Sharp thresholds for half-random games I

Jonas Groschwitz
Universität Potsdam

Tibor Szabó
Freie Universität Berlin

Abstract

We study biased Maker-Breaker positional games between two players, one of whom is playing randomly against an opponent with an optimal strategy. In this paper we consider the scenario when Maker plays randomly and Breaker is "clever", and determine the sharp threshold bias of classical graph games, such as connectivity, Hamiltonicity, and minimum degree-k. We treat the other case, that is when Breaker plays randomly, in a separate paper.

The traditional, deterministic version of these games, with two optimal players playing, are known to obey the so-called probabilistic intuition. That is, the threshold bias of these games is asymptotically equal to the threshold bias of their random counterpart, where players just take edges uniformly at random. We find, that despite this remarkably precise agreement of the results of the deterministic and the random games, playing randomly against an optimal opponent is not a good idea: the threshold bias tilts significantly more towards the random player. An important qualitative aspect of the probabilistic intuition carries through nevertheless: the bottleneck for Maker to occupy a connected graph is still the ability to avoid isolated vertices in her graph.

1 Introduction

Let us be given a finite hypergraph $\mathcal{F} \subseteq 2^X$ on a vertex set X. In the Maker-Breaker positional game $\mathcal{F}$ two players, Maker and Breaker, alternately take turns in occupying free elements of X, with Maker going first, until no free element is left. Maker is the winner if he completely occupied a hyperedge of the hypergraph $\mathcal{F}$, otherwise Breaker wins. Such a game is of perfect information with no chance moves, so one of the players has a winning strategy. That which one, depends on the hypergraph $\mathcal{F}$. A standard method, introduced by Chvátal and Erdős [9], to measure the robustness of this winning strategy is to give the "disadvantaged" player a *bias*, that is to allow him to occupy more than one element of X in each turn. In an $(a : b)$ *biased game* Maker occupies a elements of X in each turn and Breaker occupies b elements.

For our investigation we will be concerned mostly with graph games, where the *board* X is the edge set $E(K_n)$ of the complete graph and the game hypergraph $\mathcal{P} \subseteq 2^{E(K_n)}$ describes a graph property. In the present paper we study properties fundamental both in terms of graph theory and positional games. These include connectivity, having a perfect matching, Hamilton cycle or minimum degree k. For this, let $\mathcal{C}(n)$, $\mathcal{H}(n)$, $\mathcal{D}_k(n)$ denote the family of edge sets of n-vertex graphs that are connected, contain a Hamiltonian cycle, have minimum degree k, respectively. Subsequently we suppress the parameter n in the notation.

1.1 Threshold bias and probabilistic intuition

As it turns out, many of the natural graph games are relatively easy wins for Maker if the game is played $(1 : 1)$. Chvátal and Erdős [9] were the first to study how large of a bias b Breaker needs in various graph games in order to win the $(1 : b)$ biased game. For a game hypergraph $\mathcal{F}$ we define $b_{\mathcal{F}}$ to be the smallest integer b such that Breaker has a winning strategy in the $(1 : b)$ biased game $\mathcal{F}$ and $b_{\mathcal{F}}$ is called the *threshold bias* of the game.

Chvátal and Erdős [9] determined the order of magnitude of the threshold bias of the connectivity game $\mathcal{C}$ and the triangle building game $\mathcal{K}_{K_3}$. They have shown that $b_{\mathcal{C}} = \Theta\left(\frac{n}{\ln n}\right)$ and $b_{\mathcal{K}_{K_3}} = \Theta\left(\sqrt{n}\right)$. The constant factor in the lower bound for $b_{\mathcal{C}}$ was first improved by Beck [2]. Later Gebauer and Szabó [11] established $b_{\mathcal{C}} = (1 + o(1))\frac{n}{\ln n}$, showing that the upper bound of Chvátal and Erdős is asymptotically tight. For the Hamiltonicity game $\mathcal{H}$ Chvátal and Erdős only showed that $b_{\mathcal{H}} > 1$. This was subsequently improved in a series of papers by Bollobás and Papaioannou [8], Beck [3], Krivelevich and Szabó [19], until Krivelevich [16] proved that $b_{\mathcal{H}} = (1 + o(1))\frac{n}{\ln n}$. In other words, building a Hamiltonian cycle is possible for Maker against essentially the same bias as building just a connected graph.

Erdős and Chvátal's winning strategy for Breaker in the connectivity game actually isolates a vertex of Maker's graph, and thus wins the minimum degree-1 game as well. Further, since a win for Maker in the connectivity game also is a win for him in the minimum degree-1 game, the results for Maker's win of the connectivity game carry over. Thus, in the minimum degree-1 game too, the threshold bias is asymptotically equal to $\frac{n}{\ln n}$. The message of this is that in positional games, having an isolated vertex turns out to be the bottleneck for having a connected graph. This phenomenon is familiar from the theory of random graphs, where Erdős and Rényi [10] established that the sharp threshold edge number to have a connected graph in the uniform random graph model $G(n, m)$ is the same as the one to have a graph with minimum degree 1.

In fact, as already Chvátal and Erdős realized, the similarities between random graphs and positional games are even closer. In a positional game players are playing "cleverly", according to optimal strategies and exactly one of the players has a deterministic winning strategy, which wins against *any* strategy of the other player. The situation is different if both players play "randomly", that is, if both Maker and Breaker determine their moves by picking a uniformly random edge out of the currently free edges; then we can only talk about the "typical" result of the game. The graph of this `RandomMaker` will be a uniform random graph $G \sim G(n,m)$ with $m = \left\lceil \frac{\binom{n}{2}}{b+1} \right\rceil$ edges. Therefore `RandomMaker` wins a particular game involving graph property $\mathcal{P}$ asymptotically almost surely (a.a.s.) if and only if the random graph $G \sim G(n,m)$ possesses property $\mathcal{P}$ a.a.s. Hence the classic theorem of Erdős and Rényi about the sharp connectivity threshold in random graphs can be reformulated in positional game theoretic terms.

Theorem 1.1 (Erdős-Rényi, [10])**.** *For every $\epsilon > 0$, the following holds in the connectivity game $\mathcal{C}$ between* `RandomMaker` *and* `RandomBreaker`*.*

(i) $Pr\left[\texttt{RandomMaker} \text{ wins the } \left(1 : (1-\epsilon)\frac{n}{\ln n}\right) \text{ connectivity game } \mathcal{C}\ \right] \to 1,$

(ii) $Pr\left[\texttt{RandomBreaker} \text{ wins the } \left(1 : (1+\epsilon)\frac{n}{\ln n}\right) \text{ connectivity game } \mathcal{C}\right] \to 1.$

By this theorem the threshold biases of both the random connectivity game and the clever connectivity game are $(1+o(1))\frac{n}{\ln n}$. This remarkable agreement means that for most values of the bias the *result* of the random and the clever game is the same a.a.s. This phenomenon is referred to as the *probabilistic intuition.* Since similar random graph theorems also hold true for the properties of Hamiltonicity [1] and having minimum degree 1 [10], these games are also instances where the probabilistic intuition is valid. One of the main directions of research in positional game theory constitutes of understanding what games obey the probabilistic intuition.

1.2 Half-Random Games

The meaning of the probabilistic intuition is that given any bias $b \leq (1-\epsilon)b_{\mathcal{P}}$ or $b \geq (1+\epsilon)b_{\mathcal{P}}$, one could predict the winner of the "clever" $(1:b)$-game $\mathcal{P}$ just by running random experiments with two random players playing each other: whoever wins in the majority of these random games is very likely to have the winning strategy in the deterministic game between the clever players.

When learning about this interpretation, it is natural to inquire whether it is just the success of the randomized strategy in the clever game what is behind the whole phenomenon. Could it be that when Maker plays uniformly at random against Breaker, who plays with a bias near to the threshold, then this `RandomMaker` wins with high probability? In this paper we give precise quantitative evidence that the answer to this question is negative. We will see that in all the games discussed above, a player puts herself in serious disadvantage by playing randomly instead of following a clever strategy.

In what follows we investigate *half-random positional games*, where one of the players plays according to the uniform random strategy against an optimal player. There are two versions: either Maker follows a strategy and Breaker's moves are determined randomly, or the other way around. We refer to the players as `CleverMaker`/ `RandomBreaker`, and `RandomMaker`/ `CleverBreaker`, respectively. In this paper we focus on the `RandomMaker` versus `CleverBreaker` setup. Our approach to the other case requires mostly different combinatorial methods and is treated in a separate paper [13].

Below we define the notion of a sharp threshold bias for `RandomMaker`/ `CleverBreaker` games. For this, when we talk about a game, we actually mean a *sequence of games*, parametrized with the size n of the vertex set of the underlying graph. Similarly, when we refer to a strategy of `CleverBreaker`, we mean a *sequence of strategies*.

It will turn out that when Maker plays randomly, the disadvantage of making random moves outweighs his huge advantage inherent in the $(1:1)$ games, and the half-random bias needs to tilt in his favor. This motivates the following definition.

Definition 1.2. We say a function $k : \mathbb{N}_0 \mapsto \mathbb{N}_0$ is a *sharp threshold bias* of the $(a:1)$ half-random positional game between `RandomMaker` and `CleverBreaker`, if for every $\epsilon > 0$ it satisfies the following two conditions

(a) `RandomMaker` wins the $((1+\epsilon)k(n) : 1)$-biased game a.a.s. against any strategy of `CleverBreaker`, and

(b) `CleverBreaker` has a strategy against which `RandomMaker` loses the $((1-\epsilon)\, k(n) : 1)$-biased game a.a.s.

Remarks. 1. Our paper is mostly about the failure of the uniformly random strategy against a clever player in various classical graph games. There are other natural games where the situation is completely different and the uniformly random strategy is close to being optimal. Bednarska and Łuczak [5] consider the H-building game $\mathcal{K}_H$, where Maker's goal is to occupy a copy of a fixed graph H. Even though their paper is about the classical game scenario with clever players, their results also imply that the half-random $(1:b)$ H-game $\mathcal{K}_H$ between `RandomMaker` and `CleverBreaker` has a threshold bias around $n^{\frac{1}{m_2(H)}}$, where $m_2(H) = \max_{K\subseteq H, v(K)\geq 3} \frac{e(K)-1}{v(K)-2}$. Bednarska and Łuczak not only prove that `RandomMaker` succeeds against a bias $cn^{\frac{1}{m_2(H)}}$ for some small constant c a.a.s., but also that even a `CleverMaker` would not be able to do much better. That is, they give a strategy for `CleverBreaker` playing with a bias $Cn^{\frac{1}{m_2(H)}}$, where C is some large constant, to prevent the creation of H by `CleverMaker`. The H-game is an instance of a game where the threshold bias for the clever game is of the same order of magnitude as for the half-random game — very much unlike the games we consider in this paper.

