Sharp thresholds for half-random games II

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Abstract

We study biased Maker-Breaker positional games between two players, one of whom is playing randomly against an opponent with an optimal strategy. In this work we focus on the case of Breaker playing randomly and Maker being "clever". The reverse scenario is treated in a separate paper. We determine the sharp threshold bias of classical games played on the edge set of the complete graph K_n, such as connectivity, perfect matching, Hamiltonicity, and minimum degree-1 and -2. In all of these games, the threshold is equal to the trivial upper bound implied by the number of edges needed for Maker to occupy a winning set. Moreover, we show that `CleverMaker` can not only win against asymptotically optimal bias, but can do so very fast, wasting only logarithmically many moves (while the winning set sizes are linear in n).

1 Introduction

In a Maker-Breaker positional game, two players take turns occupying a free element of a vertex set X, called *the board*. The game is defined by a finite hypergraph $\mathcal{F} \subset 2^X$. One player, called Maker, is called the winner if he occupies all vertices of a hyperedge in $\mathcal{F}$. Otherwise the other player, called Breaker, wins. We focus on graph games, where the board X is the edge set of a complete graph K_n, and $\mathcal{F}$ consists of all subgraphs with a certain graph property. Here we focus on the game hypergraphs $\mathcal{C}(n)$, $\mathcal{PM}(n)$, $\mathcal{H}(n)$, $\mathcal{D}_1(n)$, $\mathcal{D}_2(n)$ denoting the edge sets of n-vertex graphs that are connected, have a perfect matching, have a hamilton cycle, have minimum degree at least one and two, respectively. Mostly, n will be clear from the context and we omit it from the notation.

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Since the game has perfect information and no chance elements, one player has a winning strategy – which of the two players, depends on the game. A standard method, suggested by Chvátal and Erdős [?], to compensate this imbalance inherent to the game, is to introduce a *bias*, that is, to allow the "disadvantaged" player to occupy more than one element per turn. In an $(a : b)$ *biased positional game* Maker occupies a elements per turn and Breaker b elements.

1.1 Threshold Bias and Half-Random Games

In all the mentioned graph games, Maker wins rather easily with a $(1 : 1)$ bias. But what bias is necessary to allow Breaker to win? Given a game $\mathcal{F}$ we define the *threshold bias* $b_\mathcal{F}$ to be the smallest integer b such that Breaker has a winning strategy in the $(1 : b)$ biased game $\mathcal{F}$. The threshold bias of the connectivity game was already studied by Chvátal and Erdős [?] in 1978, who determined it to be $b_\mathcal{C} = \Theta\left(\frac{n}{\ln n}\right)$. They also showed that $b_\mathcal{H} > 1$. Subsequently the lower bounds on the threshold bias of the connectivity game, as well as the one of the Hamiltonicity game was improved in a series of papers by Beck and several other researchers. Gebauer and Szabó [?] and Krivelevich [?], respectively, showed that both threshold are $(1 + o(1))\frac{n}{\ln n}$.

An instrumental way, suggested implicitly by Chvátal and Erdős [?], to gain insight into particular positional games is to study what happens when both players occupy a uniformly random free edge. Interestingly, as an immediate consequence of classic theorems from the theory of random graphs, these random connectivity and Hamiltonicity games exhibit the very same threshold asymptotics as their deterministic counterparts. This phenomenon is called the *probabilistic intuition* and is a driving force behind much of the research in positional games. For a more detailed discussion of the relevant history of biased graph games and the probabilistic intuition, see the first part of our work [?].

A natural problem arising from the desire for better understanding of the probabilistic intuition is to examine the games with exactly one player playing randomly and the other following a (clever) strategy. There are of course two possible scenarios for these *half-random positional game*s: either Maker or Breaker is the one who plays randomly. In this paper we focus on the latter; our work on the first scenario is contained in [?]. To signify their strategies, we call our players `RandomBreaker` and `CleverMaker`. We show that playing randomly puts `RandomBreaker` at a serious disadvantage against her clever opponent: the threshold bias of the game is much higher than the $\frac{n}{\ln n}$ of the purely clever and purely random game.

Another aspect of positional games we emphasize in this paper is the efficiency of Maker's winning strategy. The question of winning fast has recently been the subject of vigorous research. Among others, Hefetz et al. [?] and later Hefetz and Stich [?] established optimally fast Maker strategies for unbiased non-random Maker-Breaker games, in particular Hamiltonicity and perfect matching. Ferber and Hefetz [?, ?] used fast winning strategies to obtain positive results for certain *strong* positional games. In a strong positional game, as in a Maker-Breaker game, two players alternately occupy elements of a set X,

but the winner is the player whoever occupies a winning set *first*. This symmetric rule forces both players to play as Maker *and* Breaker simultaneously, which often makes strong games quite inaccessible by standard methods. For *Avoider-Enforcer* games (the misère version of Maker-Breaker games, where Avoider wins if she does *not* occupy any winning set), the optimal speed of strategies was studied by Hefetz et al. [**?**] and Barát and Stojakovic [**?**], among others.

In order to state our results we define the precise notion of sharp threshold bias of a `CleverMaker`/`RandomBreaker` half-random games. In what follows, when we talk about a game, we actually mean a *sequence of games* parametrized by n, and similarly by strategy we mean a *sequence of strategies*.

Definition 1.1. We say a function $k : \mathbb{N}_0 \mapsto \mathbb{N}_0$ is a *sharp threshold bias* of the $(1 : b)$ half-random positional game between `CleverMaker` and `RandomBreaker`, if for every $\epsilon > 0$ the following two conditions are satisfied:

(a) `RandomBreaker` wins the $(1 : (1 + \epsilon)k(n))$-biased game with probability tending to 1 against any strategy of `CleverMaker`, and

(b) `CleverMaker` has a strategy against which `RandomBreaker` loses the $(1 : (1 - \epsilon)k(n))$-biased game with probability tending to 1.

1.2 Results

We establish that both the perfect matching and the Hamiltonicity game have a sharp threshold bias. In both cases the sharp threshold turns out to match the trivial upper bound derived from the observation that a large `RandomBreaker` bias makes it impossible for `CleverMaker` to occupy as many edges throughout the game as there are in just a single winning structure.

