \geq \frac{2m}{n}. \tag{6}$$

Let $d = d(y)$ be the degree of y. By symmetry, we can assume that $d \leq \frac{n-1}{2}$. For such d we bound from below the probability p that $y \notin X$ and $|\Gamma(y) \cap X| \leq m/2 - \epsilon m$, which equals

$$p = \sum_{i < m/2 - \epsilon m} \binom{d}{i} \binom{n-1-d}{m-i} \binom{n}{m}^{-1}.$$

The combinatorial meaning of p implies that it decreases with d, so it is enough to bound p for $d = \lfloor \frac{n-1}{2} \rfloor$ only. Let us consider the summands s_h corresponding to $i = m/2 - h$ with, say, $\epsilon m < h \leq \epsilon m + n^{1/3}$. Let

$$f(x) = (1+x)^{\frac{1+x}{2}} (1-x)^{\frac{1-x}{2}}.$$

Its Taylor series at 0 is $1+\frac{x^2}{2}+O(x^4)$. By Stirling's formula, we obtain that each summand

$$\begin{aligned} s_h &= \Omega\left(\frac{n^{-1/3}(\log n)^{1/6}}{f^m(\frac{2h}{m})\, f^{2d-m}(\frac{2h}{2d-m})}\right) \\ &= \exp\left(-\frac{1}{3}\log n - \frac{2h^2}{m} - \frac{2h^2}{2d-m} + O(1)\right) \\ &= n^{-1/3-2c^3-o(1)}. \end{aligned}$$

Thus

$$\sum_{h=\epsilon m}^{\epsilon m+n^{1/3}} s_h = n^{-2c^3-o(1)} \geq \frac{2m}{n}.$$

It follows that there is a choice of an m-set X such that $|Y| \geq 2m$, where Y consists of the vertices for which R_x holds. By definition $Y \cap X = \emptyset$.

Assume without loss of generality that we have $d_X(y) < m - \epsilon m$ for at least half of the vertices of Y. Let $Z \subset Y$ consist of any m of these vertices. This pair (X, Z), both sets having at least ϵn elements, has the required bias.

Property P3

Here we show that Breaker can force Maker to create a co-degree of at least $\frac{n}{4} + c\sqrt{n}$. Our argument is based on a theorem of Beck [5], which states that Breaker can force Maker's graph to have maximum degree at least $n/2 + \sqrt{n}/20$. Then the following lemma shows that Breaker also succeeds in forcing a high co-degree in Maker's graph.

Lemma 5 *Assume that $c_1 > 0$ is constant. Then for sufficiently large n, the following holds: Let $G = (V, E)$ be a graph on n vertices with $n(n-1)/4$ edges. If G has a vertex of degree at least $n/2 + c_1\sqrt{n}$, then G has a pair of vertices w_1, w_2 whose co-degree is at least $n/4 + c_1\sqrt{n}/10$.*

Proof Let $c_2 = c_1/10$. Let v be a vertex of maximum degree in G. Denote $N_1 = N(v)$, $N_2 = V - N_1$. Then $|N_2| \leq n/2 - c_1\sqrt{n}$. If there is $u \in V$ such that $d(v, N_1) \geq n/4 + c_2\sqrt{n}$, we are done. Otherwise, for every u, $d(u, N_1) \leq n/4 + c_2\sqrt{n}$, implying:

$$\begin{aligned} A &\stackrel{def}{=} \sum_{u \in V} d(u, N_2) \\ &\geq \sum_{u \in V} (d(u) - d(u, N_1) - 1) \\ &\geq 2|E| - n(n/4 + c_2\sqrt{n}) - n \\ &= n^2/4 - c_2 n^{3/2} - 3n/2. \end{aligned}$$

Therefore by convexity,

$$B \stackrel{def}{=} \sum_{u\in V} \binom{d(u,N_2)}{2} \geq n\binom{A/n}{2} \geq n^3/32 - c_2 n^{5/2} - O(n^2).$$