2. Half-random versions of other positional games were also considered earlier in different context. The well-studied notion of an Achlioptas process can be cast as the `RandomWaiter-CleverClient` version of the classic Picker-Chooser games introduced by Beck [4] (and renamed recently to Waiter-Client by Bednarska-Bzdega, Hefetz, Łuczak [6]). In a $(1:1)$ Waiter-Client game the player Waiter chooses two, so far unchosen edges of K_n and offers them to the player called Client, who selects one of them into his graph. Waiter wins when Client's graph has property $\mathcal{P}$. A substantial amount of work [7, 21] was focused on determining *how long* does it take for `RandomWaiter` to win when the property $\mathcal{P}$ is to have a connected component of linear size. Bohman and

Frieze [7] gave a simple strategy for `CleverClient` to significantly delay the win of `RandomWaiter` compared to the well-known threshold from the work of Erdős and Rényi in the game where both players play randomly.

1.3 Results

We first show that if $a \leq (1-\epsilon)\ln\ln n$, then a simple and natural strategy of `CleverBreaker` allows him to isolate a vertex in `RandomMaker`'s graph a.a.s., and therefore win the degree-1 game. Then we establish that this threshold is asymptotically tight for all the games we are considering in this paper.

Theorem 1.3. *Let k be a positive integer. The sharp threshold bias for the $(a:1)$ minimum degree-k game between `RandomMaker` and `CleverBreaker` is $\ln\ln n$.*

Theorem 1.4. *The sharp threshold bias for the $(a:1)$ connectivity game between `RandomMaker` and `CleverBreaker` is $\ln\ln n$.*

Theorem 1.5. *The sharp threshold bias for the $(a:1)$ Hamiltonicity game between `RandomMaker` and `CleverBreaker` is $\ln\ln n$.*

On the one hand these theorems show that mindless random strategies are very ineffective for the games we consider here, where the goal is "global". As discussed earlier, randomized strategies are shown to be close to optimal for games where the goal of Maker is "local", for example when the goal of Maker is to build a fixed subgraph H [5]. On the other hand, these theorems establish that the bottleneck for winning connectivity and Hamiltonicity in half-random games is to be able to win the minimum degree-1 game. This is similar to the phenomenon that occurs in the fully random and the fully clever scenario.

Remarks. The results of this paper and of [13] are based on the Master thesis of the first author [12]. Recently, Krivelevich and Kronenberg [17] also studied half-random games independently (both in the `CleverMaker-RandomBreaker` and the `RandomMaker-CleverBreaker` setup). For the `RandomMaker-CleverBreaker` setup they determine the order of magnitude of the half-random threshold bias of the Hamiltonicity and the k-connectivity game. Our main results pin down the constant factors as well. (In the conclusion section we indicate how the sharp threshold result for the k-connectivity game with arbitrary $k \geq 2$ can be obtained easily by combining our proof technique for the connectivity game with the minimum degree-k game.)
In [13] we determine the sharp threshold bias of the perfect matching and the Hamiltonicity games in the `CleverMaker-RandomBreaker` setup. Krivelevich and Kronenberg [17] obtain similar results with different methods.

1.4 Terminology and organization

We will use the following terminology and conventions. A *move* consists of claiming one edge. *Turns* are taken alternately, one turn can have multiple moves. For example: With an $(a : 1)$ bias, Maker has a moves per turn, while Breaker has 1 move. A *round* consists of a turn by Maker followed by a turn by Breaker. By a strategy we mean a set of rules which specifies what the player does in any possible game scenario. For technical reasons we *always* consider strategies that last until there are no free edges. This will be so even if the player has already won, already lost, or his strategy description includes "then he forfeits"; in these cases the strategy just always occupies an arbitrary free edge, say with the smallest index. The *play-sequence* Γ of length i of an actual game between Maker and Breaker is the list $(\Gamma_1, \ldots, \Gamma_i) \in E(K_n)^i$ of the first i edges that were occupied during the game by either of the players, in the order they were occupied. We make here the convention that a player with a bias $b > 1$ occupies his b edges within one turn in succession and these are noted in the play-sequence in this order (even though in the actual game it makes no difference in what order one player's moves are occupied within one of his turns). We denote Maker's graph after t rounds with $G_{M,t}$ and similarly Breaker's graph with $G_{B,t}$. Note that these graphs have at and bt edges respectively. We will use the convention that Maker goes first. This is more of a notational convenience, since the proofs can be easily adjusted to Breaker going first, and yielding the same asymptotic results.

For a graph $G = (V, E)$, a subset $F \subseteq E$ of edges and a subset $B \subseteq V$ of vertices, we denote by $G-F$ the graph with $V(G-F) = V$ and $E(G-F) = E \setminus F$, and by $G - B$ the graph induced by G on $V \setminus B$.

Due to the asymptotic nature of our statements we can always assume n to be sufficiently large and we will routinely omit rounding signs, whenever they are not crucial.

We introduce the useful notion of the *permutation strategy* in the next section, and prove Theorems 1.3, 1.4 and 1.5 in Section 3.

2 The permutation strategy

In this section, we introduce an alternative way to think of half-random games which will be important in many of our proofs. One feature that makes half-random games more difficult to study than random games is that the graph of the random player is *not* uniformly random: the moves of the clever player affect it. Our goal is still to be able to somehow compare it to a random graph from the uniform model $G(n, m)$ with the appropriate number of edges and draw conclusions from the rich theory of random graphs.

Any of the players in a positional game can use a permutation $\sigma \in S_{E(K_n)}$, i.e., $\sigma : \left[\binom{n}{2}\right] \to E(K_n)$, of the edges of K_n for his strategy as follows. The player following the *permutation strategy* σ is scanning through the list $(\sigma(1), \ldots, \sigma(\binom{n}{2}))$ during the game and in each of his moves he occupies the next free edge on it (that is, the next edge which was not yet occupied by his opponent). The permutation strategy gives rise to a natural randomized strategy for `RandomMaker` when he selects the permutation uniformly at random. It turns out that playing according to this random permutation strategy is equivalent to playing according to the original definition of RandomMaker's strategy (i.e., always choosing uniformly at random from the remaining free edges).

The following proposition formalizes this. Intuitively it is quite clear, in [13] we give a formal proof of a more general statement. Here we only state the special case we need. Since the goal of the game is not relevant here, we state the proposition for graph games in general.

Proposition 2.1. *For every strategy S of CleverBreaker in a $(a:b)$-game on $E(K_n)$ the following is true. For every $m \le \binom{n}{2}$ and every sequence $\Gamma = (\Gamma_1, \ldots, \Gamma_m)$ of distinct edges, the probability that Γ is the play-sequence of a half random game between CleverBreaker playing according to strategy S and RandomMaker is equal to the probability that Γ is the play-sequence of the game when RandomMaker plays instead according to the random permutation strategy.*

For $1 \le m \le \binom{n}{2}$ and a permutation $\sigma \in S_{E(K_n)}$, let $G_\sigma(m) \subseteq K_n$ be the subgraph with edge set $E(G_\sigma(m)) = \{\sigma(i) : 1 \le i \le m\}$. Note that if σ is a permutation chosen uniformly at random out of all permutations, then $G_\sigma(m)$ is a graph chosen uniformly at random from all graphs with m edges, i.e., $G_\sigma(m) \sim G(n,m)$. If RandomMaker plays a particular game according to a permutation $\sigma \in S_{E(K_n)}$ and the last edge he takes in round i is $\sigma(m_i)$, then RandomMaker's graph after round i is contained in $G_\sigma(m_i)$. Here $m_i \ge ia$, but the actual value of it depends on the strategy of CleverBreaker and the permutation σ itself. Since CleverBreaker occupied ib edges so far and these are the only edges RandomMaker possibly skips from his permutation, we also have that $m_i \le i(a+b)$. Hence RandomMaker's graph after the ith round is always contained in the random graph $G_\sigma(i(a+b))$.

3 CleverBreaker vs RandomMaker

In this section, we prove Theorems 1.3, 1.4, and 1.5. We start with showing that a.a.s. CleverBreaker is able to isolate a vertex in RandomMaker's graph if the bias of RandomMaker is not too large. This provides the lower bound on the sharp thresholds in all the games we study and is the topic of the next subsection. We treat the upper bounds in Subsections 3.2 and 3.3.

3.1 CleverBreaker isolates a vertex of RandomMaker

In this subsection we prove the following theorem.

Theorem 3.1. *Let $\epsilon > 0$ and $a \le (1-\epsilon)\ln\ln n$. Then there exists a strategy for CleverBreaker, such that he a.a.s. wins the $(a:1)$-biased minimum degree-1 game against RandomMaker.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the underlying complete graph. CleverBreaker's strategy is rather simple. CleverBreaker identifies the vertex v_i of smallest index which has degree 0 in Maker's graph. Then he occupies the free edges incident to v_i, one by one, in an increasing order of the indices of their other endpoint. (We refer to this process as *CleverBreaker trying to isolate v_i*.) If he succeeds in occupying all $n-1$ edges incident to v_i, then he won the game. Otherwise, that is if `RandomMaker` occupied an edge incident to v_i while `CleverBreaker` was trying to isolate it, `CleverBreaker` iterates: he identifies a new vertex he tries to isolate. In this case we say that *CleverBreaker failed to isolate v_i*. If `CleverBreaker` fails to isolate

$$k(n) = k := (1-\epsilon)\frac{\ln n}{4\ln\ln n}$$

vertices then he forfeits.

Recall the permutation strategy for the random player of Section 2, based on a random permutation of the edges of $E(K_n)$. Let us denote by $\mathcal{W}$ the set of those permutations for `RandomMaker` which would result in a win for `CleverBreaker` using this described strategy. Note that for k tries, Breaker spends at most $(n-1)k < nk$ edges (and therefore turns) and hence the presence of a permutation σ in $\mathcal{W}$ is determined by its first $(a+1)nk$ edges.

Let $\mathcal{A}$ denote the set of those permutations σ for which the graph $G_\sigma\left((a+1)nk\right)$ of the first $(a+1)nk$ edges has an isolated vertex. Since

$$(a+1)nk \le (\ln\ln n + 1)n\frac{(1-\epsilon)\ln n}{4\ln\ln n} \le (1-\epsilon)\frac{1}{2}n\ln n,$$

the classic result of Erdős and Rényi [10] on the sharp threshold in $G(n,m)$ for the minimum degree being at least 1 implies the following.

Lemma 3.2. *$\mathcal{A}$ occurs a.a.s.*

The following lemma guarantees that, conditioned on $\mathcal{A}$, `CleverBreaker` tries to isolate k vertices or wins already earlier.