If the bias of `RandomBreaker` is at least n then `CleverMaker` occupies at most $\binom{n}{2}/(n + 1) < \frac{n}{2}$ edges and hence can occupy neither a perfect matching nor a graph with minimum degree 1. In our first theorem we show that this trivial upper bound on the threshold biases of the games $\mathcal{PM}$ and $\mathcal{D}_1$ is essentially tight. We achieve this by providing `CleverMaker` with a strategy that occupies a perfect matching very fast, in just $O(\log n)$ more rounds than the absolute necessary $\frac{n}{2}$.

Theorem 1.2. *For every $\epsilon > 0$, CleverMaker has a strategy in the $(1 : (1-\epsilon)n))$ half-random game $\mathcal{PM}$ that is winning in $\frac{n}{2} + O(\log n)$ moves asymptotically almost surely (a.a.s.). In particular the sharp threshold bias for both the $(1 : b)$ perfect matching, and the $(1 : b)$ minimum degree-1 half-random game between CleverMaker and RandomBreaker is n.*

For our next theorem observe that if the bias of `RandomBreaker` is more than $\frac{n}{2}$ then `CleverMaker` occupies less than $\binom{n}{2}/(n/2) = n - 1$ edges and hence can build neither a connected graph nor a Hamilton cycle nor a graph with minimum degree 2. It turns out that this trivial upper bound on the threshold biases is essentially tight for all three games. Again, we prove this by providing `Clever-Maker` with a very fast strategy to build a Hamiltonian cycle, succeeding in just $n + O(\log n)$ moves.

Theorem 1.3. *For every $\epsilon > 0$, `CleverMaker` has a strategy in the $(1 : (1 - \epsilon)\frac{n}{2}))$ half-random game $\mathcal{H}$ that is winning in $n + O(\log n)$ moves a.a.s. In particular the sharp threshold bias for the $(1 : b)$ connectivity, minimum-degree-2, and Hamiltonicity half-random games between `CleverMaker` and `RandomBreaker` is $\frac{n}{2}$.*

Note that all the games discussed in Theorems 1.2 and 1.3 have a significantly higher half-random threshold than the threshold bias $\frac{n}{\ln n}$ in their fully deterministic and fully random version.

Remark. The results of this paper are based on the Master thesis of the first author [**?**]. Recently, Krivelevich and Kronenberg [**?**] also studied the same problem independently and used different strategies to obtain the same sharp thresholds. In their paper they also deal with the k-connectivity game for arbitrary constant k, which we only consider here for $k = 1$. For the perfect matching and Hamiltonicity games our strategy for `CleverMaker` succeeds much faster, with wasting only $O(\log n)$ extra moves above the size of a winning set, as opposed to the $O(n^{\alpha})$ in [**?**].

1.3   Terminology and organization

We will use the following terminology and conventions. A *move* consists of claiming one edge. *Turns* are taken alternately, one turn can have multiple moves. For example: With a $(1 : b)$ bias, Maker has 1 move per turn, while Breaker has b moves. A *round* consists of a turn by Maker followed by a turn by Breaker. By a strategy we mean a set of rules which specifies what the player does in any possible game scenario. For technical reasons we *always* consider games to last until there are no free edges, even if one of the players "has already won" (say Maker already occupied a winning set). Consequently we also define strategies till the end; if otherwise not specified (e.g. we say that "a player forfeits") the strategy just keeps occupying an arbitrary free edge, say one with the smallest index. The *play-sequence* Γ of length i of a game between Maker and Breaker is the list $(\Gamma_1, \ldots, \Gamma_i) \in E(K_n)^i$ of the first i edges that were occupied during the game by either of the players, in the order they were occupied. We make here the convention that a player with a bias $b > 1$ occupies his b edges within one turn in succession and these are noted in the play-sequence in this order (even though in the actual game it makes no difference in what order one player's moves are occupied within one of his turns). We denote Maker's graph after t turns with $G_{M,t}$ and similarly Breaker's graph with $G_{B,t}$. Note that in an $(a : b)$-game, these graphs have at and bt edges respectively. We will use the convention that Maker goes first. This is more of a notational convenience, since the proofs can be easily adjusted to Breaker going first, and yielding the same asymptotic results. We will routinely omit rounding signs, whenever they are not crucial in affecting our asymptotic statements.

We first introduce the notion of a *permutation strategy* in the next section, and continue to prove Theorems 1.2 and 1.3 in Section 3.

2 The permutation strategy

In this section, we discuss an alternative way to think of half-random games that will simplify our reasoning in many proofs. This discussion does not depend on the game's win conditions, so we will refer to the random player and the clever player as RP and CP respectively, regardless of who is Breaker and Maker. This allows us to also use Proposition 2.1 in [**?**]. We refer to RP's and CP's bias as r and b respectively.

One reason why half-random games are more difficult to study than fully random games, is that RP's graph is in fact not fully random. This is because the edges occupied by CP can no longer be claimed by RP, the deterministic and random aspects of the game interact. Our goal in this section is to relate RP's graph to a fully random auxiliary graph, so we can apply results from the rich theory of random graphs still.

Given a permutation $\sigma \in S_{E(K_n)}$ of the edges of K_n, i.e. $\sigma : \left[\binom{n}{2}\right] \to E(K_n)$, a player can use σ to determine a strategy as follows. We say that he follows the *permutation strategy* σ if for every move, he scans through the edges in σ and occupies the first one that is free. That is, he occupies the edges in their order in σ, skipping the ones occupied by his opponent. This naturally leads to a randomized strategy for RP: in the beginning, she picks σ uniformly at random out of all permutations, and then follows the corresponding permutation strategy. As it turns out, this is equivalent to the original method by which RP chooses her moves (i.e., always choosing a uniformly random free edge). We formalize this in the following proposition and include a proof for completeness.

Proposition 2.1. *For every strategy S of CP in a $(r : c)$-game on $E(K_n)$ the following is true. For every $m \leq \binom{n}{2}$ and every sequence $\Gamma = (\Gamma_1, \ldots, \Gamma_m)$ of distinct edges, the probability that Γ is the play-sequence of a half random game between CP playing according to strategy S and RP is equal to the probability that Γ is the play-sequence of the game when RP plays instead according to the random permutation strategy.*

Proof. Let $R \subseteq [m]$ and $C = [m] \setminus R$ be the subsets of coordinates in any play-sequence of length m, which belong to RP's and CP's moves in an $(r : c)$-biased game, respectively. Note that these sets are determined by m, r, and c and by who starts the game (and independent of the play-sequence).

