Hamilton cycles in highly connected and expanding graphs

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Abstract

In this paper we prove a sufficient condition for the existence of a Hamilton cycle, which is applicable to a wide variety of graphs, including relatively sparse graphs. In contrast to previous criteria, ours is based on only two properties: one requiring expansion of "small" sets, the other ensuring the existence of an edge between any two disjoint "large" sets. We also discuss applications in positional games, random graphs and extremal graph theory.

1 Introduction

A Hamilton cycle in a graph G is a cycle passing through all vertices of G. A graph is called *Hamiltonian* if it admits a Hamilton cycle. Hamiltonicity is one of the most central notions in Graph Theory, and many efforts have been devoted to obtain sufficient conditions for the existence of a Hamilton cycle (a "nice" necessary and sufficient condition should not be expected however, as deciding whether a given graph contains a Hamilton cycle is known to be NP-complete). In this paper we will mostly concern ourselves with establishing a sufficient condition for Hamiltonicity which is applicable to a wide class of sparse graphs.

One of the first Hamiltonicity results is the celebrated theorem of Dirac [8], which asserts that if the minimum degree of a graph G on n vertices is at least $n/2$ then G is Hamiltonian.

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Since then, many other sufficient conditions that deal with dense graphs, were obtained (see e.g. [11] for a comprehensive reference). However, all these conditions require the graph to have $\Theta(n^2)$ edges whereas for a Hamilton cycle, only n edges are needed. Chvátal and Erdős [6] proved that if $\kappa(G) \geq \alpha(G)$ (that is, the vertex connectivity of G is at least as large as the size of a largest independent set in G) then G is Hamiltonian. Note that if G is a d-regular graph, then $\kappa(G) \leq d$ and $\alpha(G) \geq \frac{n}{d+1}$; hence the Chvátal-Erdős criterion cannot be applied if $d \leq c\sqrt{n}$ for an appropriate constant c.

When looking for sufficient conditions for the Hamiltonicity of sparse graphs, it is natural to look at random graphs with an appropriate edge probability. Erdős and Rényi [9] raised the question of what is the threshold for Hamiltonicity in random graphs. After a series of efforts by various researchers, including Korshunov [13] and Pósa [16], the problem was finally solved by Komlós and Szemerédi [14], who proved that if $p = (\log n + \log\log n + \omega(1))/n$, where $\omega(1)$ tends to infinity with n arbitrarily slowly, then $G(n,p)$ is a.s. Hamiltonian. Note that this is best possible since for $p \leq (\log n + \log\log n - \omega(1))/n$ almost surely there are vertices of degree at most one in $G(n,p)$.

The next natural step is to look for Hamilton cycles in relatively sparse pseudo-random graphs. During the last few years, several such sufficient conditions were found (see e.g. [10, 15]). These are quite complicated at times as they rely on many properties of pseudo-random graphs. Furthermore, one can argue that these conditions are not the most natural, as Hamiltonicity is a monotone increasing property, whereas pseudo-randomness is not. Our main result is a natural and simple (at least on the qualitative level) sufficient condition based on expansion and high connectivity. Before stating the result we introduce and discuss the following properties of a graph $G = (V,E)$ where $|V| = n$. As usual, the notation $N(S)$ stands for the *external neighborhood* of S, that is, $N(S) = \{v \in V \setminus S : \exists u \in S,\ (u,v) \in E\}$. Let $d = d(n)$ be a parameter.

P1 For every $S \subset V$, if $|S| \leq \frac{n \log\log n \log d}{d \log n \log\log\log n}$ then $|N(S)| \geq d|S|$;

P2 There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \geq \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$.

From now on, for the sake of convenience, we denote

$$m = m(n,d) = \frac{\log n \cdot \log\log\log n}{\log\log n \cdot \log d} .$$

Let us give an informal interpretation of the above conditions. Condition P1 guarantees *expansion*: every sufficiently small vertex subset (of size $|S| \leq \frac{n}{dm}$) expands by a factor of d. Condition P2 is what can be classified as a *high connectivity* condition of some sort: every two disjoint subsets $A, B \subseteq V$ which are relatively large (of size $|A|, |B| \geq \frac{n}{4130m}$) are connected by at least one edge. Note that properties P1 and P2 together guarantee some expansion for *every* $S \subset V(G)$ of size $o(n)$. Indeed, if $|S| \le \frac{n}{dm}$ then $|N(S)| \ge d|S|$ by property P1. If $\frac{n}{dm} < |S| < \frac{n}{4130m}$ (assuming $d > 4130$) then S contains a subset of size exactly $\frac{n}{dm}$ and so by property P1 expands at least to a size of $\frac{n}{m}$, that is it expands by a factor of at least 4130. Finally, if $|S| \ge \frac{n}{4130m}$ then $N(S) \ge (1 - o(1))n$ as, by property P2, the number of vertices of $V \setminus S$ that do not have any neighbor in S is strictly less than $\frac{n}{4130m}$.

We can now state our main result:

Theorem 1.1 *Let $12 \le d \le e^{\sqrt[3]{\log n}}$ and let G be a graph on n vertices satisfying properties P1, P2 as above; then G is Hamiltonian, for sufficiently large n.*

The lower bound on d in the theorem above can probably be somewhat improved through a more careful implementation of our arguments. As for the upper bound $d \le e^{\sqrt[3]{\log n}}$, it is a mere technicality, as one expects that proving that denser graphs (that is, graphs for which d is larger) are Hamiltonian should in fact be easier. The requirement $d \le e^{\sqrt[3]{\log n}}$ makes sure (in particular) that $\frac{n}{4130m} = o(n)$ and so P2 is a non-trivial condition. We can obtain a sufficient condition for Hamiltonicity, similar to that of Theorem 1.1, and applicable to graphs with larger values of $d = d(n)$ as well; more details are given in Section 2.4.

It is instructive to observe that neither P1 nor P2 is enough to guarantee Hamiltonicity by itself, without relying on its companion property (unless of course they degenerate to something trivial). Indeed, for property P1 observe that the complete graph $K_{n,n+1}$ is a very strong expander locally, yet it obviously does not contain a Hamilton cycle. As for property P2, the graph G formed by a disjoint union of a clique of size $n - \frac{n}{4130m} + 1$ and $\frac{n}{4130m} - 1$ isolated vertices clearly meets P2, but is obviously quite far from being Hamiltonian. Thus, P1 and P2 complement each other in an essential way.