Lemma 3.3. *For every $\sigma \in \mathcal{A}\setminus\mathcal{W}$ CleverBreaker tries to isolate k vertices.*

Proof. For any permutation $\sigma \in \mathcal{A}$, the graph $G_\sigma\left((a+1)nk\right)$ contains the graph of `RandomMaker` up to the point when `CleverBreaker` tries and fails to isolate at most k vertices. On the other hand $G_\sigma\left((a+1)nk\right)$ does have an isolated vertex by the definition of $\mathcal{A}$, so `CleverBreaker` did not run out of isolated vertices by the time he failed to isolate his $(k-1)$st vertex. □

The main ingredient of our proof is an estimation of the probability that `CleverBreaker` fails to isolate his jth vertex, given that he already failed to isolate the first $j-1$ vertices. Let $\mathcal{D}_0 := S_{E(K_n)}$ be the set of all permutations, and for $1 \le j \le k$, let $\mathcal{D}_j$ denote the event (set of permutations) that induces a game where `CleverBreaker` tries and fails to isolate at least the first j vertices. Notice that $\mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \cdots \supseteq \mathcal{D}_k$. Our eventual goal is to show that $\mathcal{D}_k \cap \mathcal{A}$ is very small. To achieve this we bound $|\mathcal{D}_j \cap \mathcal{A}|$ in terms of $|\mathcal{D}_{j-1} \cap \mathcal{A}|$.

Proposition 3.4. *For every n large enough and every $j, 1 \le j \le k$, we have*

$$|\mathcal{D}_j \cap \mathcal{A}| \le \left(1 - \frac{1}{\ln^{1-\epsilon^2/2} n}\right)|\mathcal{D}_{j-1} \cap \mathcal{A}|.$$

Before we prove the proposition, let us show how it implies our theorem. Following the strategy defined above, `CleverBreaker` forfeits either if he fails every one of his first k tries to isolate a vertex, or if there is no more vertex of Maker-degree 0. We saw in Lemma 3.3, that for any permutation in $\mathcal{A}$ the latter one is not an option: `CleverBreaker` has to fail at least k times before he runs out of vertices he can try. Therefore, using Proposition 3.4, we obtain

$$\begin{aligned}
\frac{|\mathcal{D}_k \cap \mathcal{A}|}{|\mathcal{A}|} &\leq \left(1 - \frac{1}{(\ln n)^{1-\epsilon^2/2}}\right) \frac{|\mathcal{D}_{k-1} \cap \mathcal{A}|}{|\mathcal{A}|} \\
&\leq \left(1 - \frac{1}{(\ln n)^{1-\epsilon^2/2}}\right)^k \\
&\leq e^{-k(\ln n)^{-(1-\epsilon^2/2)}} \\
&\leq e^{-(1-\epsilon)\frac{(\ln n)^{\epsilon^2/2}}{4 \ln \ln n}} \to 0.
\end{aligned}$$

Finally, since $\mathcal{A}$ holds a.a.s. by Lemma 3.2, we also have

$$Pr\left[\texttt{CleverBreaker}\text{ wins}\right] \geq Pr\left[\overline{\mathcal{D}_k} \;\middle|\; \mathcal{A}\right] Pr\left[\mathcal{A}\right] \;\to\; 1.$$

□

To complete the proof of Theorem 3.1 we need to prove Proposition 3.4. For that there is a subtle technicality that we have to take care of. If we assume that $\mathcal{A}$ holds, we use knowledge of the first $(a+1)nk$ random edges of our permutation and thus knowledge of `RandomMaker`'s moves up until the turn nk. Therefore, if we consider the distribution of the next move of `RandomMaker` among the free edges *before* turn nk, conditioned under $\mathcal{A}$, this distribution might not be uniform anymore. For example, if there is only one vertex $\tilde{v}$ left with degree 0 in `RandomMaker`'s graph, then the probability that `RandomMaker` chooses an edge incident to $\tilde{v}$, under the condition that $\mathcal{A}$ holds, is 0. However, while some edges may have very low probability to be chosen by `RandomMaker`, we can show that there are no edges that have a particularly high probability to be picked.

For a starting edge sequence $\pi \in S^{(m)}_{E(K_n)}$ of length m, let $\mathcal{A}(\pi) \subseteq \mathcal{A}$ denote the set of permutations $\sigma \in \mathcal{A}$ with initial segment equal to π. Given an edge sequence $\eta \in S_{E(K_n)}$ and a strategy S of `CleverBreaker`, we say an edge $e \in E(K_n)$ to be (S, η)-*Maker* if it is taken by `RandomMaker` when he plays according to η against strategy S. Let $\mathcal{A}(\pi; S; e) \subseteq \mathcal{A}(\pi)$ denote the set of permutations $\eta \in S_{E(K_n)}$ which start with π and after that the next (S, η)-Maker edge is e.

Lemma 3.5. *For every $\epsilon > 0$ the following holds for large enough n. For every strategy S of `CleverBreaker`, positive integer $m \leq (a+1)nk$, starting permutation $\pi \in S^{(m)}_{E(K_n)}$ of length m and edge $e \in E(K_n)$ we have that*

$$|\mathcal{A}(\pi; S; e)| \leq (1+\epsilon)\frac{2}{n^2}|\mathcal{A}(\pi)|.$$

Proof. We can assume that e is still unoccupied after the permutation strategy has been played according to π, otherwise the statement is trivial (since the set $\mathcal{A}(\pi; S; e)$ is empty).

We partition the sets $\mathcal{A}(\pi)$ and $\mathcal{A}(\pi; S; e)$ according to the sequence of edges that come after π in a permutation $\eta \in \mathcal{A}(\pi)$ until the first (S, η)-Maker edge. Let π' be an arbitrary extension of π with a sequence τ containing only such edges which were occupied by `CleverBreaker` when the permutation strategy was played until round m according to π. ($\tau = \emptyset$, i.e., $\pi' = \pi$ is also a possibility.) Note that the length of π' is at most $m + \frac{m}{a} \leq 2(a+1)nk = o(n^2)$.

Let $\hat{\mathcal{A}}(\pi'; S) \subseteq \mathcal{A}(\pi')$ be the set of those permutations η which start with π' and continue with an (S, η)-Maker edge. Let $\hat{\mathcal{A}}_e(\pi', S) \subseteq \hat{\mathcal{A}}(\pi', S)$ be the set of permutations where the edge e comes immediately after π'. Unless otherwise stated, from now on we consider π' fixed and suppress it in the arguments of $\hat{\mathcal{A}}_e$ and $\hat{\mathcal{A}}$.

To show the upper bound of the lemma, we will find for any permutation $\eta \in \hat{\mathcal{A}}_e$ many different permutations in $\hat{\mathcal{A}}$. For any such η and edge $f \in E(K_n)$ we denote by η_e^f the edge permutation with the positions of e and f interchanged. Let $\mathcal{M}(\eta)$ be the set of those permutations η_e^f which are in $\hat{\mathcal{A}}$. That is,

$$\mathcal{M}(\eta) := \left\{ \eta_e^f : f \in E(K_n), \eta_e^f \in \hat{\mathcal{A}} \right\}.$$

There are three possible reasons why a permutation η_e^f would not be in $\mathcal{M}(\eta)$:

1. Any permutation in $\hat{\mathcal{A}}$ must start with π', hence we are not allowed to swap e with any edge that comes up in π'. The number of these forbidden edges is $m \leq (a+1)nk = o(n^2)$.

2. In any permutation $\eta \in \hat{\mathcal{A}}$ the edge following π' must be (S, η)-Maker, hence we cannot swap e with any edge claimed by `CleverBreaker` up to this point. There are at most $\frac{m}{a} = o(n^2)$ such edges.

3. Finally, the graph formed by the first $(a+1)nk$ edges of any edge permutation in $\hat{\mathcal{A}}$ must have an isolated vertex. So if $G_\eta\left((a+1)nk\right)$ had only one isolated vertex $\tilde{v}$, we might not be able to swap e with an edge f incident to $\tilde{v}$, since then $G_{\eta_e^f}\left((a+1)nk\right)$ might not have an isolated vertex anymore. So we forbid a swap with the $n-1 = o(n^2)$ incident edges to the last isolated vertex of $G_\eta\left((a+1)nk\right)$.

Swapping e with any edge that is *not* in these three categories leads to an edge permutation in $\hat{\mathcal{A}}$. Therefore, $|\mathcal{M}(\eta)| \geq \binom{n}{2} - o(n^2)$. By definition $\mathcal{M}(\eta) \subseteq \hat{\mathcal{A}}$ for every permutation $\eta \in \hat{\mathcal{A}}_e$. The sets $\mathcal{M}(\eta)$ and $\mathcal{M}(\zeta)$ are disjoint for $\eta \neq \zeta$, as clearly $\eta_e^f = \zeta_e^{f'}$ is only possible if $f = f'$ and $\eta = \zeta$. Hence for the cardinalities we have

$$\left|\hat{\mathcal{A}}\right| \geq \sum_{\eta \in \hat{\mathcal{A}}_e} |\mathcal{M}(\eta)| \geq \left|\hat{\mathcal{A}}_e\right| \left(\binom{n}{2} - o(n^2) \right). \tag{1}$$

Now recall that $\hat{\mathcal{A}} = \hat{\mathcal{A}}(\pi'; S)$ and $\hat{\mathcal{A}}_e = \hat{\mathcal{A}}_e(\pi', S)$ where π' was an arbitrary, but fixed extension of π with an edge sequence τ containing only edges `CleverBreaker` took up to playing according to π.

Our focus of interest, the sets $\mathcal{A}(\pi)$ and $\mathcal{A}(\pi; S; e)$ are disjoint unions of the sets $\hat{\mathcal{A}}(\pi', S)$ and $\hat{\mathcal{A}}_e(\pi', S)$, respectively, where the disjoint union is taken over all extensions π' of π with distinct edges which were occupied by `CleverBreaker` in the game played according to π. Therefore Equation (1) is also valid for them and hence,

$$|\mathcal{A}(\pi; S; e)| \leq \frac{1}{\binom{n}{2} - o(n^2)} |\mathcal{A}(\pi)| \leq \frac{2(1+\epsilon)}{n^2} |\mathcal{A}(\pi)|$$

for n large enough, which is the statement of the lemma. $\square$

With Lemma 3.5 proven, we can return to the main line of reasoning.