Let $\Gamma = (\Gamma_1, \ldots, \Gamma_m)$ be a sequence of distinct edges which can be realized as a play-sequence provided CP plays according to strategy S (otherwise the probability of Γ is 0 in both games). In other words, for every $j \in C$, if $(\Gamma_1, \ldots, \Gamma_{j-1})$ is a play-sequence of the $(r : c)$-game then the next edge CP chooses according to S is Γ_j.

Clearly, the probability that this particular Γ is the play-sequence of the half-random game is

$$\prod_{j \in R} \frac{1}{\binom{n}{2} - j + 1}. \tag{1}$$

Let us now turn to the game generated by the random permutation strategy and let us define $\mathcal{N}(\Gamma, S) = \mathcal{N}$ to be the set of those permutations $\sigma \in S_{E(K_n)}$

which produce the play-sequence Γ when RP plays with the permutation strategy σ against CP's strategy S. Then $\mathcal{N}$ consists exactly of those permutations σ for which

(i) the relative order of the edges in $\{\Gamma_i : i \in R\}$ agrees in σ and Γ

(ii) the edges in $\{\Gamma_i : i \in R\}$ precede in σ the edges in $E(K_n) \setminus \{\Gamma_i : i \in [m]\}$.

(iii) For every $j \in C$, the Clever-edge Γ_j comes after all the Random-edges $\{\Gamma_i : i \in R, i < j\}$ in σ.

We can obtain every such permutation by starting exactly with the restriction of Γ to R, so (i) is satisfied. Then we append the edges from $E(K_n) \setminus \{\Gamma_i : i \in [m]\}$ in an arbitrary order, so (ii) holds. Finally we insert the Clever-edges Γ_j, $j \in C$, one by one, in decreasing order, making sure that (iii) is maintained. When inserting the edge Γ_j, the number of possible places is exactly $\binom{n}{2} - j + 1$, since all the edges Γ_l with $l > j$ are already there and all the edges of index $l < j$ which are already there are contained in R (and hence must precede Γ_j). Hence the number of permutations in $\mathcal{N}$ is

$$\left(\binom{n}{2} - m \right)! \prod_{j \in C} \left(\binom{n}{2} - j + 1 \right).$$

Hence the probability of $\mathcal{N}$ is equal to (1) since C and R partition $[m]$. $\qquad\square$

In the following, for $1 \leq m \leq \binom{n}{2}$ and a permutation $\sigma \in S_{E(K_n)}$, we let $G_\sigma(m) \subseteq K_n$ be the subgraph with edge set $E(G_\sigma(m)) = \{\sigma(i) : 1 \leq i \leq m\}$. Note that if the permutation σ is chosen uniformly at random, then $G_\sigma(m)$ is distributed like the random graph $G(n, m)$.

Let us now switch back to the CleverMaker/ RandomBreaker setup. Assume RandomBreaker plays a particular game according to a permutation $\sigma \in S_{E(K_n)}$, and let m_i be the index in σ of the last edge he takes in round i, i.e. that edge is $\sigma(m_i)$. Then RandomBreaker's graph after round i is contained in $G_\sigma(m_i)$. Note that $m_i \geq ir$, but since RandomBreaker maybe skipped some edges occupied by CleverMaker, the actual value depends on the strategy of CleverMaker and the permutation σ itself. However, since CleverMaker occupied only ic edges so far, we also have that $m_i \leq i(r + c)$. Hence RandomBreaker's graph after the ith round is always a subgraph of the random graph $G_\sigma(i(r + c))$. This line of reasoning leads to the following proposition.

Proposition 2.2. *Let b and i be positive integers such that $i \leq \frac{\binom{n}{2}}{b+1}$. Then for every monotone increasing graph property $\mathcal{P}$ and strategy S of CleverMaker for a $(1 : b)$ half-random game the following holds. The probability that in a half-random game against strategy S of CleverMaker the graph of RandomBreaker after the ith round has property $\mathcal{P}$ is at most $Pr[G(n, i(b+1))$ has property $\mathcal{P}]$.*

Proof. Consider all play sequences of length $i(1+b)$ that are possible with CleverBreaker playing according to S so that by round i his graph has property $\mathcal{P}$.

By the previous proposition the probability of these play sequences in the half-random game is equal to $\frac{|\mathcal{M}|}{\binom{n}{2}!}$, where $\mathcal{M} = \mathcal{M}(\mathcal{P}, i, S)$ is the set of permutations σ of $E(K_n)$ having the property that if `RandomBreaker` plays according to the permutation strategy σ against strategy S of `CleverMaker`, then by the end of round i `RandomBreaker`'s graph has property $\mathcal{P}$.

Now recall that all edges of `RandomBreaker`'s graph in the first i rounds while playing according to an arbitrary permutation strategy σ are among the first $i(b + 1)$ elements of σ. Therefore, since $\mathcal{P}$ is monotone increasing, for any permutation $\sigma \in \mathcal{M}$, the graph $G_\sigma(i(b + 1))$ has property $\mathcal{P}$. Since for a uniform random permutation σ, the graph $G_\sigma(i(b + 1))$ is a uniform random graph $G(n, i(b + 1))$, the statement follows. $\qquad\square$

3 CleverMaker vs RandomBreaker

In this section we prove the non-trivial parts of Theorems 1.2 and 1.3 involving `CleverMaker`'s strategy.

For our proofs we fix an $\epsilon > 0$ sufficiently small and set the following values:

$$p \quad := \quad 1 - \frac{\epsilon}{2} \tag{2}$$

$$k = k(n) \quad := \quad 4\left\lceil \frac{\ln n}{\ln(1/p)} \right\rceil \tag{3}$$

$$l \quad := \quad 8\left\lceil \frac{1}{\epsilon \ln(1/p)} \right\rceil \tag{4}$$

Note that $k = k(n)$ is of the order $\ln n$ and l is constant depending only on ϵ.

We will need a few properties of `RandomBreaker`'s graph, which are borrowed from the uniformly random model $G(n, m)$.