On the other hand,

$$B = \sum_{w_1\neq w_2\in N_2} \text{co-degree}(w_1,w_2),$$

and thus there is a pair $w_1, w_2 \in N_2$ such that:

$$\begin{aligned}
\text{co-degree}(w_1,w_2) &\geq |B|/\binom{|N_2|}{2} \\
&\geq \frac{n^3/32 - c_2 n^{5/2} - O(n^2)}{\binom{n/2-c_1 n^{1/2}}{2}} \\
&\geq n/4 + c_2\sqrt{n}.
\end{aligned}$$

□

4 Consequences

As we have already mentioned in the introduction, Maker's ability to create a pseudo-random graph of density about $\frac{1}{2}$ allows him to win quite a few other combinatorial games. We will describe some of them below. All these games are played on the complete graph K_n unless stated otherwise, Maker and Breaker choose one edge alternately, Maker's aim is to create a graph that possesses a desired graph property.

Edge-disjoint Hamilton cycles. In this game Maker's aim is to create as many pairwise edge disjoint Hamilton cycles as possible. Lu proved [13] that Maker can always produce at least $\frac{1}{16}n$ Hamilton cycles and conjectured that Maker should be able to make $(\frac{1}{4}-\epsilon)n$ for any fixed $\epsilon > 0$. This conjecture follows immediately from our Theorem 1 and Theorem 2 of [11]. In [11], Frieze and Krivelevich show that a 2ϵ-regular graph contains at least $(\frac{1}{2}-6.5\epsilon)n$ edge disjoint Hamilton cycles, for all $\epsilon > 10(\log n/n)^{1/6}$. Our argument applies equally to the bipartite version of the problem where the game is played on the complete bipartite graph $K_{n,n}$. Thus Maker can always produce at least $(\frac{1}{4}-\epsilon)n$ edge disjoint Hamilton cycles, verifying another conjecture of Lu [14], [15]. Finally, there is an analogous game that can be played on the complete digraph D_n and here Maker can always produce at least $(\frac{1}{2}-\epsilon)n$ edge disjoint Hamilton cycles.

Vertex-connectivity. Theorem 2 can be used to show that Maker can always force an $(n/2 - 3\sqrt{n\log n})$-vertex-connected graph. Indeed, let Maker's graph M have minimum degree at least $n/2-3\sqrt{n\log n}$ and maximum co-degree at most $n/4+3\sqrt{n\log n}$. Suppose that the removal of some set R disconnects M, say $V(M) \setminus R = A \cup B$ with $|A| \le |B|$. If $|A| = 1$, then obviously all neighbours of $a \in A$ are in R, implying $|R| \ge \delta(M) \ge n/2 - 3\sqrt{n \log n}$. If $|A| \ge 2$, let a_1, a_2 be two distinct vertices in A. Then all neighbours of a_1, a_2 lie in $A \cup R$, and therefore

$$|A| + |R| \ge \deg_M(a_1) + \deg_M(a_2) - \text{co-deg}_M(a_1, a_2) \ge \frac{3n}{4} - 9\sqrt{n \log n}\,.$$

If $|A| \ge n/4 - 6\sqrt{n \log n}$, then $|B| \ge |A| \ge n/4 - 6\sqrt{n \log n}$ as well, and by the $o(1)$-regularity of M there is an edge between A and B – a contradiction. We conclude that $|A| \le n/4 - 6\sqrt{n \log n}$, implying $|R| \ge n/2 - 3\sqrt{n \log n}$, as required.

The result of Beck [3] showing that Breaker can force a vertex which has degree at most $n/2 - \Omega(\sqrt{n})$ in Maker's graph indicates that the error term in our result about the connectivity game is tight up to a logarithmic factor.

$c \log n$-Universality. A graph G is called *r-universal* if it contains an induced copy of every graph H on r vertices. We can show the following result.

Theorem 6 *Let $r = r(n)$ be an integer, which satisfies*

$$\frac{n - r + 1}{r} \left(\frac{1}{2} - \epsilon\right)^{r-1} \ge \frac{2 \log n}{\epsilon^2},$$

for some $\epsilon = \epsilon(n) \to 0$. Then for all sufficiently large n Maker can ensure that his graph M is r-universal.