Proof of Proposition 3.4. Let $\sigma \in \mathcal{A} \cap \mathcal{D}_{j-1}$ and let $w_1, \ldots, w_{j-1}, w_j$ be the vertices `CleverBreaker` tries to isolate, in this order, when he plays against σ. Note that the first $j-1$ of these vertices do exist because $\sigma \in \mathcal{D}_{j-1}$ and then w_j also exists by Lemma 3.3. We define $\pi = \pi(\sigma)$ to be the initial segment of σ, which ends with the last edge `RandomMaker` takes in the round where he occupies his first edge incident to w_{j-1}. The length of π is at most $(a+1)(n-1)(j-1)$, so we will be able to use Lemma 3.5. Let Π be the set of all such π, i.e.,

$$\Pi = \{\pi(\sigma) : \sigma \in \mathcal{A} \cap \mathcal{D}_{j-1}\}.$$

We classify the permutations $\sigma \in \mathcal{D}_{j-1} \cap \mathcal{A}$ according to these initial segments. We will prove that for all $\pi \in \Pi$,

$$|(\mathcal{D}_j \cap \mathcal{A})(\pi)| \leq \left(1 - \frac{1}{\ln^{1-\epsilon^2/2} n}\right) |(\mathcal{D}_{j-1} \cap \mathcal{A})(\pi)|$$

and the statement follows since $(\mathcal{D}_j \cap \mathcal{A})$ is the disjoint union of the $(\mathcal{D}_j \cap \mathcal{A})(\pi)$ when the disjoint union is taken over all $\pi \in \Pi$. The union is disjoint since no element of Π is the prefix of another.

Let us fix an arbitrary initial segment $\pi \in \Pi$. After π has been played out, `CleverBreaker` immediately identifies the next vertex w_j he will try to isolate. Suppose that $r \leq n-1$ edges incident to w_j are free, that is `CleverBreaker` occupied already $n-1-r$ edges incident to w_j during his previous tries to isolate a vertex, while `RandomMaker` occupied none. In the next round `CleverBreaker` occupies the free edge from w_j to the vertex with the smallest index. Then `RandomMaker` has a random edges and the question is whether he hits any of the remaining $r-1$ free edges incident to w_j.

Note that all permutations starting with π are in $\mathcal{D}_{j-1}$, and thus $(\mathcal{D}_{j-1} \cap \mathcal{A})(\pi) = \mathcal{A}(\pi)$. Therefore, the number of such permutations where `RandomMaker` in his next move hits one of these edges is at most

$$(r-1)\frac{2(1+\epsilon)}{n^2} |(\mathcal{D}_{j-1} \cap \mathcal{A})(\pi)|$$

by Lemma 3.5. Then the number of permutations where `RandomMaker` did not play any of these edges is at least

$$\left(1-(r-1)\frac{2(1+\epsilon)}{n^2}\right)|(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)|.$$

We repeat the process for the a moves of `RandomMaker`, always taking a new set Π, letting the initial segment π run until `RandomMaker`'s last move each time, always conditioning that `RandomMaker` has not yet claimed an edge incident to w_j (i.e., allowing only such σ). Applying Lemma 3.5 iteratively, the number of permutations where none of `RandomMaker`'s a edges are incident to v_j is at least

$$\left(1-(r-1)\frac{2(1+\epsilon)}{n^2}\right)^a|(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)|.$$

In order to estimate the number of permutations in which `RandomMaker` does not take any edges incident to w_j and hence `CleverBreaker` isolates w_j, we repeat the above process over the relevant $r-1$ turns. The calculation is identical for each turn, except that the number of vacant edges incident to w_j decreases. Taking the product over these $r-1$ turns, we obtain

$$\begin{aligned}
|(\overline{\mathcal{D}_j}\cap\mathcal{A})(\pi)| &\geq \prod_{\ell=1}^{r-1}\left(1-\frac{2(1+\epsilon)\ell}{n^2}\right)^a|(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)| \\
&\geq e^{-a\left(\sum_{\ell=1}^{r-1}\frac{2(1+\epsilon)\ell}{n^2}\right)-a\left(\sum_{\ell=1}^{r-1}(\frac{2(1+\epsilon)\ell}{n^2})^2\right)}(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)| \\
&\geq e^{-a(1+\epsilon)-O(\frac{a}{n})}(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)| \\
&\geq e^{-(1-\epsilon^2)\ln\ln n-O(\frac{\ln\ln n}{n})}(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)| \\
&\geq \left(\ln^{-(1-\epsilon^2/2)}n\right)|(\mathcal{D}_{j-1}\cap\mathcal{A})(\pi)|
\end{aligned}$$

using $r\leq n-1$ in the third inequality.

□

3.2 `RandomMaker` builds a connected graph with minimum degree at least k

The proofs of the upper bound for all games in Theorems 1.3, 1.4 and 1.5 all start out the same. First we establish that the vertices with many incident edges occupied by `CleverBreaker` (called "bad vertices") are well-connected to the rest of the graph (called "good vertices"). Then we go on to show that the graph of Maker on the good vertices is very close to being a uniformly random graph. Then the upper bound in the min-degree-k and connectivity game follows easily. To prove the existence of a Hamilton cycle is somewhat more technical and is presented in its separate section.

Let $\epsilon>0$ fixed and $a\geq(1+\epsilon)\ln\ln n$. We consider the $(a:1)$-game on K_n between `RandomMaker` and `CleverBreaker` playing according to an arbitrary but fixed strategy S. First we introduce some notation. Let us fix a parameter, α, $0 < \alpha < \frac{1}{2}$ sufficiently small, such that

$$(1+\epsilon)(1-3\alpha)^2 > 1 + \frac{\epsilon}{2}. \tag{2}$$

We will consider the first $t := \frac{\alpha}{2} n \ln n$ rounds of the game and show that `RandomMaker` finishes his job within his first t turns, a.a.s.
The key idea of the proof is to divide the vertices in categories, based on how many incident edges `CleverBreaker` claims. We call a vertex *α-bad*, if its degree in `CleverBreaker`'s graph is $3\alpha n$ or more and otherwise we call it *α-good*. Since throughout this section α is fixed, we suppress it and talk about bad and good vertices.

An important observation is that during the first t rounds the total degree in `CleverBreaker`'s graph does not exceed $2t$ hence there cannot be more than

$$\frac{2t}{3\alpha n} = \frac{2}{3\alpha n} \cdot \frac{\alpha n \ln n}{2} \leq \ln n$$

bad vertices. In other words, the vast majority of vertices, namely $n - \ln n$, are still good after t turns.

3.2.1 Connecting the bad vertices

To grade the transition of a good vertex into a bad one we define the concept of a candidate vertex. We say that a vertex u is

(i) an *early candidate* if `CleverBreaker` claimed his αn-th edge incident to u before round $t-(1-\alpha)n$, and

(ii) a *late candidate*, if u is not an early candidate and `CleverBreaker` claims his $2\alpha n$-th edge incident to u in a turn s with $t-(1-\alpha)n \leq s \leq t - \alpha n$.

Observe that every vertex that is bad at turn t had to become (early or late) candidate in a turn $s \leq t - \alpha n$. Indeed, if a vertex u is bad then it must have had degree at least $2\alpha n$ in `CleverBreaker`'s graph at round $t - \alpha n$. If u is not an early candidate then it got its αn-th edge and hence also its $2\alpha n$-th edge after round $t-(1-\alpha)n$, so it is a late candidate.

Note that only good vertices can become candidates and once a vertex becomes candidate it stays that way till the end. This also means in particular that *every* bad vertex is *also* a candidate.

Let us now fix an integer $k \geq 1$. In most definitions and statements that follow k appears as a parameter, but we will suppress it if this creates no confusion. Let us define an auxiliary digraph $D_k = D$ which is built throughout the first t rounds of the game on the vertex set $[n]$ of the K_n the game is played on. For this, we imagine that `RandomMaker` occupies the a edges within each of his turn one after another, so we can talk, without ambiguity, about an edge being occupied *before* another. The digraph D has no edges at the beginning of the game. During the game we add edges to D in the following two scenarios:

(1) whenever a good vertex u becomes (early or late) candidate at some turn of `CleverBreaker`, we immediately add to D up to k arbitrary arcs (u, v), such that uv is occupied by `RandomMaker` already, the vertex v is not a candidate, and $d_D^-(v) = 0$. If there are less than k such edges incident to u we add them all.

(2) whenever `RandomMaker` occupies an edge uv in the game where the vertex u is a candidate, we add the arc (u, v) to D if the vertex v is not a candidate, $d_D^+(u) < k$, and $d_D^-(v) = 0$.

We call the edges of D *saviour edges*. At any point of the game an arc (u, v) is called a *potential savior edge for* u if the edge uv is unoccupied in the game, the vertex u turned candidate already, the vertex v did not, $d_D^+(u) < k$, and $d_D^-(v) = 0$.

Lemma 3.6. *For the maximum in- and out-degree of D we have $\Delta^-(D) \leq 1$ and $\Delta^+(D) \leq k$. The underlying graph of D is an acyclic subgraph of Maker's graph.*

Proof. The bounds on the maximum in- and out-degree immediately follow from the rules in (1) and (2), as it does that the underlying graph is a subgraph of Maker's graph. For acyclicity it is enough to check that at the time an arc (u, v) is added to D, the tail vertex v is isolated in D. For this first note that $d_{D_k}^-(v) = 0$, so v has no incoming arc. For v to have some out-going arc (v, w) in D, v has to be a candidate already. But adding the arc (u, v) requires that v did not turn candidate yet, a contradiction. Hence v was isolated before the addition of (u, v). $\square$

In the remainder of this section we show that a.a.s. every bad vertex has out-degree k in D.

Lemma 3.7. *For every vertex $u \in [n]$, the following holds:*

(i) If u is an early candidate, then $(1-3\alpha)n$ rounds after it turned candidate, $d_D^+(u) = k$ with probability $1 - o(\ln^{-1} n)$.

(ii) If u is a late candidate, then αn rounds after it turned candidate, $d_D^+(u) = k$ with probability $1 - o(1)$.