Lemma 3.1. *Let $n, b, t \in \mathbb{N}$ such that $(b + 1)t \le m := p\binom{n}{2}$ and let S be a strategy of* `CleverMaker` *in a $(1 : b)$ half-random game. Then a.a.s. the graph $G_{B,t}$ of* `RandomBreaker` *after t rounds in a game against* `CleverMaker` *playing according to S has the following properties.*

(i) $G_{B,t}$ has maximum degree at most $\left(1 - \frac{\epsilon}{4}\right) n$.

(ii) There is no set of k vertices in $G_{B,t}$ inducing at least $\binom{k}{2} - \frac{k}{2}$ edges.

(iii) $G_{B,t}$ contains no complete bipartite graph of size $\frac{\epsilon}{8} n \times l$.

(iv) $G_{B,t}$ contains no complete bipartite graph of size $\frac{\epsilon}{32l} n \times \frac{\epsilon}{32l} n$.

Proof. We show the properties for $G(n, m)$ and then transfer them to $G_{B,t}$ using Proposition 2.2. To estimate, we repeatedly use that $\frac{\binom{N-q}{m-q}}{\binom{N}{m}} = \prod_{i=0}^{q-1} \frac{m-i}{N-i} \le \left(\frac{m}{N}\right)^q = p^q$, where $N = \binom{n}{2}$.

For part (i) see e.g. [?], Theorem 10.

For part (ii) we have that the probability that there exists a k-element set K such that $G(n,m)$ has at least $\binom{k}{2} - \frac{k}{2}$ edges in K is, by the union bound, at most

$$\binom{n}{k}\binom{\binom{k}{2}}{\binom{k}{2} - \frac{k}{2}}\frac{\binom{N-\binom{k}{2}+\frac{k}{2}}{m-\binom{k}{2}+\frac{k}{2}}}{\binom{N}{m}} \leq \left(\frac{en}{k}\right)^k (e(k-1))^{k/2} p^{\binom{k}{2}-\frac{k}{2}}$$

$$\leq e^{k(3/2+\ln n - \frac{1}{2}\ln k - \frac{k-2}{2}\ln(1/p))} = o(1)$$

We prove parts (iii) and (iv) similarly, by observing that the probability of the event that there is a complete bipartite graph of size $r \times q$ in $G(n,m)$ is upper bounded by

$$\binom{n}{r}\binom{n}{q}\frac{\binom{N-rq}{m-rq}}{\binom{N}{m}}.$$

For (iii), we set $r = l$ and $q = \frac{\epsilon}{8}n$ and estimate by $n^r 2^n p^{qr} = e^{l \ln n + (\ln 2)n - \frac{\epsilon}{8}nl \ln(1/p)}$. This tends to 0 by the choice of l. For (iv) we set $r = q = \frac{\epsilon}{32l}n$ and estimate with $2^n 2^n p^{\frac{\epsilon^2}{1024l^2}n^2} = o(1)$. $\qquad\square$

Towards the end of both of his strategies, `CleverMaker` occasionally sets out to make a *double move* or a *triple move*. By this we mean that `CleverMaker` identifies two or three free edges which he intends to occupy immediately in the next two or three rounds, respectively. To occupy the first edge is of course no problem since it is free, but in order to be able to occupy the second or third edge, it is also necessary that `RandomBreaker` did not occupy them in his turn(s) in between. The next simple lemma states that this is very likely if there are still many free edges.

Lemma 3.2. *The probability that* `CleverMaker` *is not able to complete a double move (or a triple move) within the first t rounds of a* $(1 : b)$ *half-random game with* $b \leq n$ *is at most* $\frac{4}{\epsilon n}$ *(or* $\frac{12}{\epsilon n}$*), provided the number of free edges is at least* $\binom{n}{2} - (b+1)t \geq \frac{\epsilon}{4}n^2$.

Proof. The probability, that out of the still at least $\frac{\epsilon}{4}n^2$ free edges `RandomBreaker` occupies exactly the second edge of the double move `CleverMaker` has just started, is at most $\frac{4}{\epsilon n^2}$. He has at most n chances before `CleverMaker` completes his double move, hence the upper bound follows. For triple moves `RandomBreaker` has n chances to occupy the second edge of the triple move and $2n$ chances for the third edge. $\qquad\square$

3.1   `CleverMaker` builds a perfect matching

In this section we prove the non-trivial part of Theorem 1.2. We consider the $(1 : b)$ perfect matching game between `CleverMaker` and `RandomBreaker` on $E(K_n)$, where n is even and $b \leq (1-\epsilon)n$ for an arbitrary but fixed ϵ, $0 < \epsilon < 1/2$. Throughout the proof when we say that a vertex is *isolated*, we always mean that it is isolated in `CleverMaker`'s graph at the current point in the game.

During the game `CleverMaker` maintains a matching M of his graph. He starts with $M = \emptyset$ and then eventually achieves that M is perfect, at which point `CleverMaker` wins the game. Let us call an edge of K_n *vacant* if it is neither occupied by `RandomBreaker` nor used by `CleverMaker` in his matching M. In contrast, we call an edge *free* if it is not occupied at all. For an isolated vertex a, we define

$$X_a := \{u \in V : au \text{ is vacant}\}$$

to be the set of vertices with a vacant edge to a. Further, let

$$X_a^+ := \{v \in V : vu \in M, u \in X_a\}.$$

We now define strategy S_{PM} for `CleverMaker`. If S_{PM} calls `CleverMaker` to take an edge he has already occupied, he takes an arbitrary free edge. If anytime during a game `CleverMaker` is not able to make a move according to the directions below, we say that he *forfeits*. (Recall that for technical reasons, `CleverMaker` continues to play in this case by always claiming the free edge with the smallest index until the board is full.) We use the values $p = 1 - \frac{\epsilon}{2}$, $k(n) = 4 \left\lceil \frac{\ln n}{\ln (1/p)} \right\rceil$ and $l = 8 \left\lceil \frac{1}{\epsilon \ln(1/p)} \right\rceil$, as defined above. The strategy consists of three stages.

Stage 1. This stage lasts while $|M| < \frac{n - k(n)}{2}$. `CleverMaker` iteratively occupies an arbitrary free edge e between two isolated vertices, and adds e to M.