Proof. Let $t = \lfloor \frac{2 \log n}{\epsilon^2} \rfloor$. Let n be sufficiently large so that the conclusion of Lemma 4 is valid. Let M be an arbitrary graph satisfying this property, that is, any pair of disjoint subsets of $V(M)$, both of size at least t, is ϵ-unbiased. Let G be any graph on $[r]$. We will show that G is an induced subgraph of M.

Partition $V(M) = \cup_{i=1}^{r} V_i$ into r parts, each having at least $\frac{n-r+1}{r}$ vertices. Initially, let $A_i = V_i$, $i \in [r]$. We define $f : [r] \to V(M)$ with $f(i) \in A_i$ inductively.

Suppose we have already defined f on $[i-1]$. It will be the case that $|A_j| \ge \frac{n-r+1}{r} \eta^{i-1}$ for any $j \ge i$, where for brevity $\eta = \frac{1}{2} - \epsilon$. We will choose $f(i) = v \in A_i$ so that for any $j > i$ we have

$$|A_{ji}(v)| \ge \eta |A_j|, \tag{7}$$

where we define $A_{ji}(v) = A_j \cap \Gamma_M(v)$ if $\{i, j\} \in E(G)$ and $A_{ji}(v) = A_j \setminus \Gamma_M(v)$ otherwise. (Here $\Gamma_M(v)$ is the set of neighbours of v in M.)

Let B_{ji} be the set of vertices of A_i violating (7), i.e. $\{v \in A_i : \ |A_{ji}(v)| < \eta |A_j|\}$. Then $|B_{ji}| < t$ as the pair (B_{ji}, A_j) is not ϵ-unbiased. (Observe that $|A_j| \ge \frac{n-r+1}{r} \eta^{r-1} \ge t$.) Update A_i by deleting B_{ji} for all $j \in [i+1, r]$. Thus at least $\frac{n-r+1}{r} \eta^{i-1} - (r-i)t \ge t$

vertices still remain in A_i. This inequality is true for $i = r$ by our assumption and for any other i, because $\eta \le \frac{1}{2}$. So a suitable $f(i)$ can always be found. Now, replace A_j with $A_{ji}(f(i))$ for $j > i$. This completes the induction step. At the end of the process $f([r])$ induces a copy of G in M. $\square$

It follows from Theorem 6 that Maker can create an r-universal graph with $r = (1 + o(1)) \log_2 n$. On the other hand, Maker cannot achieve $r = 2\log_2 n - 2\log_2\log_2 n + C$ because, as was shown by Beck [4, Theorem 4], Breaker can prevent K_r in Maker's graph.

There is a remarkable parallel between random graphs and Maker-Breaker games, see e.g. Chvátal and Erdős [8], Beck [3, 4], Bednarska and Łuczak [6]. As shown by Bollobás and Thomason [7], the largest r such that a random graph of order n is almost surely r-universal is around $2\log_2 n$. We conjecture that games have the same universality threshold (asymptotically).

Conjecture 7 *Maker can claim an r-universal graph with $r = (2 + o(1)) \log_2 n$.*

The following related result improves the unbiased case of Theorem 4 in Beck [3]. (His assumption $n \ge 100r^3v3^{r+1}$ is stronger than ours.)

Theorem 8 *Let integers r, v and a real $\epsilon > 0$ (all may depend on n) satisfy $\epsilon \to 0$ and*

$$\frac{n-r+1}{r}\left(\frac{1}{2}-\epsilon\right)^{r-1} \ge v + \frac{2\log n}{\epsilon^2}.$$

Then for sufficiently large n, Maker can ensure that any graph G of order at most v and maximum degree less than r is a subgraph (not necessarily induced) of Maker's graph M.

Outline of Proof. Use the method of Theorem 6 with the following changes. Take a proper colouring $c : V(G) \to [r]$. The desired f will map $i \in V(G)$ into $A_{c(i)}$. The proof goes the same way except that when choosing $f(i)$ we have to worry only about those $j \ge i$ which are neighbours of i in G and make sure that there are at least v good choices for $f(i) \in A_{c(i)}$ (so that we can ensure that f is injective). The details are left to the Reader. $\square$

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