Proof. Let u be a vertex which turns candidate in round t_u and assume that $d_D^+(u) < k$ at that point (otherwise we are done). Let $\ell := k - d_D^+(u)$ be the number of saviour edges u must still collect. For $i = 1, \ldots, (1-3\alpha)n$ let E_i be the event that no saviour edge from u is added to D at turn $t_u + i$ of `Random-Maker`. If at least ℓ of the events E_i do not hold, then u has out-degree k.
We are interested in the probability

$$p_i := Pr\left[E_i \;\middle|\; \text{there are less than } \ell \text{ rounds } t_u + j,\ j < i, \text{ s.t. } E_j \text{ holds}\right].$$

How many potential saviour out-edges are there for u? Since u turns candidate in round t_u and `CleverBreaker` claims at most one incident edge per turn, by round $t_u + i$ `CleverBreaker` has claimed at most $2\alpha n + i$ edges incident to u (this holds for both early and late candidates). There are at most $2t/\alpha n = \ln n$ vertices that turned candidate before round t and each of these might have at most k outneigbors. These are at most $(k+1)\ln n$ further vertices to where there is no potential saviour edge from u. Thus, there are at least $n-1-(k-1)-(2\alpha n+i)-(k+1)\ln n \geq (1-3\alpha)n - i$ potential k-savior edges from u. Hence every edge `RandomMaker` claims in round $t_u + i$ has probability at least $\frac{(1-3\alpha)n-i}{\binom{n}{2}}$ to be a savior edge for u. This implies that p_i, the probability that `RandomMaker` does not claim any savior edge in his a moves this round can be estimated as

$$p_i \leq \left(1 - \frac{(1-3\alpha)n-i}{\binom{n}{2}}\right)^a < e^{-\frac{2a}{n}\left(1-3\alpha-\frac{i}{n}\right)}.$$

Conversely, there are at least $\binom{n}{2} - (a+1)t = \binom{n}{2} - o(n^2)$ free edges in total, therefore

$$p_i \geq \left(1 - \frac{n-1}{\binom{n}{2} - o(n^2)}\right)^a = \left(1 - \frac{2+o(1)}{n}\right)^a > e^{-\frac{2a}{n}(1+o(1))}.$$

We now consider the first Cn rounds after t_u, for a constant $0 < C < 1$ (for (i) we choose $C = 1-3\alpha$, for (ii) we choose $C = \alpha$). Fix now an integer j, $0 \leq j \leq \ell - 1$ and let q_j be the probability that there are exactly j rounds where a saviour edge from u is occupied by `RandomMaker`. Then the probability that $d^+_D(u) < k$ after round $t + Cn$ is $\sum_{j=0}^{\ell-1} q_j$.

We classify these bad events according to the set $J \in \binom{[Cn]}{j}$ for which a saviour edge for u was occupied by `RandomMaker` exactly in rounds $t_u + h$, $h \in J$, and apply the union bound:

$$\begin{aligned}
q_j \leq & \sum_{J \in \binom{[Cn]}{j}} \prod_{h \in J} (1 - p_h) \prod_{i \in [Cn] \setminus J} p_i \\
\leq & \binom{Cn}{j} \left(1 - e^{-\frac{2a}{n}(1+o(1))}\right)^j \prod_{i \in [Cn]\setminus[j]} e^{-\frac{2a}{n}\left(1-3\alpha-\frac{i}{n}\right)} \\
\leq & n^j \left(\frac{2a}{n}(1+o(1))\right)^j e^{-\frac{2a}{n}\left((1-3\alpha)(Cn-j) - \sum_{i\in[Cn]\setminus[j]} \frac{i}{n}\right)} \\
\leq & O((\ln\ln n)^j) e^{-2a\left(C(1-3\alpha) - \frac{C^2}{2} + o(1)\right)}.
\end{aligned}$$

In the second line we use that $e^{-\frac{2a}{n}\left(1-3\alpha-\frac{i}{n}\right)}$ is monotone increasing in i. For (i), we choose $C = 1-3\alpha$ and obtain

$$\begin{aligned}
q_j &\leq O\left((\ln\ln n)^j \, e^{-a\left((1-3\alpha)^2 + o(1)\right)}\right) \\
&\leq O\left((\ln\ln n)^j \ln^{-(1+\epsilon)\left((1-3\alpha)^2+o(1)\right)} n\right) \\
&= O\left(\ln^{-1-\frac{\epsilon}{2}} n\right).
\end{aligned}$$

For (ii), we choose $C = \alpha$ and obtain

$$q_j \leq O\left((\ln\ln n)^j \ln^{-2(1+\epsilon)\alpha(1-\frac{7\alpha}{2})+o(1)} n\right)$$
$$= o(1).$$

As $\ell < k$ is constant, summing over these estimates for $j = 0, 1 \ldots, \ell - 1$ gives the result in both cases (i) and (ii). □

Corollary 3.8. *For every $\epsilon > 0$ there exists an $\alpha > 0$, such that for every k and every strategy S of* `CleverBreaker` *the following holds a.a.s. In the $(a : 1)$-biased* `RandomMaker-CleverBreaker` *game with $a = (1 + \epsilon)\ln\ln n$ and* `CleverBreaker` *playing with strategy S, we have $d^+_{D_k}(u) = k$ for every α-bad vertex u by the end of round t.*

Proof. Recall that every bad vertex is an early or late candidate.

By Lemma 3.7(i) the probability that any early candidate vertex does not have out-degree k in D_k by round $t - 2\alpha n$ is $o(\ln^{-1} n)$. Since there are at most $\frac{2t}{\alpha n} = \ln n$ early candidates, the union bound gives that a.a.s. all early candidates have out-degree k by round t.

By Lemma 3.7(ii) the probability that a late candidate vertex does not have out-degree k in D_k by round t is $o(1)$. Now we claim that the number of late candidate vertices is $O(1)$ and hence applying the union bound again we get that they all have out-degree k by round t a.a.s. Indeed, since each late candidate has at most degree αn in `CleverBreaker`'s graph at round $t - (1 - \alpha)n$, `CleverBreaker` needed to claim at least αn incident edges at each late candidate in the next $(1 - 2\alpha)n$ rounds. Thus, there can be at most $2\frac{(1-2\alpha)}{\alpha} = O(1)$ late candidate vertices, as promised. □

Corollary 3.9. *For every $\epsilon > 0$ there exists $\alpha > 0$, such that for every strategy S of* `CleverBreaker` *the following holds a.a.s. In the $(a : 1)$-biased* `RandomMaker-CleverBreaker` *game with $a = (1 + \epsilon)\ln\ln n$ and* `CleverBreaker` *playing with strategy S, there are vertex-disjoint paths $P_1, \ldots, P_\kappa$ in* `RandomMaker`*'s graph that cover all α-bad vertices and have their start- and endpoints, and only these, among the α-good vertices.*

Proof. Let us use Corollary 3.8 with $k = 2$, so we can assume that every bad vertex has out-degree 2 in D_2. We start at an arbitrary bad vertex v having no incoming edge (such a vertex exists, since D_2 is acyclic by Lemma 3.6). We follow both of its outgoing edges in D_2 to create two vertex disjoint directed paths from v. If we reach a good vertex we stop and choose it as the endpoint of our path. Otherwise, i.e., if the reached vertex v' is bad, it has out-degree 2 in D_2, and we continue along one of the out-going edges. Since D_2 is acyclic and the number of bad vertices is finite, we must reach a good vertex eventually. Once both directed paths from v are completed, their union in the underlying undirected graph of `RandomMaker` forms a path P_1 with good endpoints and bad interior vertices. We remove the vertices of P_1 from D_2 and continue iteratively with a bad vertex that does not have an incoming edge, until there are no bad vertices left. Note that, crucially, after the iterative removal of such rooted paths, all remaining vertices still have all their out-going edges, hence all remaining bad vertices still have out-degree 2. Indeed, all vertices have in-degree at most 1 and those with in-degree exactly 1 that were removed also had their ancestor removed. □

3.2.2 On the good vertices

Now that we have "anchored" the bad vertices, let us turn to the good vertices. We show that the graph spanned by them is close enough to a truly random graph and make use of the strong expander properties of the latter. To make these notions more precise, we switch to the point of view, where `RandomMaker`'s turns are determined by a random permutation σ.

We consider the first at random edges of σ which surely were all "tried" to be played by `RandomMaker` in the first t rounds. However he might not actually own all of these, because `CleverBreaker` might have taken some of them by the time they were tried by `RandomMaker`. In the greatest generality, to be able to do multi-round exposure later, we consider subsets $M \subset [at]$ of coordinates of σ and we will be interested in the truly random graph $G_\sigma(M) = G(M)$ that consist of the edges exactly at these coordinates, that is,

$$E(G(M)) = \{\sigma(m) : m \in M\}.$$

Note that the notion of $G_\sigma([i])$ coincides with the notion of $G_\sigma(i)$ defined earlier.

We define now a set of edges that will be "forbidden" for our analysis. Recall that we fixed a strategy S for `CleverBreaker`. Let $H_{\sigma,S}(M) = H(M)$ be the graph defined on the vertex set $[n]$ containing those edges uv for which $uv \in \sigma(M)$ and for both u and v the edge uv was among the first $3\alpha n$ incident edges which `CleverBreaker`, playing according to S, claimed in the first t rounds, when the permutation game according to σ was played.

The crucial point of this definition is the following simple lemma:

Lemma 3.10. *Let σ be an arbitrary permutation of the edges of K_n. Then for every subset $M \subseteq [at]$ the graph $G(M) - E(H(M)) - B$, with B being the set of α-bad vertices after t rounds, is a subgraph of `RandomMaker`'s graph.*

Proof. Let uv be an edge of $G(M) - E(H(M)) - B$. Then u and v are both good vertices after t rounds and hence `CleverBreaker`'s degree at both of them is at most $3\alpha n$. Thus, since $uv \in \sigma(M)$, if uv would have been claimed by `CleverBreaker` up to round t then uv would be in $E(H(M))$. Consequently the edge uv was not claimed by `CleverBreaker` in the first t rounds. Now, since $uv \in \sigma(M) \subseteq \sigma([at])$ and `RandomMaker` did try to claim the first at least at edges of σ in the first t rounds, he must have claimed uv by that time. □

The following lemma ensures that not too many edges of cuts $(X, \overline{X}) := \{xy : x \in X, y \in V \setminus X\}$ of $G([at])$ are "blocked" by `CleverBreaker` as one of its first $3\alpha n$ edges at the endpoints. In particular, every vertex has small degree in $H([at])$.

Lemma 3.11. *The following is true a.a.s. For every subset $X \subseteq [n]$, we have that*

$$\left|E(H([at])) \cap (X, \overline{X})\right| \leq \frac{8e\alpha at|X|}{n}.$$

Proof. We write $H = H([at])$. We create a random permutation σ coordinatewise. The crucial observation is that whether $\sigma(j) \in E(H)$ for some $j \in [at]$ depends only on the initial segment of the first $j-1$ edges of σ. Indeed, for $\sigma(j)$ to be in $E(H)$, we need that at both of its endpoints $\sigma(j)$ is one of the first $3\alpha n$ incident edges which `CleverBreaker` claims when Maker plays according to σ. After Maker swiped through the first $j-1$ edges of σ, two things can happen: either $\sigma(j)$ was taken by `CleverBreaker` in the game and hence it was already decided whether it is one of the first $3\alpha n$ `CleverBreaker`-edges at both of its endpoints. If $\sigma(j)$ was not taken in the game, then Maker takes it in its next move and hence $\sigma(j)$ will not become part of H later either.