Next, we discuss several applications of our main result. Theorem 1.1 was first used by the authors (see [12]) to address a problem of Beck [3]: they proved that Enforcer can win the $(1, q)$ Avoider-Enforcer Hamilton cycle game, played on the edges of K_n, for every $q \le \frac{cn \log\log\log\log n}{\log n \log\log\log n}$ where c is an appropriate constant; this is presently the best known bound. A similar result can be obtained for Maker in the corresponding Maker-Breaker game. The latter result falls short of the true value of the critical bias of $\frac{cn}{\log n}$ obtained by Beck (see [2]), but our proof is conceptually simpler and shorter. [A brief background: both Maker-Breaker and Avoider-Enforcer games mentioned above are played on the edge set of the complete graph K_n. In every move, Maker (resp. Avoider) claims one unoccupied edge, then Breaker (resp. Enforcer) responds by claiming q unoccupied edges. The game ends when all edges have been claimed by one of the players. In the Maker-Breaker Hamiltonicity game Maker wins if he creates a Hamilton cycle, otherwise Breaker wins. In the Avoider-Enforcer version, Avoider wins if he avoids creating a Hamilton cycle by the end of the game, otherwise Enforcer wins.] More details can be found in [12].

In this paper we prove several other corollaries of Theorem 1.1.

A graph $G = (V, E)$ is called *Hamilton-connected* if for every $u, v \in V$ there is a Hamilton path in G from u to v.

Theorem 1.2 *Let $G = (V, E)$ be a graph that satisfies properties P1 and P2; then G is Hamilton-connected.*

Remark. An immediate consequence of Theorem 1.2 is that for every edge $e \in E$ there is a Hamilton cycle of G that includes e.

A graph G is called *pancyclic* if it admits a cycle of length k for every $3 \leq k \leq n$. We prove that a graph which satisfies property P2 is "almost pancyclic".

Theorem 1.3 *Let $G = (V, E)$, where $|V| = n$ is sufficiently large, be a graph, satisfying property P2; more precisely, for every disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \geq n/t$, where $t = t(n) \geq 2$, there is an edge between a vertex of A and a vertex of B. Then G admits a cycle of length exactly k for every $\frac{8n \log n}{t \log \log n} \leq k \leq n - 3n/t$.*

Remark. The upper bound on k in Theorem 1.3 is tight up to a constant factor in the second order term, as shown by a disjoint union of $K_{n+1-n/t}$ and $n/t - 1$ isolated vertices. On the other hand, we believe that the lower bound can be improved to $\frac{c \log n}{\log t}$ for some constant c. Methods recently utilized by Verstraëte [18] and by Sudakov and Verstraëte [17] can possibly be used to establish this conjecture.

Theorem 1.1 (with minor changes to the proof) can be used to prove the following classic result (see [14]).

Theorem 1.4 *$G(n, p)$, where $p = (\log n + \log \log n + \omega(1))/n$, is a.s. Hamiltonian.*

Let $G = (V, E)$, where $|V| = n$, and let $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$. A pair (A, B) of proper subsets of V is called a *separation* of G if $A \cup B = V$ and there are no edges in G between $A \setminus B$ and $B \setminus A$. The graph G is called f-connected if $|A \cap B| \geq f(|A \setminus B|)$, for every separation (A, B) of G with $|A \setminus B| \leq |B \setminus A|$. In [5] it was proved that if $f(k) \geq 2(k + 1)^2$ for every $k \in \mathbb{N}$ then G is Hamiltonian for every $n \geq 3$. It was also conjectured that there exists a function f which is linear in k and is enough to ensure Hamiltonicity. Using Theorem 1.1, we can get quite close to proving this conjecture for sufficiently large n:

Theorem 1.5 *If $G = (V, E)$, where $|V| = n$, is f-connected for $f(k) = k \log k + O(1)$, then it is Hamiltonian for sufficiently large n.*

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. All of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. We say that some event holds *almost surely*, or a.s. for brevity, if the probability it holds tends to 1 as n tends to infinity. Our graph-theoretic notation is standard and follows that of [7].

The rest of the paper is organized as follows: in Section 2 we prove and discuss Theorem 1.1, in Section 3 we prove its corollaries: Theorems 1.2, 1.3, 1.4 and 1.5.

2 Proof of the main result

The proof of Theorem 1.1 is based on the ingenious rotation-extension technique, developed by Pósa [16], and applied later in a multitude of papers on Hamiltonicity (mostly of random graphs). Our proof technique borrows some technical ideas from the paper of Ajtai, Komlós and Szemerédi [1].

Before diving into fine details of the proof, we would like to compare our Hamiltonicity criterion and its proof with its predecessors. Several previous papers, including [1], [10], [15], state, explicitly or implicitly, sufficient conditions for Hamiltonicity applicable in principle to sparse graphs. Usually criteria of this sort are carefully tailored to be applied to random or pseudo-random graphs, and are therefore rather complicated and not always natural. Moreover, such criteria are sometimes fragile in the sense that they can be violated by adding more edges to the graph – a somewhat undesirable feature. Our criterion in Theorem 1.1 is (on a qualitative level, at least) quite natural and easily comprehensible, and can be potentially applied to a very wide class of graphs. As for our proof, due to the relative simplicity of the conditions we use, the argument is perhaps more involved than some of the previous proofs; there are however similarities. A novel ingredient, relying heavily on Property P2, is the part presented in Section 2.2 (finding many good initial rotations).

In order to be able to refer to the proof of our criterion while proving some of the corollaries we break the proof into four parts, each time indicating which property is needed for which part.

Proposition 2.1 *Let G satisfy properties P1 and P2. Then G is connected.*

Proof If not, let C be the smallest connected component of G. Then by P1, $|C| > \frac{n}{m}$, but then by P2, $E(C, V \setminus C) \neq \emptyset$ – a contradiction. □

2.1 Constructing an initial long path

In this subsection we show that a graph which satisfies some expansion properties (that is, property P1 and some expansion of larger sets, implied by property P2) contains a long path, and even more, it has many paths of maximum length starting at the same vertex.

Let $P_0 = (v_1, v_2, \ldots, v_q)$ be a path of maximum length in G. If $1 \leq i \leq q-2$ and (v_q, v_i) is an edge of G then $P' = (v_1 v_2 \ldots v_i v_q v_{q-1} \ldots v_{i+1})$ is also of maximum length. P' is called a *rotation* of P_0 with *fixed endpoint* v_1 and *pivot* v_i. The edge (v_i, v_{i+1}) is called the *broken* edge of the rotation. We say that the segment $v_{i+1} \ldots v_q$ of P_0 is reversed in P'.

In case the new endpoint, v_{i+1}, has a neighbor v_j such that $j \notin \{i, i+2\}$, then we can rotate P' further to obtain more paths of maximum length. We use rotations and extensions together with property P1 to find a path of maximum length with large rotation endpoint sets (see for example [4], [10], [14], [15]).