Stage 2. This stage lasts until $\frac{n-k(n)}{2} \leq |M| < \frac{n-l}{2}$ and consists of $\frac{k(n)-l}{2}$ double moves, each increasing the size of M by one (using augmenting paths of length 3). For each of his double moves `CleverMaker` identifies an arbitrary edge $uv \in M$ such that there exists isolated vertices $a \in X_u$ and $b \in X_v$, with $a \neq b$, and then he occupies au and bv in his next two turns. Finally `CleverMaker` removes uv from M, and adds au and bv instead.

Stage 3 This stage lasts until $\frac{n-l}{2} \leq |M| < \frac{n}{2}$ and consists of $\frac{l}{2}$ triple moves, each increasing the size of M by one (using augmenting paths of length 5). For each of his triple moves `CleverMaker` first identifies two arbitrary isolated vertices a, b and then an arbitrary vacant edge wz with $w \in X_a^+, z \in X_b^+$. Let $u \in X_a$ and $v \in X_b$ be the vertices with $uw \in M$ and $zv \in M$, respectively. In his next three turns, `CleverMaker` occupies au, wz and vb. He then adds these three edges to M, while removing uw and zv.

Throughout this process M remains a matching, and with each single-/double- or triple move increases in size by one. Thus, after all 3 stages are complete, M is a matching of size $\frac{n}{2}$, i.e. a perfect matching.

Therefore, what remains to show is that `CleverMaker` can a.a.s. execute strategy S_{PM} without forfeiting.

Proof of Theorem 1.2. Let $\epsilon > 0$ be fixed and let $b \leq (1 - \epsilon)n$ be a positive integer. We prove that `CleverMaker`, playing against `RandomBreaker` in the

$(1 : b)$ half-random game, can execute the strategy S_{PM} without forfeiting, a.a.s. This in particular, will imply that CleverMaker wins the $(1 : b)$ half-random perfect matching game (and thus also the degree-1 game) within $\frac{n}{2} + O(\ln n)$ moves, a.a.s.

First note that strategy S_{PM} takes at most

$$t := \frac{n - k(n)}{2} + k(n) - l + \frac{3l}{2} = \frac{n + k(n) + l}{2} = \frac{n}{2} + O(\ln n)$$

rounds. This is because in Stage 1, $\frac{n-k(n)}{2}$ edges are added to M, and in Stage 2 $\frac{k(n)-l}{2}$ edges, taking two rounds each. This leaves $\frac{l}{2}$ edges to be added in Stage 3, which takes $\frac{3l}{2}$ rounds.

Observe also that for the total number of edges claimed by either player we have

$$(b+1)t = ((1 - \epsilon)n + 1)\frac{n + o(n)}{2} \leq p\binom{n}{2} = m.$$

This has two important consequences. On the one hand the conditions of Lemma 3.1 are satisfied, so a.a.s. all properties $(i) - (iv)$ hold for the graph RandomBreaker occupies by turn t, and since the properties are decreasing, also at all previous points in the game. On the other hand $\binom{n}{2} - (b+1)t \geq \frac{\epsilon}{4}n^2$, so by Lemma 3.2 the probability that any double or triple move CleverMaker has started cannot be completed is at most $\frac{12}{\epsilon n}$. Since the number of double moves is $O(\ln n)$ and the number of triple moves is $O(1)$, this will occur only with probability $O(\frac{\ln n}{n})$. In other words, a.a.s. CleverMaker can complete every double or triple move he starts.

We now assume that indeed these two events hold, i.e. RandomBreaker's graph has properties $(i) - (iv)$ of Lemma 3.1 up to at least round t, and CleverMaker can complete every double or triple move he starts. We go through the three stages and show that under these conditions, the strategy can be carried through without forfeiting.

First let $|M| < \frac{n-k(n)}{2}$, so we are in Stage 1. Since in Stage 1 there are $\frac{n-k(n)}{2}$ rounds, there must be at least $k(n)$ isolated vertices left in CleverMaker's graph. By Property (ii) of Lemma 3.1, RandomBreaker has no clique of size $k(n)$ occupied, and thus there must be at least one vacant edge between two isolated vertices.

Let now $\frac{n-k(n)}{2} \leq |M| < \frac{n-l}{2}$, so we are in Stage 2. Let a be an arbitrary isolated vertex and consider the edges of K_n between X_a^+ and the set L of isolated vertices different from a. If any of these edges is vacant, CleverMaker can start his double move. Otherwise RandomBreaker's graph contains a complete bipartite graph of size $|L| \times |X_a^+|$. Since Stage 2 is not yet over, we have $|L| \geq l$. For the other side, $|X_a^+| = 2|M| - \deg_B(a) \geq n - k(n) - \deg_B(a) \geq \frac{\epsilon}{8}n$ by Property (i) of Lemma 3.1 and since $k(n) = O(\ln n)$. Since by Property (iii) of Lemma 3.1 RandomBreaker's graph contains no complete bipartite graph $K_{\frac{\epsilon}{8}n,l}$, one of the edges between L and X_a^+ must be vacant, allowing CleverMaker to start his double move.

Let now $\frac{n-l}{2} \leq |M| < \frac{n}{2}$, so we are in Stage 3. Here `CleverMaker` has to select triplets of moves. For this there needs to be two isolated vertices a, b such that there is a vacant edge wz with $w \in X_a^+, z \in X_b^+$. Since M is not yet perfect and n is even, there must be two isolated vertices a and b. As in Stage 2, we also have that both the sizes $|X_a^+|$ and $|X_b^+|$ are at least $2|M| - \Delta(G_B) \geq n - l - \Delta(G_B) \geq \frac{\epsilon}{8}n$ (recall that $l = O(1)$). In particular, there are disjoint sets $Y_a^+ \subset X_a^+$ and $Y_b^+ \subset X_b^+$ of size at least $\frac{\epsilon}{16}n$ each. By Property (iv) of Lemma 3.1, `RandomBreaker`'s graph contains no complete bipartite graph $K_{\epsilon n/16, \epsilon n/16}$, which means that indeed there is a vacant edge wz with $w \in X_a^+, z \in X_b^+$. $\qquad\square$

3.2 `CleverMaker` builds a Hamilton cycle

We now turn towards the Hamiltonicity game and show the non-trivial direction of Theorem 1.3. Recall the values p, $k(n)$ and l as defined above.