Hence, conditioning on any initial segment $\pi \in S^{j-1}_{E(K_n)}$, the probability that the next edge $\sigma(j)$ is in $E(H) \cap \left(X, \overline{X}\right)$ depends only on whether it is one of the at most $3\alpha n|X|$ edges that are already in $H([i-1])$ and go between X and its complement. Furthermore, given that σ starts with π, $\sigma(j)$ can take at least $\binom{n}{2} - at$ different values, each equally likely. Thus,

$$Pr\left[\sigma(j) \in E(H) \cap \left(X, \overline{X}\right) \;\middle|\; \sigma|_{[j-1]} = \pi\right] \leq \frac{3\alpha n|X|}{\binom{n}{2} - at} \leq \frac{7\alpha|X|}{n},$$

for large n. For our main estimate we can classify according to the set L of coordinates where the corresponding edges of σ are from $E(H) \cap \left(X, \overline{X}\right)$ and apply the union bound:

$$\begin{aligned}
Pr\left[E(H) \cap \left(X, \overline{X}\right) \geq \frac{8e\alpha at|X|}{n}\right] &\leq \sum_{\substack{L \subset [at], \\ |L| = \frac{8e\alpha at|X|}{n}}} Pr\left[\forall j \in L : \sigma(j) \in E(H) \cap \left(X, \overline{X}\right)\right] \\
&\leq \binom{at}{\frac{8e\alpha at|X|}{n}} \left(\frac{7\alpha|X|}{n}\right)^{\frac{8e\alpha at|X|}{n}} \\
&\leq \left(\frac{eat}{\frac{8e\alpha at|X|}{n}}\right)^{\frac{8e\alpha at|X|}{n}} \left(\frac{7\alpha|X|}{n}\right)^{\frac{8e\alpha at|X|}{n}} \\
&= \left(\frac{7}{8}\right)^{\frac{8e\alpha at|X|}{n}}.
\end{aligned}$$

Taking the union bound over all cuts $(X, \overline{X})$, we see that

$$\sum_{s=1}^{n/2} \binom{n}{s} \left(\frac{7}{8}\right)^{\frac{8e\alpha ats}{n}} \leq \sum_{s=1}^{n/2} \left(n \left(\frac{7}{8}\right)^{4e\alpha^2 \ln n \ln \ln n}\right)^s = o(1).$$

□

We also need the following standard fact from random graph theory; for completeness we include a proof in the Appendix.

Lemma 3.12. *Let $\delta > 0$. The following holds a.a.s in a random graph $G \sim G(n,m)$ with $m = \delta n \ln n \ln \ln n$. For every vertex set $X \subset [n]$ of size $|X| \le \frac{n}{2}$, we have*

$$E(G) \cap \left(X, \overline{X}\right) \ge |X| \frac{m}{2n}.$$

3.2.3 `CleverMaker` builds a connected graph and achieves a large minimum degree

We now have all the necessary tools to conclude the theorems about the min-degree k game and the connectivity game.

Proof of Theorems 1.3 and 1.4. By Theorem 3.1 here we need to take care of the upper bounds only.

Let $\alpha < \frac{1}{32e}$ be arbitrary such that (2) is satisfied. Define $\delta = (1+\epsilon)\frac{\alpha}{2}$, so $at = m = \delta n \ln n \ln \ln n$. We show that by round t `RandomMaker`'s graph is connected and has minimum degree at least k a.a.s.

Recall that by Corollary 3.8 and Lemma 3.6 all bad vertices have degree at least k in `RandomMaker`'s graph by round t a.a.s. Moreover, by Corollary 3.9 and Lemma 3.6 every bad vertex is connected to some good vertex via a path in `RandomMaker`'s graph a.a.s.

It is enough to show that `RandomMaker`'s graph induced by the set of good vertices is connected and has minimum degree at least k. We will use Lemma 3.10.

Let $X \subseteq [n]$ be an arbitrary subset of good vertices, of size $|X| \le \frac{n}{2}$. By Lemma 3.12 there are at least $|X|\frac{at}{2n}$ edges in $G([at])$ between X and $\overline{X}$. At most $\frac{8e\alpha at|X|}{n}$ of these $\frac{at|X|}{2n}$ edges are in $H([at])$ by Lemma 3.11, and at most another $|X| \ln n$ of them are going to an α-bad vertex (since there are at most $\ln n$ bad vertices).

The rest of these edges is in `RandomMaker`'s graph by Lemma 3.10. That means that at least $\left(\left(\frac{1}{2} - 8e\alpha\right)\frac{at}{n} - \ln n\right)|X| = \Omega(|X| \ln n \ln \ln n) \ge k$ edges of `RandomMaker`'s graph leave X to its complement among the good vertices. In particular, each good vertex v has degree at least k in `RandomMaker`'s graph. □

3.3 `RandomMaker` builds a Hamilton cycle

We now turn to the Hamiltonicity game. The plan is the following: We use Corollary 3.9 to find paths covering the bad vertices. Then we connect them to one long path, using short paths on the good vertices. Finally, we show that the rest is Hamilton connected, which allows us to close the loop using all remaining vertices. To find the short paths and prove Hamilton connectivity, we turn away from the game for a while, and look at random graphs in general.

3.3.1 Short Paths

The following precise notion of expansion from [18] will be central to our proofs. Here $N(X)$ denotes the set of vertices which have a neighbour in X.

Definition 3.13. Let λ and r be positive reals. A graph G is a *half-expander with parameters* λ *and* r if the following properties hold:

1. For every set X of vertices of size $|X| \leq \frac{\lambda n}{r}$, $|N(X)| \geq r\,|X|$,
2. for every set X of vertices of size $|X| \geq \frac{n}{\lambda r}$, $|N(X)| \geq \left(\frac{1}{2} - \lambda\right) n$, and
3. for every pair of disjoint sets X,Y such that $|X|, |Y| \geq \left(\frac{1}{2} - \lambda^{1/5}\right) n$, $e(X, Y) > 2n$.

The following tail estimates for the *hypergeometric distribution* will be very convenient. Let F, f and l be positive integers such that $f, l \leq F$. The value of the random variable X is the size of the intersection of fixed f-element subset $M \subseteq [F]$ with a uniformly chosen l-subset M^*. Note that the expected value of X is $\frac{fl}{F}$. For the following standard estimates see e.g. [15] Theorem 2.10.

Theorem 3.14. *Let X have the hypergeometric distribution with parameters F, f and l. Then*

$$Pr\left[X \geq 2\frac{fl}{F}\right] \leq e^{-\frac{fl}{3F}}, \tag{3}$$

$$Pr\left[X \leq \frac{fl}{2F}\right] \leq e^{-\frac{fl}{8F}}. \tag{4}$$

We will use the theorem to estimate how many edges of a "good" edge set of size f are realized in $G(n, m)$.

The following useful properties of the random graph are consequences of Theorem 3.14; a proof is included in the Appendix.

Lemma 3.15. *Let $\delta > 0$. The following three properties hold with probability at least $1 - e^{-\Omega(\ln n \ln \ln n)}$ in a random graph $G \sim G(n, m)$ with $m = \delta n \ln n \ln \ln n$.*

(a) Every vertex set X of size at most $|X| \leq \frac{n^2}{m}$ has a neighborhood of size at least $|N_G(X)| \geq |X| \frac{m}{8n}$.

(b) for every pair of vertex sets $X \subset [n]$ of size $\frac{n}{4} \geq |X| \geq \frac{64n^2}{m}$, and $N \subset [n]$ of size $|N| \leq \frac{n}{2}$ there are at least $|X| \frac{m}{8n}$ edges between X and $[n] \setminus (X \cup N)$ in G.

(c) for every pair of disjoint vertex sets $X, Y \subset [n]$ of size at least $|X|, |Y| \geq \frac{n}{4}$ there are at least $m\,|X|/8n$ edges between X and Y in G.

First we show that random graphs are half-expanders, with some resilience to edge and vertex removal. This will be useful in particular with respect to Lemma 3.11. We state the lemma in a bit more general form than is need in this section in order to provide us with some leeway later.

Lemma 3.16. *Let $0 < \lambda < 2^{-11}$, $\delta > 0$, and let $\mathcal{D} \subseteq \binom{[n]}{\le \ln^2 n}$ be a family of $n^{3\ln n}$ vertex subsets such that each set $D \in \mathcal{D}$ has size at most $|D| \le \ln^2 n$. Then in a random graph $G \sim G(n,m)$ with $m = \delta n \ln n \ln\ln n$ the following holds with probability at least $1 - e^{-\Omega(\ln n \ln\ln n)}$: for all $D \in \mathcal{D}$ and all graphs $H \subset K_n$ with maximum degree at most $\Delta(H) \le \frac{m}{32n}$, the graph $G - E(H) - D$ is a half-expander with parameters λ and $r = \frac{m}{16n} = \frac{\delta}{16} \ln n \ln\ln n$.*

Proof. We first show the following.

Claim. *With probability at least $1 - e^{-\Omega(\ln^2 n \ln\ln n)}$, for all $D \in \mathcal{D}$ and all $v \in [n]$, v has at most $\frac{m}{32n}$ G-neighbors in D.*

Proof. For a fixed vertex v, set $D \in \mathcal{D}$ and subset $Q \subseteq D$ of size $|Q| = q = \frac{m}{32n}$, the probability that all vertices in Q are G-neighbors of v is $\frac{\binom{N-q}{m-q}}{\binom{N}{m}} = \prod_{i=0}^{q-1} \frac{m-i}{N-i} \le \left(\frac{m}{N}\right)^q$, where $N = \binom{n}{2}$.
Taking the union bound over all v, D, and Q, yields that the failure probability of the event in the claim is at most

$$n|\mathcal{D}| \binom{|D|}{q} \left(\frac{m}{N}\right)^q \le n \cdot n^{3\ln n} \left(\frac{200\ln^2 n}{n}\right)^q = n^{-\Omega(\ln n \ln\ln n)}$$

and the claim is proved. ◻

Since the events in the claim and Lemma 3.15 hold with probability at least $1 - e^{-\Omega(\ln n \ln\ln n)}$, it is enough to show that they imply the event in our lemma.

Let $D \in \mathcal{D}$ be an arbitrary set from the family $\mathcal{D}$ and let H be an arbitrary graph with maximum degree $\Delta(H) \le \frac{m}{32n}$. First note that by the property of the claim, removing D and $E(H)$ from G removes at most $\frac{m}{16n}$ incident edges at any vertex in $V(G) \setminus D$.