Claim 2.2 *Let $G = (V, E)$ be a graph on n vertices that satisfies property P1 and moreover any subset of V of size $n/4130m$ has at least $n - o(n)$ external neighbors. Let $P_0 = (v_1, v_2, \ldots, v_q)$ be a path of maximum length in G. Then there exists a set $B(v_1) \subseteq V(P_0)$ of at least $n/3$ vertices, such that for every $v \in B(v_1)$ there is a $v_1 v$-path of maximum length which can be obtained from P_0 by at most $\frac{2 \log n}{\log d}$ rotations with fixed endpoint v_1. In particular $|V(P_0)| \geq n/3$.*

Proof Let t_0 be the smallest integer such that $\left(\frac{d}{3}\right)^{t_0 - 2} > \frac{n}{md}$. Note that $t_0 \leq 2\frac{\log n}{\log d}$.

We prove that there exists a sequence of sets $S_0, \ldots, S_{t_0} = B(v_1) \subseteq V(P_0) \setminus \{v_1\}$ of vertices such that for every $0 \leq t \leq t_0$, every $v \in S_t$ is the endpoint of a path, obtainable from P_0 by t rotations with fixed endpoint v_1, such that for every $0 \leq i \leq t$, after the ith rotation the non-v_1-endpoint of the path is in S_i, and moreover $|S_t| = \left(\frac{d}{3}\right)^t$ for every $t \leq t_0 - 3$, $|S_{t_0-2}| = \frac{n}{dm}$, $|S_{t_0-1}| = \frac{n}{4130m}$, and $|S_{t_0}| \geq n/3$.

First we construct the sets by induction on t. For $t = 0$, one can choose $S_0 = \{v_q\}$ and all requirements are trivially satisfied.

Induction step: let $0 < t \leq t_0 - 2$ and assume that the appropriate sets $S_0, \ldots, S_{t-1}$ with the appropriate properties were already constructed. We will now construct S_t. Let first

$$T = \{v_i \in N(S_{t-1}) : v_{i-1}, v_i, v_{i+1} \notin \bigcup_{j=0}^{t-1} S_j\} \ .$$

be the set of potential pivots for the tth rotation. Assume now that $v_i \in T$, $y \in S_{t-1}$ and $(v_i, y) \in E$. Then a $v_1 y$-path Q can be obtained from P_0 by $t-1$ rotations such that after the jth rotation, the non-v_1-endpoint is in S_j for every $j \leq t-1$. Each such rotation breaks an edge incident with the new endpoint. Since v_{i-1}, v_i, v_{i+1} are not endpoints after any of these $t-1$ rotations, both edges (v_{i-1}, v_i) and (v_i, v_{i+1}) of the original path P_0 must be unbroken and thus must be present in Q.

Hence, rotating Q with pivot v_i will make either v_{i-1}, or v_{i+1} an endpoint (which one, depends on whether the unbroken segment $v_{i-1} v_i v_{i+1}$ is reversed or not after the first $t-1$

rotations). Assume w.l.o.g. it is v_{i-1}. We add v_{i-1} to the set $\hat{S}_t$ of new endpoints and say that v_i *placed* v_{i-1} *in* $\hat{S}_t$. The only other vertex that can place v_{i-1} in $\hat{S}_t$ is v_{i-2} (if it exists). Thus,

$$
\begin{aligned}
|\hat{S}_t| &\geq \frac{1}{2}|T| \geq \frac{1}{2}\left(|N(S_{t-1})| - 3(1 + |S_1| + \ldots + |S_{t-1}|)\right) \\
&\geq \frac{d}{2}\left(\frac{d}{3}\right)^{t-1} - \frac{3}{2}\frac{(d/3)^t - 1}{d/3 - 1} \geq \left(\frac{d}{3}\right)^t
\end{aligned}
$$

where the last inequality follows since $d \geq 12$. Clearly we can delete arbitrary elements of $\hat{S}_t$ to obtain S_t of size exactly $\left(\frac{d}{3}\right)^t$ if $t \leq t_0 - 3$ and of size exactly $\frac{n}{dm}$ if $t = t_0 - 2$. So the proof of the induction step is complete and we have constructed the sets $S_0, \ldots, S_{t_0-2}$.

To construct S_{t_0-1} and S_{t_0} we use the same technique as above, only the calculation is slightly different. Since $|N(S_{t_0-2})| \geq d \cdot \frac{n}{dm}$, we have

$$
\begin{aligned}
|\hat{S}_{t_0-1}| &\geq \frac{1}{2}|T| \geq \frac{1}{2}\left(|N(S_{t_0-2})| - 3(1 + |S_1| + \ldots + |S_{t_0-3}| + |S_{t_0-2}|)\right) \\
&\geq \frac{n}{2m} - \frac{3}{2}\left(\frac{(d/3)^{t_0-3} - 1}{(d/3) - 1} + 2\frac{n}{dm}\right) \geq \frac{n}{2m} - \frac{3}{2} \cdot \left(\frac{d}{3}\right)^{t_0-3} - 3\frac{n}{dm} \\
&\geq \frac{n}{2m} - \frac{3}{2} \cdot \frac{n}{dm} - 3\frac{n}{dm} \geq \frac{n}{4130m},
\end{aligned}
$$

where the last inequality follows since $d \geq 12$.

For S_{t_0} the difference in the calculation comes from using the expansion guaranteed by property P2 rather than the one guaranteed by property P1, that is, $|N(S_{t_0-1})| \geq n - o(n)$. We have

$$
\begin{aligned}
|S_{t_0}| &\geq \frac{1}{2}|T| \geq \frac{1}{2}\left(|N(S_{t_0-1})| - 3(1 + |S_1| + \ldots + |S_{t_0-2}| + |S_{t_0-1}|)\right) \\
&\geq \frac{n}{2}(1 - o(1)) - \frac{3}{2}\left(\frac{(d/3)^{t_0-3} - 1}{(d/3) - 1} + \frac{2n}{dm} + \frac{n}{4130m}\right) \\
&\geq \frac{n}{2}(1 - o(1)) - \frac{3}{2}\left(\frac{3n}{dm} + \frac{n}{4130m}\right) \\
&\geq \frac{n}{3},
\end{aligned}
$$

where the last inequality follows since $n \geq 2m$ and $d \geq 12$.

The set S_{t_0} can be chosen to be $B(v_1)$ and satisfies all the requirements of the Claim. Note that since $S_{t_0} \subseteq V(P_0)$, we have $|V(P_0)| > n/3$. This concludes the proof of the claim. $\Box$

Remark Note that, although we do not need it here, the rotations which create these paths always brake an edge of the original path P_0.