First let us describe `CleverMaker`'s strategy informally. The analysis uses many ideas from the perfect matching game. Actually, first `CleverMaker` follows the strategy S_{PM} to build a perfect matching M in $\frac{n}{2} + O(\ln n)$ moves. Next, `CleverMaker` performs another sequence of similar steps, using the matching M as a starting point, and connecting its edges first to a Hamilton path and then a cycle. The central structure `CleverMaker` maintains will be a sequence $\mathcal{P}_i$, $i = \frac{n}{2}, \ldots, 1$, of families of paths of Maker's graph, such that the paths of each family $\mathcal{P}_i$ partition the vertex set. To start `CleverMaker` sets $\mathcal{P}_{\frac{n}{2}} := M$ to be a set of $\frac{n}{2}$ paths of length 1. Then `CleverMaker` performs a sequence of $\frac{n}{2}$ single, double, or triple moves. Each of these moves reduces the number of paths in the family by one, hence $\mathcal{P}_1$ contains a single Hamilton path. In his last triple move `CleverBreaker` closes this path to a Hamilton cycle. Similarly to the perfect matching game, the number of double moves of `CleverBreaker` will be $O(\ln n)$ and the number of triple moves will only be $O(1)$. Hence the game lasts at most $n + O(\ln n)$ rounds. For convenience in notation we will assume that n is even, the odd case can be handled similarly: `CleverMaker` first occupies a matching of size $\frac{n-1}{2}$, then connects the lone isolated vertex to an arbitrary matching edge and thus builds his initial family of nontrivial paths $\mathcal{P}_{\frac{n-1}{2}}$ covering the vertex set.

In the following, for a path $\gamma \in \mathcal{P}_i$, we write γ as a sequence of vertices $\gamma_0, \ldots, \gamma_{s(\gamma)}$, where $s(\gamma)$ denotes the length of γ. We use a fixed direction on the ordering, for example, we can demand that $\gamma_0 < \gamma_{s(\gamma)}$, when seeing the vertices as elements in $[n]$. For a vertex a, we define the following helpful set, consisting of those vertices which are followed by a vertex of X_a on a path of $\mathcal{P}_i$ (recall that X_a denotes the set of vertices with a vacant edge to a).

$$X_a^{\leftarrow} := \{\gamma_{j-1} : \gamma \in \mathcal{P}_i, \gamma_j \in X_a\} \setminus \{a\}$$

We now describe `CleverMaker`'s strategy S_{HAM}.

Stage 0 Build a perfect matching M in $\frac{n}{2} + O(\ln n)$ moves using strategy S_{PM}. Set $\mathcal{P}_{\frac{n}{2}} = M$.

Stage 1 Let $\frac{n}{2} \geq i > k(n)$. To construct $\mathcal{P}_{i-1}$ from $\mathcal{P}_i$ CleverMaker uses a single move to occupy a vacant edge e between two endpoints a and b that belong to two different paths α and $\beta \in \mathcal{P}_i$. He obtains $\mathcal{P}_{i-1}$ by removing α and β from $\mathcal{P}_i$, and adding the new path obtained by connecting α and β with e. (Again, with vacant we mean neither occupied by RandomBreaker nor used by CleverMaker on the paths of $\mathcal{P}_i$; if CleverMaker has the edge previously occupied but is not using it, he just starts using it and occupies an arbitrary edge somewhere else.)

Stage 2 Let $k(n) \geq i > l$. To construct $\mathcal{P}_{i-1}$ from $\mathcal{P}_i$ CleverMaker uses a double move. Let us a fix the starting vertex $a := \alpha_0$ of an arbitrary path $\alpha \in \mathcal{P}_i$. Let $B := \{\beta_{s(\beta)} : \beta \in \mathcal{P}_i \setminus \{\alpha\}\}$ be the set of endpoints of the paths in $\mathcal{P}_i$ other than α. CleverBreaker then identifies a vertex $v \in X_a^{\leftarrow}$ and a vertex $b \in B$ such that the edge bv is currently vacant. Let $u \in X_a$ be the neighbor that follows v on the path $\gamma \in \mathcal{P}_i$ which contains v, say $u = \gamma_j$, $v = \gamma_{j-1}$. CleverMaker now occupies the edges au and bv in his next two turns. The new family $\mathcal{P}_{i-1}$ depends on which of the following three cases hold. (Recall that $\alpha \neq \beta$).

Case 1: $\gamma \neq \alpha, \beta$. Then CleverMaker obtains $\mathcal{P}_{i-1}$ by removing α, β and γ from $\mathcal{P}_i$, and adding

$$\alpha_{s(\alpha)} \dots \alpha_0 \gamma_j \gamma_{j+1} \dots \gamma_{s(\gamma)} \text{ and } \gamma_0 \gamma_1 \dots \gamma_{j-1} \beta_{s(\beta)} \dots \beta_0.$$

Case 2: $\gamma = \alpha$. I.e. $u = \alpha_j$, $v = \alpha_{j-1}$. Then CleverMaker obtains $\mathcal{P}_{i-1}$ by removing α and β from $\mathcal{P}_i$, and adding

$$\alpha_{s(\alpha)} \dots \alpha_{j+1} \alpha_j \alpha_0 \alpha_1 \dots \alpha_{j-1} \beta_{s(\beta)} \dots \beta_0.$$

Case 3: $\gamma = \beta$. I.e. $u = \beta_j$, $v = \beta_{j-1}$. Then CleverMaker obtains $\mathcal{P}_{i-1}$ by removing α and β from $\mathcal{P}_i$, and adding

$$\alpha_{s(\alpha)} \dots \alpha_0 \beta_j \beta_{j+1} \dots \beta_{s(\beta)} \beta_{j-1} \beta_{j-2} \dots \beta_0.$$

Stage 3 Let $l \geq i > 1$. To construct $\mathcal{P}_{i-1}$ from $\mathcal{P}_i$ CleverMaker uses a triple move. He first identifies two arbitrary paths $\alpha, \beta \in \mathcal{P}_i$, $\alpha \neq \beta$. Let $a := \alpha_0$, $b := \beta_0$. Next, he sets $\gamma^{\mathsf{a}} \in \mathcal{P}_i$ to be a path such that $|X_a \cap \gamma^{\mathsf{a}}|$ is maximal, and defines γ^{b} similarly. Then he constructs vertex sets X_a^* and X_b^* depending on two cases:

Case 1: Neither $\gamma^{\mathsf{a}} = \gamma^{\mathsf{b}} = \alpha$ **nor** $\gamma^{\mathsf{a}} = \gamma^{\mathsf{b}} = \beta$. Then we simply define $X_a^* := X_a^{\leftarrow} \cap \gamma^{\mathsf{a}}$ and $X_b^* := X_b^{\leftarrow} \cap \gamma^{\mathsf{b}}$.