To show the first property of Definition 3.13, fix $X \subset V(G) \setminus D$ such that $|X| \le \frac{\lambda(n-|D|)}{r}$. Note that then $|X| \le \frac{n^2}{m}$, so by Lemma 3.15(a) the neighborhood of X in G has size at least $|N_G(X)| \ge |X| \frac{m}{8n}$. Removing D and $E(H)$ eliminates at most $\frac{|X|m}{16n}$ edges incident to X, which means that after the removal, the neighborhood of X has size at least $\frac{|X|m}{16n} = r|X|$.

For the second property, let us fix a set X of size $|X| = \frac{n-|D|}{\lambda r}$. Note that $\frac{64n^2}{m} \le |X| \le \frac{n}{4}$. Assume that the neighborhood N of X in $G - E(H) - D$ has size less than $(\frac{1}{2} - \lambda)(n - |D|)$. Then by the property in Lemma 3.15(b), there are at least $|X| \frac{m}{8n}$ edges between X and $[n] \setminus (X \cup N)$ in G. Removing D and $E(H)$ removes at most $|X| \frac{m}{16n}$ edges. Thus, there is an edge from X to outside of N in $G - E(H) - D$, a contradiction to the definition of N.

For the third property, fix two disjoint vertex sets $X, Y \subset [n] \setminus D$ of size at least $\left(\frac{1}{2} - \lambda^{1/5}\right)(n - |D|)$. Note that $|X|, |Y| \ge \frac{n}{4}$ for n large enough, so we can apply Lemma 3.15(c) and conclude that there are at least $\frac{m|X|}{8n}$ edges between X and Y in G. Therefore, at least $\left(\frac{1}{8} - \frac{1}{16}\right)|X| \frac{m}{n} \ge 2n$ edges remain after removing D and $E(H)$, for n sufficiently large. ◻

Since we now know we are working with a half-expander, we can do the first step towards Hamiltonicity by connecting vertices with short paths.

Theorem 3.17. *There is a $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$, the following holds: Let G be a half-expander on n vertices with parameters λ and $r \geq \frac{8}{\lambda^2} \ln n$, and let $k \leq \ln n$. Then for all pairwise distinct points $a_1, \ldots, a_k, b_1, \ldots, b_k$, there are vertex disjoint paths $P_1, \ldots, P_k$, each of length at most $\ln n$, such that P_i connects a_i to b_i.*

Proof. We build the paths simultaneously, starting at both ends and keeping sets of possible vertices at the different positions in the paths, from which we then can choose to connect the two partial paths we built. Throughout the proof, let $q := \frac{r}{8 \ln n} \geq \frac{1}{\lambda^2}$. Note that $q \geq 2$ for λ_0 sufficiently small.
Let $j_0 = \left\lceil \frac{\ln \frac{\lambda n}{r}}{\ln q} \right\rceil$. For $0 \leq j \leq j_0 + 1$ and $1 \leq i \leq k$ we will define vertex sets $D^+_{i,j}$ and $D^-_{i,j}$, such that $D^+_{i,0} = \{a_i\}$ and $D^-_{i,0} = \{b_i\}$, all the $2(j_0+2)k$ sets $D^+_{i,j}$ and $D^-_{i,j}$ are pairwise disjoint, for every i, j we have $D^+_{i,j} \subseteq N\left(D^+_{i,j-1}\right)$ and $D^-_{i,j} \subseteq N\left(D^-_{i,j-1}\right)$, and $\left|D^-_{i,j}\right| = \left|D^+_{i,j}\right| = f(j)$ where

$$f(j) = \begin{cases} q^j & \text{if } j < j_0 \\ \frac{\lambda n}{r} & \text{if } j = j_0 \\ \frac{n}{\lambda r} & \text{if } j = j_0 + 1. \end{cases}$$

We define the sets iteratively over j, where in each step, we iterate over i.
First, let $D^+_{i,0} := \{a_i\}$ and $D^-_{i,0} := \{b_i\}$ for $i = 1, \ldots, k$.
Now let us fix $1 \leq j \leq j_0 + 1$ and $1 \leq i \leq k$, and assume that for all $j' < j$ and all $1 \leq i' \leq i$, the sets $D^+_{i',j'}$ and $D^-_{i',j'}$ are constructed, and for all $i'' < i$, the sets $D^+_{i'',j}$ and $D^-_{i'',j}$ are constructed.
We first define

$$A^{\pm}_{i,j} := N\left(D^{\pm}_{i,j-1}\right) \setminus \left(\left(\bigcup_{i''<i} D^+_{i'',j} \cup D^-_{i'',j}\right) \cup \left(\bigcup_{1 \leq i' \leq k, j' < j} D^+_{i',j'} \cup D^-_{i',j'}\right)\right).$$

We show that we can find the $D^{\pm}_{i,j} \subseteq A^{\pm}_{i,j}$ with the required properties by proving

$$\left|A^{\pm}_{i,j}\right| \geq f(j).$$

Let us first consider the case $j \leq j_0$. Then for all $j' < j$ and $1 \leq i' \leq k$, we have $\left|D^{\pm}_{i',j'}\right| = f(j') = q^{j'}$. Further, since G is a half-expander and $\left|D^{\pm}_{i,j-1}\right| \leq \frac{\lambda n}{r}$, we have $\left|N\left(D^{\pm}_{i,j-1}\right)\right| \geq r \left|D^{\pm}_{i,j-1}\right| = (8 \ln n) q^j$. Finally note that $\left|D^{\pm}_{i'',j}\right| \leq q^j$ for all $i'' < i$ (this holds also if $j = j_0$, since $f(j_0) = \frac{\lambda n}{r} \leq q^{j_0}$).

Therefore,

$$\begin{aligned}\left|A^{\pm}_{i,j}\right| &\geq \left|N(D^{\pm}_{i,j-1})\right| - \sum_{i''=1}^{i-1}\left|D^{+}_{i'',j}\cup D^{-}_{i'',j}\right| - \sum_{j'=0}^{j-1}\sum_{i'=1}^{k}\left|D^{+}_{i',j'}\cup D^{-}_{i',j'}\right| \\ &\geq (8\ln n)q^{j} - (\ln n - 1)(2q^{j}) - \ln n\sum_{j'=0}^{j-1}2q^{j'} \\ &\geq (4\ln n + 2)q^{j} \geq q^{j} \geq f(j)\end{aligned}$$

where we used that $i \leq k \leq \ln n$, and $\sum_{j'=0}^{j-1} q^{j'} \leq q^{j}$ as $q \geq 2$.

In the case $j = j_0 + 1$, note that

$$\left|\bigcup_{1\leq i'\leq k} D^{+}_{i',j_0}\cup D^{-}_{i',j_0}\right| = 2k\frac{\lambda n}{r} \tag{5}$$

and

$$\left|\bigcup_{1\leq i'\leq k, j'<j_0} D^{+}_{i',j'}\cup D^{-}_{i',j'}\right| = 2k\sum_{j'=0}^{j_0-1} q^{j'} \leq 4kq^{j_0-1} \leq 4k\frac{\lambda n}{r}, \tag{6}$$

by the definition of j_0. Again using the half-expander property of G, we have that

$$\begin{aligned}\left|A^{\pm}_{i,j_0+1}\right| &\geq r\frac{\lambda n}{r} - 2(i-1)\frac{n}{\lambda r} - \left(2k\frac{\lambda n}{r} + 4k\frac{\lambda n}{r}\right) \\ &\geq \left(\lambda^2 r - 2(\ln n - 1) - 6\lambda^2\ln n\right)\frac{n}{\lambda r} \\ &\geq \frac{n}{\lambda r}\end{aligned}$$

using that $r = 8q\ln n$, $q \geq \frac{1}{\lambda^2}$ and $i \leq k \leq \ln n$. This concludes the proof that we can construct the sets $D^{\pm}_{i,j}$ with the properties described above. We now find paths for all i, using the $D^{\pm}_{i,j}$.

Suppose we have constructed appropriate paths $P_1, \ldots, P_{i-1}$ already. To construct P_1, let us first define

$$D^{\pm}_{i,j_0+2} = N\left(D^{\pm}_{i,j_0+1}\right) \setminus \left(\bigcup_{\substack{1\leq i'\leq k\\ 0\leq j\leq j_0+1}}\left(D^{+}_{i',j}\cup D^{-}_{i',j}\right) \cup \bigcup_{i''=1}^{i-1} V(P_{i''})\right).$$

Since $|D^{\pm}_{i,j_0+1}| = \frac{n}{\lambda r}$ we can use the second half-expander property for G. This, together with the estimates (5), (6), and $\frac{n}{r} \leq \frac{\lambda^2 n}{8\ln n}$ implies

$$\left|D^{\pm}_{i,j_0+2}\right| \geq \left(\frac{1}{2} - \lambda\right)n - \ln n\left(2\frac{n}{\lambda r} + 2\frac{\lambda n}{r} + 4\frac{\lambda n}{r} + \ln n\right) \geq \left(\frac{1}{2} - \lambda^{1/5}\right)n,$$

for λ small enough. If the sets D^+_{i,j_0+2} and D^-_{i,j_0+2} are disjoint, then using the third half-expander property we can conclude that there is an edge e between them. Retracing a path from each endpoint of e through the $D^{\pm}_{i,j}$ back to $D^+_{i,0} = \{a_i\}$ and $D^-_{i,0} = \{b_i\}$, respectively, and concatenating them with e gives us the required a_i, b_i-path P_i. The length of P_i then is $j_0 + 3 \leq \ln n$, indeed. If D^+_{i,j_0+2} and D^-_{i,j_0+2} are not disjoint, we can trace back a path to a_i and b_i from any vertex in the intersection and then P_i is of length $j_0 + 2$. $\square$

The next corollary is a direct consequence of Lemma 3.16 and Theorem 3.17.

Corollary 3.18. *Let $\delta > 0$. The following holds in a random graph $G \sim G(n,m)$ with $m = \delta n \ln n \ln \ln n$ a.a.s. For all vertex sets B with $|B| \leq \ln n$, all sequences of pairwise distinct points $a_1, \ldots, a_k, b_1, \ldots, b_k \in V \setminus B$, $k \leq \ln n$, and all graphs H with $\Delta(H) \leq \frac{m}{32n}$, there are vertex disjoint paths $P_1, \ldots, P_{k-1}$ in $G - E(H) - B - \{a_1, b_k\}$, each of length at most $\ln n$, such that P_i connects a_{i+1} to b_i.*

Proof. Let $\lambda > 0$ be small enough such that Lemma 3.16 and Theorem 3.17 both hold. Further let $\mathcal{D}$ be the family of all vertex sets $B \cup \{a_1, b_k\}$ where $|B| \leq \ln n$. Note that $|\mathcal{D}| = \binom{n}{\ln n} \leq n^{\ln n}$. Then by Lemma 3.16, a.a.s. for every $B \in \mathcal{D}$ and every graph $H \subset K_n$ with $\Delta(H) \leq \frac{m}{32n}$, the graph $G - E(H) - B - \{a_1, b_k\}$ is a half-expander with parameters λ and $r = \frac{m}{16n} = \frac{\delta}{16} \ln n \ln \ln n \geq \frac{8}{\lambda^2} \ln n$. Applying Theorem 3.17 to these graphs concludes the proof. $\square$

3.3.2 Hamilton Connectivity

We now turn towards Hamilton connectivity. This section relies heavily on the works of Lee and Sudakov [20] and Krivelevich, Lee, and Sudakov [18]. The following properties prove to be a valuable criterion for Hamiltonicity.