2.2 Finding many good initial rotations

In this subsection we prove an auxiliary lemma, which will be used in the next subsection to conclude the proof of Theorem 1.1.

Let H be a graph with a spanning path $P = (v_1, \ldots, v_l)$. For $2 \leq i < l$ let us define the auxiliary graph H_i^+ by adding a vertex and two edges to H as follows: $V(H_i^+) = V(H) \cup \{w\}$, $E(H_i^+) = E(H) \cup \{(v_l, w), (v_i, w)\}$. Let P_i be the spanning path of H_i^+ which we obtain from the path $P \cup \{(v_l, w)\}$ by rotating with pivot v_i. Note that the endpoints of P_i are v_1 and v_{i+1}.

For a vertex $v_i \in V(H)$ let S^{v_i} be the set of those vertices of $V(P) \setminus \{v_1\}$, which are endpoints of a spanning path of H_i^+ obtained from P_i by a series of rotations with fixed endpoint v_1.

A vertex $v_i \in V(P)$ is called a *bad initial pivot* (or simply a *bad vertex*) if $|S^{v_i}| < \frac{l}{43}$ and is called a *good initial pivot* (or a *good vertex*) otherwise. We can rotate P_i and find a large number of endpoints provided v_i is a good initial pivot.

Using an argument similar to the one used in the proof of Claim 2.2, we can show that H has many good initial pivots provided that a certain condition, similar to property P2, is satisfied.

Lemma 2.3 *Let H be a graph with a spanning path $P = (v_1, \ldots, v_l)$. Assume that every two disjoint sets A, B of vertices of H of sizes $|A|, |B| \geq l/43$ are connected by an edge. Then we have*

$$|R| \leq 7l/43,$$

where $R = R(P) \subseteq V(P)$ is the set of bad vertices.

Proof We will create a set $U \subseteq V(H)$, whose size is at least $|R|/7$, but does not expand enough, that is, $|U \cup N_H(U)| \leq 21|U|$. This in turn will imply that the set R of bad vertices cannot be big.

Let $R = \{v_{i_1}, \ldots, v_{i_r}\}$. We *process* the vertices of R one after the other. We will maintain subsets U and X of $V(H)$ where initially $U = X = \emptyset$. Whenever we finish processing a vertex of R we update the sets U and X. The following properties will hold after the processing of v_{i_j}.

$$U \subseteq X\,, \quad N_H(U) \subseteq ext(X)\,, \quad |U| \geq \frac{1}{7}|X|\,, \quad \{v_{i_1+1}, \ldots, v_{i_j+1}\} \subseteq X\ , \qquad (1)$$

where $ext(X)$ denotes the set containing the vertices of X together with their left and right neighbors on P. Clearly $|ext(X)| \leq 3|X|$.

Suppose the current vertex to process is v_{i_j}. If $v_{i_j+1} \in X$, then we do not change U and X and so the conditions of (1) trivially hold by induction.

Otherwise, we will create sets $W_t \subseteq S^{v_{i_j}}$ inductively, such that for every t the following hold.

- (a) $W_t \subseteq S_t^{v_{i_j}}$;
- (b) $|W_t| = 2^t$;
- (c) $W_t \cap \left(\cup_{s=0}^{t-1} W_s \cup X\right) = \emptyset$,

where $S_t^{v_{i_j}}$ contains those vertices y of $S^{v_{i_j}}$ for which a spanning path of H_i^+ ending at y can be produced from P_i by t rotations with fixed endpoint v_1, such that after the sth rotation the new endpoint is in W_s, for every $s < t$.

We begin by setting $W_0 = \{v_{i_j+1}\}$. Conditions (a) and (b) trivially hold, for condition (c) note that $v_{i_j+1} \notin X$.

Assume now that we have constructed $W_0, \ldots, W_t$ with properties $(a) - (c)$. If $|N_H(W_t) \setminus ext((\cup_{i=1}^t W_i \cup X))| > 5|W_t|$, then we create W_{t+1} with properties $(a) - (c)$, otherwise we finish the processing of v_{i_j} by updating U and X.

Let $T_t = N_H(W_t) \setminus ext((\cup_{i=1}^t W_i \cup X))$ and assume first that $|T_t| > 5|W_t|$. We use an argument similar to the one used in Claim 2.2 to create W_{t+1} with properties $(a) - (c)$.

Let $v_i \in T_t$, $v_i \neq v_1, v_l$, and suppose that v_i is adjacent to $y \in W_t$. Recall, that by property (a) a spanning path Q of H_i^+ ending at y can be produced from P_i by t rotations, such that for every $s < t$, after the sth rotation the new endpoint is in W_s. Since the vertices v_{i-1}, v_i and $v_{i+1} \notin \bigcup_{s=0}^t W_s$, they are not endpoints after any of these t rotations. Each rotation breaks an edge incident with the new endpoint, hence both edges (v_{i-1}, v_i) and (v_i, v_{i+1}) of the original path P_i must be present in Q. Rotating Q with pivot v_i will brake one of them. Such a rotation also makes one of v_{i-1} and v_{i+1} into an endpoint, and as such, into an element of $S_{t+1}^{v_{i_j}}$. Denote this vertex by v_i'. We define $W_{t+1} = \{v_i' : v_i \in T_t\}$. We say that *$v_i'$ is placed in W_{t+1} by v_i*. Observe that besides v_i the only other vertex that can place v_i' in W_{t+1} is its other neighbor on the path P_i. Thus,

$$|W_{t+1}| \geq \left\lceil \frac{1}{2}(|T_t| - 2) \right\rceil \geq 2|W_t|.$$

Deleting arbitrarily some vertices from W_{t+1} we can make sure that its cardinality is exactly $2|W_t|$. Properties (a) and (b) are then naturally satisfied. Property (c) is satisfied because, by the definition of T_t we have $v_i \notin ext\,(\cup_{s=0}^{t} W_s \cup X)$ and so none of its neighbors on P_i, in particular v_i', is an element of $(\cup_{s=0}^{t} W_s \cup X)$.

Property (b) ensures that $|\cup_{s=0}^{t} W_s|$ is strictly increasing so the processing of the vertex v_{i_j} is bound to reach a point in which $|T_k| \le 5|W_k|$ for some index k. At that point we update U and X by adding W_k to U and adding $W_1 \cup \cdots \cup W_k \cup T_k$ to X. We have to check that the conditions of (1) hold.

Observe that $|W_1 \cup \cdots \cup W_k| < 2|W_k|$, so the number of vertices added to X is at most seven times more than the number of vertices added to U. Also, property (c) and $U \subseteq X$ made sure that W_k was disjoint from U, so indeed the property $|U| \ge |X|/7$ remains valid. The other conditions in (1) follow easily from the definition of the "new" U and X. Hence the processing of v_{i_j} is complete.