Case 2: $\gamma^{\mathsf{a}} = \gamma^{\mathsf{b}} = \alpha$ **or** $\gamma^{\mathsf{a}} = \gamma^{\mathsf{b}} = \beta$. Let us assume $\gamma^{\mathsf{a}} = \gamma^{\mathsf{b}} = \alpha$, the other case is treated similarly. First, we write

$$X_a \cap \alpha = \{\alpha_{e_0}, \dots, \alpha_{e_q}\}$$
$$X_b \cap \alpha = \{\alpha_{f_0}, \dots, \alpha_{f_r}\}$$

Then, we define

$$X_a^* := \begin{cases} \{\alpha_{e_1-1}, \ldots, \alpha_{e_{\frac{q}{2}}-1}\} & \text{if } e_{\frac{q}{2}} < f_{\frac{r}{2}} \\ \{\alpha_{e_{\frac{q}{2}}-1}, \ldots, \alpha_{e_q-1}\} & \text{if } e_{\frac{q}{2}} \geq f_{\frac{r}{2}} \end{cases}$$

$$X_b^* := \begin{cases} \{\alpha_{f_{\frac{r}{2}}+1}, \ldots, \alpha_{f_{r-1}+1}\} & \text{if } e_{\frac{q}{2}} < f_{\frac{r}{2}} \\ \{\alpha_{f_1-1}, \ldots, \alpha_{f_{\frac{r}{2}}-1}\} & \text{if } e_{\frac{q}{2}} \geq f_{\frac{r}{2}} \end{cases}$$

Now let $w \in X_a^*$ and $z \in X_b^*$ be such that wz is a vacant edge. Then let $u \in X_a$ be the neighbor following w on γ^{a} and $v \in X_b$ be the neighbor following z on γ^{b}. In his next three moves `CleverMaker` claims the edges au, bv and wz. He updates his paths by adding these three edges, and removing the edges uw and vz. It is now easy to verify that this indeed reduces the number of paths in $\mathcal{P}_i$ by one in each case (see Figure 1).
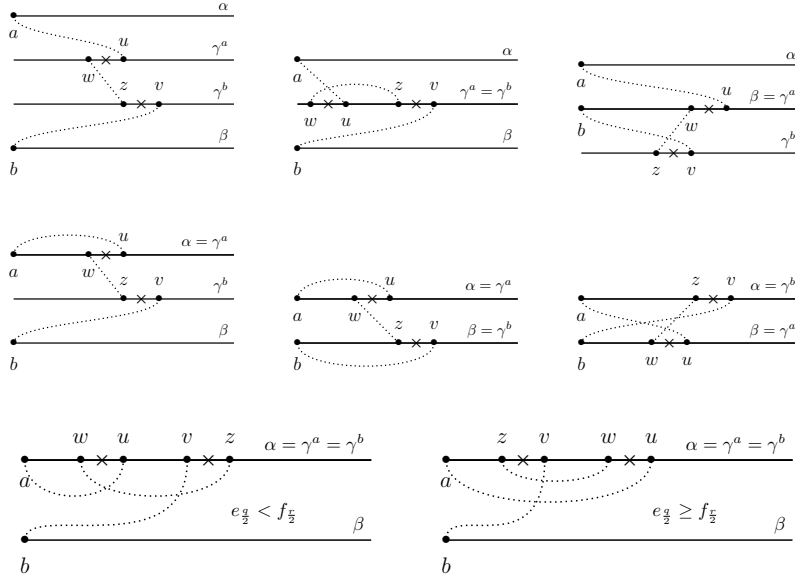

Figure 1: The different sub-cases of Stage 3, with the two sub-cases of Case 2 on the bottom. Symmetric cases where the roles of α and β are swapped are not pictured.

Stage 4 Assume now there is only one path $\gamma_0 \ldots \gamma_n$ left that covers all vertices, and has endpoints $a = \gamma_0$ and $b = \gamma_n$. We can then write $X_a = \{\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_s}\}$ and $X_b = \{\gamma_{j_0}, \gamma_{j_1}, \ldots, \gamma_{j_t}\}$ with $i_x < i_{x+1}$ and $j_y < j_{y+1}$ for all x, y. Then, if $i_{\frac{s}{2}} < j_{\frac{t}{2}}$, we take $u = \gamma_i \in X_a$ and $v = \gamma_j \in X_b$

such that $i \leq \frac{s}{2}$, $j \geq \frac{t}{2}$, and the edge $\gamma_{i-1}\gamma_{j+1}$ is vacant. Then Clever-Maker claims the edges au, $\gamma_{i-1}\gamma_{j+1}$ and bv to complete a Hamilton cycle (deleting the edges $u\gamma_{i-1}$ and $v\gamma_{j+1}$), see Figure 2. If $i_{\frac{s}{2}} \geq j_{\frac{t}{2}}$, CleverMaker instead chooses $i \geq i_{\frac{s}{2}}$, $j < j_{\frac{t}{2}}$ such that the edge $\gamma_{i+1}\gamma_{j-1}$ is vacant and proceeds accordingly.
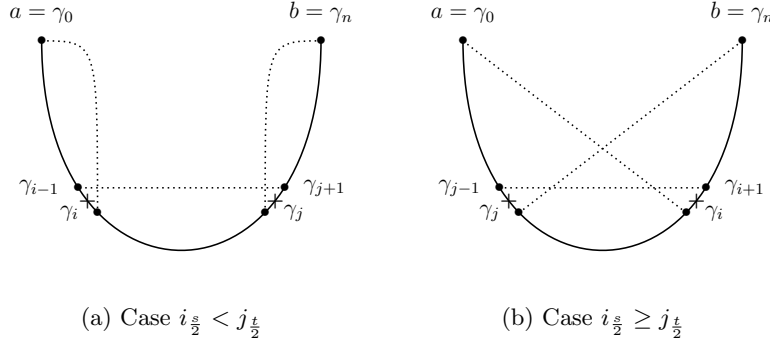

(a) Case $i_{\frac{s}{2}} < j_{\frac{t}{2}}$ (b) Case $i_{\frac{s}{2}} \geq j_{\frac{t}{2}}$

Figure 2: Closing the Hamilton Cycle in Stage 4

If CleverMaker can not make a move according to these directions, he forfeits.