Claim $|U| \le l/43$.

Proof Assume the contrary and let j be the smallest index, such that $|U| > l/43$ after the processing of v_{i_j}.
Observe that $|U| \le 2l/43$. Indeed, after the processing of v_{i_j} the set U received at most $|S^{v_{i_j}}|$ vertices, which is at most $l/43$, due to the fact that v_{i_j} is a bad vertex. We thus have $l/43 < |U| \le 2l/43$, $U \subseteq X$, $N_H(U) \subseteq ext(X)$ and $|ext(X)| \le 3|X| \le 21|U|$. Then $|V(P) \setminus ext(X)| \ge l/43$, and there are no edges of H between U and $V(P) \setminus ext(X)$. This contradicts our assumption on H. □

To conclude the proof of the Lemma we note that after processing all vertices of R, we have $R^+ := \{v_{i_1+1}, \ldots, v_{i_r+1}\} \subseteq X$ and $|U| \ge |X|/7$ by (1). Since $|U| \le l/43$, it follows that $|R| = |R^+| \le 7l/43$. □

2.3 Closing the maximal path

Lemma 2.4 *Let G be a connected graph that satisfies property P2. Let the conclusion of Claim 2.2 be also true for G, that is, for every path $P_0 = (v_1, v_2, \ldots, v_q)$ of maximum length in G there exists a set $B(v_1) \subseteq V(P_0)$ of at least $n/3$ vertices, such that for every $v \in B(v_1)$ there is a v_1v-path of maximum length which can be obtained from P_0 by at most $t_0 \le \frac{2\log n}{\log d}$ rotations with fixed endpoint v_1. Then G is Hamiltonian.*

Proof We will prove that there exists a path of maximum length which can be closed into a cycle. This, together with connectedness implies that the cycle is Hamiltonian. To find such a path of maximum length we will create two sets of vertices, large enough to satisfy property P2, such that between any two vertices (one from each) there is a path of maximum length.

Let $P_0 = (v_1, v_2, \ldots, v_q)$ be a path of maximum length in G. Let $A_0 = B(v_1)$. For every $v \in A_0$ fix a $v_1 v$-path $P^{(v)}$ of maximum length and, using our assumption, construct sets $B(v)$, $|B(v)| \geq n/3$, of endpoints of maximum length paths with endpoint v, obtained from a $P^{(v)}$ by at most t_0 rotations. In summary, for every $a \in A_0$, $b \in B(a)$ there is a maximum length path $P(a,b)$ joining a and b, which is obtainable from P_0 by at most $\rho := 2t_0 \leq \frac{4 \log n}{\log d}$ rotations.

We consider P_0 to be directed and divided into 2ρ segments $I_1, I_2, \ldots, I_{2\rho}$ of length at least $\lfloor |P_0|/2\rho \rfloor$ each, where $|P_0| \geq n/3$. As each $P(a,b)$ is obtained from P_0 by at most ρ rotations and every rotation breaks exactly one edge of P_0, the number of segments of P_0 which occur complete on this path, although perhaps reversed, is at least ρ. We say that such a segment is *unbroken*. These segments have an absolute orientation given to them by P_0, and another, relative to this one, given to them by $P(a,b)$, which we consider to be directed from a to b. We consider sequences $\sigma = I_{i_1}, I_{i_2}, \ldots, I_{i_\tau}$ of unbroken segments of P_0, which occur in this order on $P(a,b)$, where σ also specifies the relative orientation of each segment. We call such a sequence σ a *τ-sequence*, and say that $P(a,b)$ *contains* σ.

For a given τ-sequence σ, we consider the set $L(\sigma)$ of ordered pairs (a,b), $a \in A_0$, $b \in B(a)$, such that $P(a,b)$ contains σ.

The total number of τ-sequences is $2^\tau (2\rho)_\tau$. Any path $P(a,b)$ contains at least ρ unbroken segments, and thus at least $\binom{\rho}{\tau}$ τ-sequences. The average, over τ-sequences, of the number of pairs (a,b) such that $P(a,b)$ contains a given τ-sequence is therefore at least

$$\frac{n^2}{9} \cdot \frac{\binom{\rho}{\tau}}{2^\tau (2\rho)_\tau} \geq \alpha n^2,$$

where $\alpha = \alpha(\tau) = 1/9(4\tau)^{-\tau}$. Thus, there is a τ-sequence σ_0 and a set $L = L(\sigma_0)$, $|L| \geq \alpha n^2$ of pairs (a,b) such that for each $(a,b) \in L$ the path $P(a,b)$ contains σ_0. Let $\hat{A} = \{a \in P_0 : L$ contains at least $\alpha n/2$ pairs with a as first element$\}$. Then $|\hat{A}| \geq \alpha n/2$. For each $a \in \hat{A}$ let $\hat{B}(a) = \{b : (a,b) \in L\}$. Then, by the definition of $\hat{A}$, for each $a \in \hat{A}$ we have $|\hat{B}(a)| \geq \alpha n/2$.

Let $\tau = \frac{\log \log n}{2 \log \log \log n}$ and let $\sigma_0 = (I_{i_1}, I_{i_2}, \ldots, I_{i_\tau})$. We divide σ_0 into two sub-sequences, $\sigma_0^1 = (I_{i_1}, \ldots, I_{i_{\tau/2}})$ and $\sigma_0^2 = (I_{i_{\tau/2+1}}, \ldots, I_{i_\tau})$ where both sub-sequences maintain the order and orientation of the segments of σ_0. Both sub-sequences σ_0^1 and σ_0^2 have at least $\tau/2 \cdot n/(6\rho) \geq \frac{n}{96m}$ vertices. Let x be the last vertex of $I_{i_{\tau/2}}$, and let y be the first vertex of $I_{i_{\tau/2+1}}$ (in the orientation given by σ_0). Now we define the notion of good vertices in σ_0^1 and σ_0^2. For σ_0^1 construct a graph H_1 from the segments of σ_0^1 by joining by an edge the last vertex of I_{i_j} to the first vertex of $I_{i_{j+1}}$ for every $1 \leq j < \tau/2$ and then adding the edges of G with both endpoints in the interior (that is, not endpoints) of segments of σ_0^1 to H_1. Then the segments of σ_0^1 with the edges linking them form an oriented spanning path in H_1, starting at x. We define *good* vertices in σ_0^1 to be the vertices which are not endpoints of any segment of σ_0^1 and are good vertices of H_1 as defined in Section 2.2, with $l = \sum_{j=1}^{\tau/2} |I_{i_j}|$. Due to property P2, Lemma 2.3 applies here, and so, since $\tau = o(|\sigma_0^1|)$, more than half of the vertices of σ_0^1 are good. For σ_0^2 we act similarly: construct a graph H_2 from the segments of σ_0^2 by joining the first vertex of I_{i_j} to the last vertex of $I_{i_{j-1}}$ for every $\tau/2+1 < j \le \tau$ and then adding the edges of G with both endpoints in the interior of segments of σ_0^2 to H_2. Then the segments of σ_0^2 with the edges linking them and h_2 form an oriented spanning path in H_2, starting at y. We define *good* vertices in σ_0^2 to be the vertices which are not endpoints of any segment of σ_0^2 and are good vertices of H_2 as defined in Section 2.2, with $l = \sum_{j=\tau/2+1}^{\tau} |I_{i_j}|$. Due to property P2, Lemma 2.3 applies here, and so, since $\tau = o(|\sigma_0^2|)$, more than half of the vertices of σ_0^2 are good.

Since $|\hat{A}| \ge \alpha n/2 \ge \frac{n}{4130m}$ (which is why we get the upper bound on d in Theorem 1.1) and σ_0^1 has at least $|\sigma_0^1|/2 > \frac{n}{192m}$ good vertices, there is an edge from a vertex $\hat{a} \in \hat{A}$ to a good vertex in σ_0^1. Similarly, as $|\hat{B}(\hat{a})| \ge \alpha n/2$,there is an edge from some $\hat{b} \in \hat{B}(\hat{a})$ to a good vertex in σ_0^2. Consider the path $\hat{P} = P(\hat{a}, \hat{b})$ of maximum length connecting $\hat{a}$ and $\hat{b}$ and containing σ_0. The vertices x and y split this path into three sub-paths: P_1 from $\hat{a}$ to x, P_2 from y to $\hat{b}$ and P_3 from x to y. We will rotate P_1 with x as a fixed endpoint and P_2 with y as a fixed endpoint. We will show that the obtained endpoint sets V_1 and V_2 are sufficiently large. Then by property P2 there will be an edge of G between V_1 and V_2. Since we did not touch P_3, this edge closes a maximum path into a cycle, which is Hamiltonian due to the connectivity of G.

Since there is an edge from $\hat{a}$ to a good vertex in σ_0^1, by the definition of a good vertex we can rotate P_1, starting from this edge, to get a set V_1 of at least $|\sigma_0^1|/43 > n/(4130m)$ endpoints. When doing this, we will treat the subpath that links $\hat{a}$ and the first vertex of I_{i_1} and each subpath that links two consecutive segments of σ_0^1, as single edges and ignore edges of G that are incident with an endpoint of some segment of σ_0^1 - like in H_1. This ensures that all rotations and broken edges are inside segments of σ_0^1 and so there is indeed a path of the appropriate length from x to every vertex of V_1.

Similarly, since there is an edge from $\hat{b}$ to a good vertex in σ_0^2, we can rotate P_2, starting from this edge to get a set V_2 of at least $|\sigma_0^2|/43 > n/(4130m)$ endpoints. When doing this, we will treat the subpath that links $\hat{b}$ and the last vertex of I_{i_τ} and each subpath that links two consecutive segments of σ_0^2, as single edges and ignore edges of G that are incident with an endpoint of some segment of σ_0^2 - like in H_2. This ensures that all rotations and broken edges are inside segments of σ_0^2 and so there is indeed a path of the appropriate length from y to every vertex of V_2. This concludes the proof of Theorem 1.1. □

2.4 Hamiltonicity with larger expansion

As we have mentioned, our Hamiltonicity criterion can be extended to handle graphs with a larger expansion than that postulated in Theorem 1.1 ($d \le e^{\sqrt[3]{\log n}}$). In particular, using very similar arguments, we can prove the following statement.

Theorem 2.5 *Let $12 \le d \le \sqrt{n}$ and let G be a graph on n vertices satisfying the following two properties:*

P1' *For every $S \subset V$, if $|S| \le \frac{n \log d}{d \log n}$ then $|N(S)| \ge d|S|$;*

P2' *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \ge \frac{n \log d}{1035 \log n}$.*

Then G is Hamiltonian, for sufficiently large n.

The proof of Theorem 2.5 is almost identical to that of Theorem 1.1 given above. The only notable difference is that here we can allow ourselves to take $\tau = 2$ in the proof.

3 Corollaries

In this section we prove the afore-mentioned corollaries of Theorem 1.1.

Proof of Theorem 1.2 Let $G_{uv} = (V, E \cup \{(u,v)\})$; clearly G_{uv} satisfies properties P1 and P2 and is therefore Hamiltonian by Theorem 1.1. Let $C = w_1 w_2 \dots w_n w_1$ be a Hamilton cycle in G_{uv}. If (u,v) is an edge of C, remove it to obtain the desired path in G. Otherwise, assuming that $u = w_i$ and $v = w_j$, add (u,v) to $E(C)$ and remove (u, w_{i+1}) and (v, w_{j+1}), where all indices are taken modulo n, to obtain a Hamilton path of G_{uv} that contains the edge (u,v); denote this path by P. We will close P into a Hamilton cycle that includes (u,v); removing this edge will result in the required path. The building of the cycle will be done as in the proof of Theorem 1.1 Section 2.3, with P as P_0, while making sure that (u,v) is never broken. The proof is essentially the same, except for the following minor changes:

1. When dividing P into 2ρ segments, we will make sure that (u,v) is in one of the segments; denote it by I_j.
2. When considering τ-sequences, we will restrict ourselves to those that include I_j.
3. Assume without loss of generality that $I_j \in \sigma_0^1$. When building H_1 (and later, when rotating P_1 according to the model of H_1) we will ignore I_j, that is, we will replace it by a single edge (a,b) where a is the last vertex of I_{j-1} (or h_1 if $j = 1$) and b is the first vertex of I_{j+1} (or x if $j = \tau/2$).