Theorem 3.3. *Let $\epsilon > 0$ be fixed and let $b \leq (1-\epsilon)\frac{n}{2}$ be a positive integer. Then CleverMaker, playing against RandomBreaker in the $(1:b)$ half-random game, can execute the strategy S_{HAM} without forfeiting, a.a.s. Hence CleverMaker wins the $(1:b)$ half-random Hamiltonicity game (and thus also the degree-2 game) within $\frac{n}{2} + O(\ln n)$ moves, a.a.s.*

Proof of Theorem 1.3. Let $\epsilon > 0$ be fixed and let $b \leq (1-\epsilon)\frac{n}{2}$ be a positive integer. We will prove that CleverMaker, playing against RandomBreaker in the $(1:b)$ half-random game, can execute the strategy S_{HAM} without forfeiting, a.a.s. This, in particular, will imply that CleverMaker wins the $(1:b)$ half-random Hamiltonicity game (and thus also the degree-2 game and the connectivity game) within $\frac{n}{2} + O(\ln n)$ moves, a.a.s.

First note that in Stage 0 CleverMaker can a.a.s. create a perfect matching within $\frac{n}{2} + O(\ln n)$ rounds by Theorem 1.2. Stage 1.-4. take another

$$\frac{n}{2} - k(n) + 2(k(n) - l) + 3l + 3 = \frac{n}{2} + O(\ln n)$$

rounds, for a total of $t = n + O(\ln n)$ rounds. This means that throughout the game there are at most $(b+1)t \leq \left((1-\epsilon)\frac{n}{2} + 1\right)(n + o(n)) \leq p\binom{n}{2}$ occupied edges, and there are always at least $\frac{\epsilon}{4}n^2$ free edges, so both Lemma 3.1 and Lemma 3.2 are applicable.

As in the proof of Theorem 1.2, the number of double and triple moves is $O(k(n)) = O(\ln n)$, so the overall probability of CleverMaker forfeiting because

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he could not complete a double or triple move is $O(\ln n/n)$. Again we assume that `RandomBreaker`'s graph has Properties (i)-(iv) of Lemma 3.1, and `Clever-Maker` can complete all his double and triple moves. Now we need to check that the vacant edges required by S_{HAM} for the single, double, and triple moves of `CleverMaker` do exist each time.

Stage 1: If there was no vacant edge between two endpoints of different paths, then `RandomBreaker` would have occupied a clique of size $k(n)$ minus a matching in his graph, spanned by the endpoints of the paths in $\mathcal{P}_i$ (where the matching consists of the edges between the two endpoints of each paths). However, by Property (ii), this is impossible.

Stage 2: By Property (i), $|X_a^{\leftarrow}| \geq |X_a| - |\mathcal{P}_i| - 1 \geq n - 1 - deg_B(a) - k(n) - 1 \geq \frac{\epsilon}{8}n$. Furthermore, B has one vertex from each path in $\mathcal{P}_i \setminus \{\alpha\}$. This means $|B| \geq l$ since the number of paths in Stage 2 is at least $l+1$. By Property (iii), there is no $\frac{\epsilon}{8}n \times l$ complete bipartite graph in $G_{B,t}$, and hence `Cle-verMaker` can start his double move.

Stage 3: For `CleverMaker` being able to identify its triple move there must only be a vacant edge between X_a^* and X_b^*. Both sets have linear size: Indeed, both X_a and X_b have size at least $\frac{\epsilon n}{4} - 1$ by Property (i) of Lemma 3.1, and since there are only at most l paths left in this stage, $X_a \cap \gamma^{\mathsf{a}}$ and $X_b \cap \gamma^{\mathsf{b}}$ both must have at least $\left(\frac{\epsilon n}{4} - 1\right)/l$ vertices. Furthermore, in all cases $|X_a^*| \geq \frac{1}{2}|X_a \cap \gamma^{\mathsf{a}}| - 1$ and $|X_b^*| \geq \frac{1}{2}|X_b \cap \gamma^{\mathsf{b}}| - 1$, which means that X_a^* and X_b^* are of size at least $\frac{\epsilon n}{16l}$. In particular, there are disjoint sets $Y_a^* \subset X_a^*$ and $Y_b^* \subset X_b^*$ of size at least $\frac{\epsilon n}{32l}$ each. Then by Property (iv), `RandomBreaker` could not have occupied all edges between Y_a^* and Y_b^*, i.e. one must be vacant. Note that by definition of X_a^* and X_b^*, no edge between the sets is used in a path in $\mathcal{P}_i$ (this corresponds to the edge wz in Figure 1).

Stage 4: The analysis here is very similar to stage 3.

$\square$

4 Conclusion and Open Problems

We found that in the `CleverMaker-RandomBreaker` scenario the trivial upper bound on the threshold bias, provided by the size of a winning set, gives the true asymptotics. It would be interesting to decide whether a stronger lower bound holds. For a k-uniform graph property $\mathcal{F} \subseteq 2^{E(K_n)}$ let $b_{triv} = \lfloor \binom{n}{2}/k \rfloor - 1$ be the largest bias b such that Maker occupies at least k edges in the $(1 : b)$ game. Is it true that already for `RandomBreaker`-bias $b_{triv} - \omega(1)$, where $\omega(1)$ is a function tending to infinity arbitrarily slowly, `CleverMaker` has a strategy that is winning against `RandomMaker` a.a.s.?

A possible first step in this direction could be to give a strategy for `CleverMaker` for every $\epsilon > 0$, that a.a.s occupies a winning set $F \in \mathcal{F}$ in exactly $|F|$ moves

against a `RandomBreaker` bias of $(1 - \epsilon)b_{triv}$. We are not that far away from this: our strategies for `CleverMaker` in the perfect matching and Hamiltonicity game use only $O(\log n)$ more moves than necessary.