□

Proof of Theorem 1.3

Fix some $\frac{8n\log n}{t\log\log n} \le k \le n - 3n/t$. Let $V_0 \subseteq V$ be an arbitrary subset of size $k + n/t$. We construct a sequence of subsets S_i, let $S_0 = \emptyset$. As long as $|S_i| < n/t$ and there exists a set $A_i \subseteq V_0 \setminus S_i$ such that $|A_i| \le n/t$ but $|N_{G[V_0\setminus S_i]}(A_i)| < |A_i|\frac{4\log n}{\log\log n}$, we define $S_{i+1} := S_i \cup A_i$. Let q be the smallest integer such that $|S_q| \ge n/t$ or $|N_{G[V_0\setminus S_q]}(A)| \ge |A|\frac{4\log n}{\log\log n}$ for every $A \subseteq V_0 \setminus S_q$ of size at most n/t. We claim that $|S_q| < n/t$. Indeed assume for the sake of contradiction that $|S_q| \ge n/t$. Since we halt the process as soon as this occurs, and $|A_{q-1}| \le n/t$, we have $|S_q| < 2n/t$. For every $0 \le i \le q-1$ we have $|N_{G[V_0\setminus S_i]}(A_i)| < |A_i|\frac{4\log n}{\log\log n}$ and so $|N_{G[V_0]}(S_q)| < |S_q|\frac{4\log n}{\log\log n}$. On the other hand, G satisfying property P2 together with our lower bound on k implies $|N_{G[V_0]}(S_q)| > |V_0| - n/t - |S_q| \ge |V_0| - 3n/t \ge k \ge |S_q|\frac{4\log n}{\log\log n}$, a contradiction.

Hence, $|S_q| < n/t$ and so, for $U = V_0 \setminus S_q$, $G[U]$ satisfies an expansion condition similar to **P1**, that is, for every $A \subseteq U$, if $|A| \le n/t$ then $|N_{G[U]}(A)| \ge 4|A|\frac{\log n}{\log\log n}$.

In the following we prove that with positive probability the induced subgraph of G on a random k-element subset of U also satisfies a condition similar to **P1**. Let K be a k-subset of U drawn uniformly at random. We will prove that, with positive probability, $G[K]$ satisfies the following:

P1 For every $A \subseteq K$, if $|A| \le n/t$ then $|N_{G[K]}(A)| \ge 2|A|\frac{\log n}{\log\log n}$.

Let $r = |U| - k$. Note that $0 \le r \le n/t$. Let $A \subseteq U$ be any set of size $a \le n/t$, then, as was noted above, $|N_{G[U]}(A)| \ge 4|A|\frac{\log n}{\log\log n}$. Let $N_0 \subseteq N_{G[U]}(A)$ be an arbitrary subset of size $4|A|\frac{\log n}{\log\log n}$. If $A \subseteq K$ and $|N_{G[K]}(A)| \le 2|A|\frac{\log n}{\log\log n}$, then K misses at least $2|A|\frac{\log n}{\log\log n}$ vertices from N_0. This can occur with probability at most

$$\begin{aligned}
\frac{\binom{|N_0|}{\frac{2a\log n}{\log\log n}}\binom{|U|-\frac{2a\log n}{\log\log n}}{r-\frac{2a\log n}{\log\log n}}}{\binom{|U|}{r}} &\le \binom{\frac{4a\log n}{\log\log n}}{\frac{2a\log n}{\log\log n}}\left(\frac{r}{|U|}\right)^{\frac{2a\log n}{\log\log n}} \\
&\le 2^{\frac{4a\log n}{\log\log n}}\left(\frac{\frac{n}{t}}{\frac{8n\log n}{t\log\log n}}\right)^{\frac{2a\log n}{\log\log n}} \\
&= \left(\frac{\log\log n}{2\log n}\right)^{\frac{2a\log n}{\log\log n}}.
\end{aligned}$$

Note that the latter bound is $o(\frac{1}{n})$ for $a = 1$, and $o(\frac{1}{n}\binom{n}{a}^{-1})$ for every $a \ge 2$.

It follows by a union bound argument that

$$Pr\left[\text{ there exists an } A \subseteq K \text{ such that } |A| \le n/t \text{ but } N_{G[K]}(A) < \frac{2\log n}{\log\log n}|A|\right] = o(1).$$

Hence, there exists an k-subset X of U such that for every $A \subseteq X$, if $|A| \le n/t$ then $|N_{G[X]}(A)| \ge \frac{2\log n}{\log\log n}|A|$. Moreover, if A, B are disjoint subsets of V, and $|A|, |B| \ge \frac{k \log\log k \log\left(\frac{2\log n}{\log\log n}\right)}{4130 \log k \log\log\log k} \ge n/t$ then there is an edge between a vertex of A and a vertex of B.

Thus $G[X]$ satisfies the conditions of Theorem 1.1 with $|V| = k$ and $d = \frac{2\log n}{\log\log n}$ and is therefore Hamiltonian. It follows that G admits a cycle of length exactly k.

□

Proof of Theorem 1.4 Let $G = G(n,p) = (V, E)$ and let $d = (\log n)^{0.1}$. We begin by showing that a.s. G satisfies property P2 with respect to d. Indeed

$$\begin{aligned}
Pr[G \nvDash P2] &\le \binom{n}{\frac{n\log\log n \log d}{4130\log n \log\log\log n}}^2 \left(1 - \frac{\log n + \log\log n + \omega(1)}{n}\right)^{\left(\frac{n\log\log n\log d}{4130\log n\log\log\log n}\right)^2} \\
&\le \left(\frac{4130e\log n\log\log\log n}{0.1(\log\log n)^2}\right)^{\frac{0.2n(\log\log n)^2}{4130\log n\log\log\log n}} \\
&\quad \times \exp\left\{-\frac{\log n + \log\log n + \omega(1)}{n} \cdot \frac{0.01n^2(\log\log n)^4}{4130^2(\log n)^2(\log\log\log n)^2}\right\} \\
&= o(1).
\end{aligned}$$

Next, we deal with property P1. Since a.s. there are vertices of "low" degree in G, we cannot expect every "small" set to expand by a factor of d. Therefore, to handle this difficulty, we introduce some minor changes to the proof of Theorem 1.1, in fact only to the part included in Claim 2.2. First of all, note that a.s. G is connected (this fact replaces Proposition 2.1). Let $SMALL = \{u \in V : d_G(u) \le (\log n)^{0.2}\}$ denote the set of all vertices of G that have a "low" degree. The vertices in $SMALL$ will be called *small vertices*. Standard calculations show that a.s. G satisfies the following properties:

(1) $\delta(G) \ge 2$.

(2) For every $u \ne v \in SMALL$ we have $dist_G(u,v) \ge 250$, where $dist_G(u,v)$ is the number of edges in a shortest path between u and v in G.

(3) G satisfies a weak version of P1, that is, if $A \subseteq V \setminus SMALL$ and $|A| \le \frac{n\log\log n\log d}{d\log n\log\log\log n}$ then $|N_G(A)| \ge 3d|A|$.

(4) The number of vertices of degree at most 11 is $O(\log^{11} n)$.

We will prove that, based on these properties, we can build initial long paths as in Claim 2.2 of the proof of Theorem 1.1; this will conclude our proof of Theorem 1.4, as in Subsections 2.2 and 2.3 we did not rely on property P1. The argument is essentially the same as in Claim 2.2; the main difference is that we will use roughly twice as many rotations to create the eventual endpoint set of size $n/3$. This extra factor two has no real effect on the rest of the proof.

Suppose first that the initial path of maximum length P_0 is such that, while creating the sets $S_1, \ldots, S_{120}$ as we did in the proof of Claim 2.2, no vertex from $\cup_{i=1}^{119} S_i$ is a small vertex. Then, by (3), like in the proof of Claim 2.2, after the ith rotation there are exactly $(3d/3)^i = (\log n)^{0.1i}$ new endpoints in S_i. Therefore, after 120 rotations we will have an endpoint set S_{120} with $(\log n)^{12}$ elements.

Suppose now that there is a vertex $u \in S_j \cap SMALL$ for some $j \leq 119$. Let P_u denote a path of maximum length from v_1 to u (which can be obtained from P_0 by at most 119 rotations). At this point we ignore the endpoint sets S_i, $i \leq j$ created so far and restart creating them. The first rotation is somewhat special. By property (1), u has at least one neighbor on P_u other than its predecessor. Thus we can rotate P_u once and obtain a v_1w-path P_w of maximum length, such that w is at distance two from a small vertex. We create new endpoint sets $S_1, \ldots, S_{120}$ with P_w as the initial path. Note that property (2) implies $w \notin SMALL$. Since a new endpoint is always at distance at most two from the old endpoint, we can rotate another 120 times without ever creating an endpoint which is a small vertex. Thus, property (3) applies and after the ith rotation (not including the one that turned w into an endpoint), $i \leq 120$, there are exactly $(3d/3)^i = (\log n)^{0.1i}$ new endpoints in S_i. Hence, after 120 further rotations we obtain a set S_{121} of size exactly $(\log n)^{12}$. Altogether we used up to 240 rotations.

In the following we will prove that the endpoint sets we build grow by the same multiplicative factor every at most *two* rotations.

We will prove by induction on t that there exist endpoint sets $S_{121}, S_{122}, \ldots$ such that for every $t \geq 122$, either $|S_t| = \frac{d}{3}|S_{t-1}|$ or $|S_t| = |S_{t-1}| = \frac{d}{3}|S_{t-2}|$.

Note that this implies $\sum_{i=0}^{t} |S_i| \leq \frac{4}{3}|S_t|$ if $|S_t| = \frac{d}{3}|S_{t-1}|$, provided n is large enough.

For the base case we just have to note that $\sum_{i=0}^{121} |S_i| \leq \frac{4}{3}|S_{121}|$. Suppose we have already built S_t for some $t \geq 121$ such that $\sum_{i=0}^{t} |S_i| \leq \frac{4}{3}|S_t|$ and now wish to build S_{t+1}. We will proceed as in the proof of Claim 2.2.

Assume first that $|N(S_t)| \geq d|S_t|$. Then, as in the proof of Claim 2.2

$$|\hat{S}_{t+1}| \geq \frac{1}{2}(d|S_t| - 3 \cdot \frac{4}{3}|S_t|) = \frac{d-4}{2}|S_t|.$$

Hence, a subset $S_{t+1} \subseteq \hat{S}_{t+1}$ with $|S_{t+1}| = \frac{d}{3}|S_t|$ can be selected.

Assume now that $|N(S_t)| < d|S_t|$. By (3), this must mean that for $S'_t := S_t \cap SMALL$

we have $|S'_t| \geq \frac{2}{3}|S_t|$. Since $|S'_t| \gg \log^{11} n$, property (4) implies that almost every vertex of S'_t has degree at least 12. By (3), no two small vertices have a common neighbor, so $|N(S'_t)| \geq (12 - o(1))|S'_t| \geq (8 - o(1))|S_t|$. As in the proof of Claim 2.2, we have

$$|\hat{S}_{t+1}| \geq \frac{1}{2}(|N(S'_t)| - 3 \cdot | \cup_{i=1}^{t} S_i|) \geq \frac{1}{2}((8 - o(1))|S_t| - 3 \cdot \frac{4}{3}|S_t|) \geq |S_t|.$$

Hence we can select an $S_{t+1} \subseteq \hat{S}_{t+1}$ such that $|S_{t+1}| = |S_t|$. Crucially, since we only used vertices from S'_t for further rotation, all the new endpoints in S_{t+1} are at distance two from a small vertex. It follows by property (2) that $S_{t+1} \cap SMALL = \emptyset$. Hence $|N(S_{t+1})| \geq 3d|S_{t+1}|$ by property (3), which implies that after the next rotation we will have

$$\hat{S}_{t+2} \geq \frac{1}{2}(3d|S_{t+1}| - 3(\frac{4}{3}|S_t| + |S_{t+1}|)) = \frac{3d - 7}{2}|S_t|.$$

Hence, a subset $S_{t+2} \subseteq \hat{S}_{t+2}$ with $|S_{t+2}| = \frac{d}{3}|S_t|$ can be selected.

For the last two rotations our calculations are identical to the ones in Claim 2.2 as those depend on the expansion properties implied by condition P2.

In conclusion, we created an endpoint set $B(v_1)$ of size at least $n/3$ such that for every $v \in B(v_1)$ there is a v_1v-path of maximum length which can be obtained from P_0 by at most $240 + \frac{4 \log n}{\log d}$ rotations with fixed endpoint v_1. □

Proof of Theorem 1.5

Let $G = (V, E)$ be f-connected where $f(k) = 12e^{12} + k \log k$. We prove that G satisfies conditions P1 and P2 with $d = 12$ and apply Theorem 1.1 to conclude that G is Hamiltonian for sufficiently large n. Let $A \subseteq V$ be of size at most $\frac{n}{12m}$. Either $|A| > |V \setminus (A \cup N(A))|$ and so in particular $|N(A)| \geq 12|A|$, or the pair $(A \cup N(A), V \setminus A)$ is a separation of G with $|A| \leq |V \setminus (A \cup N(A))|$ and so by our assumption $|N(A)| \geq f(|A|) \geq 12e^{12} + |A| \log |A| \geq 12|A|$. It follows that G satisfies property $P1$ with $d = 12$. Let A, B be two disjoint subsets of V such that $|B| \geq |A| \geq \frac{n}{4130m}$. Assume for the sake of contradiction that there is no edge in G between A and B; hence $(V \setminus B, V \setminus A)$ is a separation of G. By our assumption $|(V \setminus A) \cap (V \setminus B)| = |V \setminus (A \cup B)| \geq f(|A|) \geq |A| \log |A| > n$. This is clearly a contradiction and so G satisfies property $P2$ with $d = 12$. □

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