Definition 3.19. Let ξ be a positive constant. We say that a graph G has property $\mathcal{RE}(\xi)$ if it is connected, and for every path P with a fixed edge e, (i) there exists a path containing e longer than P in the graph $G \cup P$, or (ii) there exists a set of vertices S_P of size $|S_P| \geq \xi n$ such that for every vertex $v \in S_P$, there exists a set T_v of size $|T_v| \geq \xi n$ such that for every $w \in T_v$, there exists a path containing e of the same length as P that starts at v, and ends at w.

Definition 3.20. Let ξ be a positive constant and let G_1 be a graph with property $\mathcal{RE}(\xi)$. We say that a graph G_2 *complements* G_1, if for every path P with a fixed edge e, (i) there exists a path containing e longer than P in the graph $G_1 \cup P$, or (ii) there exist $v \in S_P$ and $w \in T_v$, such that $\{v, w\}$ is an edge of $G_1 \cup G_2 \cup P$ (the sets S_P and T_v are as defined in Definition3.19).

Proposition 3.21 ([18, Proposition 3.3])**.** *Let ξ be a positive constant. If $G_1 \in \mathcal{RE}(\xi)$ and G_2 complements G_1, then $G_1 \cup G_2$ is Hamilton connected.*

Again, the notion of a half-expander comes in useful.

Lemma 3.22 ([18, Lemma 3.5])**.** *There exists a positive λ_0 such that for every positive $\lambda \leq \lambda_0$, the following holds for every $r \geq 16\lambda^{-3}\ln n$: every half-expander on n vertices with parameters λ and r has property $\mathcal{RE}\left(\frac{1}{2}+\lambda\right)$.*

The next lemma and its proof are based on [20] and adapted to our situation.

Lemma 3.23. *For all $0 < \lambda \leq 1/2$ there is a $\beta > 0$ such that for all $\delta > 0$, for a random graph $G \sim G(n, m)$ with $m = \delta n \ln n \ln\ln n$ edges, the following holds with probability at least $1 - e^{-\Omega(m)}$:*
For every graph H with maximum degree $\Delta(H) \leq \frac{m}{8n}$, the graph $G - E(H)$ complements every subgraph $R \subseteq G$ with property $\mathcal{RE}(\frac{1}{2}+\lambda)$ that has at most βm edges.

Proof. Let us fix a graph $R \subseteq K_n$ with at most βm edges such that $R \in \mathcal{RE}\left(\frac{1}{2}+\lambda\right)$. We will estimate the probability that $R \subseteq G$ and there exists an H with maximum degree $\Delta(H) \leq \frac{m}{8n}$, such that the graph $G - E(H)$ does not complement R.

For this we fix a path P and an edge $e \in E(P)$ and estimate from above the probability that there exists an H with maximum degree $\Delta(H) \leq \frac{m}{8n}$, such that (i) in $R \cup P$ no path containing e is longer than P and (ii) for every $v \in S_P$ and every $w \in T_v$ we have $vw \notin E((G - E(H)) \cup R \cup P)$ (where S_P is the set of size $|S_P| \geq \left(\frac{1}{2}+\lambda\right)n$ and T_v the set of size $|T_v| \geq \left(\frac{1}{2}+\lambda\right)n$ from Definition 3.19 applied to R). Observe that (i) is not a random statement, hence we can assume that it holds for P and e, otherwise the probability is 0.

Note also that if there exists a vertex $v \in S_P$ and a $w \in T_v$ such that $\{v, w\} \in E(R)$, then the probability is 0 as well. Thus from now on we also assume that for all $v \in S_P$ and all $w \in T_v$, the edge $\{v, w\} \notin E(R)$.

Let now $S'_P = \{v_1, \ldots, v_{\lambda n}\}$ be an arbitrary subset of S_P of size $|S'_P| = \lambda n$. For all $v \in S'_P$, let T'_v be a subset of $T_v \setminus S'_P$ of size $|T'_v| = \frac{1}{2}n$. Note that the edge sets $E'_v = \{\{v, w\} : w \in T'_v\}$ for $v \in S'_P$ are all disjoint, since S'_P is disjoint from every T'_v. Hence their union

$$E' := \{\{v, w\} : v \in S'_P, w \in T'_v\}$$

has size $|E'| = \frac{\lambda}{2}n^2$.

We will show that with high probability, for every H with $\Delta(H) \leq \frac{m}{8n}$, there is a $v \in S'_P$ and a $w \in T'_v$ such that $\{v, w\}$ is an edge of $G - E(H)$. For that, it is sufficient that, independently of H, there are at least $\lambda m/4$ edges in $E(G) \cap E'$. Indeed, removing the edges of any graph H with maximum degree $\Delta(H) \leq \frac{m}{8n}$ can eliminate at most $|S'_P| \frac{m}{8n} \leq \frac{\lambda m}{8}$ edges from $E(G) \cap E'$, which means that at least $\frac{\lambda m}{8} > 0$ edges of E' are left in $(E(G) \setminus E(H)) \cap E'$.

Recall that we assumed that $E(R)$ is disjoint from E', but condition on $E(R) \subseteq E(G)$. Thus, the size of $E(G) \cap E'$ has a hypergeometric distribution with parameters $F = \binom{n}{2} \setminus |E(R)| \leq n^2/2$, $f = |E'| = \lambda n^2/2$ and $l = m - |E(R)| \geq m/2$. Hence for the expectation we have $\frac{fl}{F} \geq \frac{\lambda m}{2}$ and then Theorem 3.14 implies that

$$Pr\left[|E(G) \cap E'| \leq \lambda m/4 \;\middle|\; R \subset G\right] \leq e^{-\lambda m/16}.$$

Taking the union bound for all choices of P and $e \in E(P)$ we obtain that

$$Pr\left[G \text{ does not complement } R \,\middle|\, R \subset G\right] \leq nn!e^{-\lambda m/16}.$$

Finally, taking the union bound for all $R \subseteq K_n$ with at $k \leq \beta m$ edges and using that $Pr\left[R \subset G\right] \leq (\frac{m}{\binom{n}{2}})^k$, we obtain that our failure probability is at most

$$\sum_{R \in \mathcal{RE}(\frac{1}{2}+\lambda), |E(R)| \leq \beta m} Pr\left[G \text{ does not complement } R \,\middle|\, R \subset G\right] Pr\left[R \subset G\right]$$
$$\leq nn!e^{-\lambda m/4} \sum_{k=1}^{\beta m} \binom{\binom{n}{2}}{k} \left(\frac{m}{\binom{n}{2}}\right)^k \leq e^{-\lambda m/5} \sum_{k=1}^{\beta m} \left(\frac{em}{k}\right)^k$$
$$\leq e^{-\lambda m/5} \beta m \left(\frac{e}{\beta}\right)^{\beta m}.$$

Here we used that the terms of the last sum are monotone increasing for $k \leq \beta m$, as long as $\beta < 1$. Thus the event of the lemma fails with probability $e^{-\Omega(m)}$ provided β is sufficiently small. $\square$

The next statement wraps up this section.

Corollary 3.24. *There is a $\gamma > 0$ such that for every $\delta > 0$ and every family $\mathcal{D} \subseteq \binom{[n]}{\leq \ln^2 n}$ of at most $n^{3 \ln n}$ vertex subsets of size at most $\ln^2 n$ each, the following holds with probability at least $1 - e^{-\Omega(\ln n \ln \ln n)}$ for a random graph $G \sim G(n, m)$, with $m = \delta n \ln n \ln \ln n$:*
For every $D \in \mathcal{D}$ and every graph $H \subset K_n$ with $\Delta(H) \leq \gamma \frac{m}{n}$, the graph $G - E(H) - D$, is Hamilton connected.

Proof. Let λ_0 as in Lemma 3.22 and $0 < \lambda < \min\left(\lambda_0, 2^{-11}\right)$. Let $0 < \beta < 1$ such that Lemma 3.23 holds and let $\gamma = \frac{\beta}{32}$. Let $\mathcal{D} \subseteq \binom{[n]}{\leq \ln^2 n}$ be a family of at most $n^{3 \ln n}$ vertex subsets of size at most $\ln^2 n$ each. Then let $G \sim G(n, m)$ be a random graph and let G' be a uniformly random subgraph of G with $\frac{\beta m}{2}$ edges. Let $\mathcal{E}$ be the event that for every $D \in \mathcal{D}$ and $H \subseteq K_n$ with $\Delta(H) \leq \gamma \frac{m}{n}$, the graph $G' - E(H) - D$ is a half-expander with parameters λ and $r = \frac{\beta m}{32n}$. By definition, G' is distributed like $G\left(n, \frac{\beta m}{2}\right)$ and thus by Lemma 3.16, $\mathcal{E}$ holds with probability at least $1 - e^{-\Omega(\ln n \ln \ln n)}$.

We now fix a $D \in \mathcal{D}$. Let $\mathcal{A}_D$ be the event that $G - D$ has at least $m/2$ edges. We show that $\mathcal{A}_D$ fails with probability at most $e^{-\Omega(n)}$. Let $N := \binom{n}{2}$. Note that there are at most $n|D|$ edges incident to D. Then the probability that removing D from G removes at least $k = \frac{1}{2}m$ edges is at most

$$\frac{\binom{n|D|}{k}\binom{N-k}{m-k}}{\binom{N}{m}} \leq \left(\frac{en|D|}{k}\right)^k \left(\frac{m}{N}\right)^k \leq \left(\frac{4e \ln^2 n}{n-1}\right)^k \leq e^{-\Omega(m \ln n)}.$$

From now on we condition on $\mathcal{A}_D$ holding.

Let $\mathcal{B}_D$ be the event that for every $H \subseteq K_{n-|D|}$ with $\Delta(H) \leq \frac{m}{16(n-|D|)}$, $G - E(H) - D$ complements every one of its subgraphs with at most $\frac{\beta m}{2}$ edges that has property $\mathcal{RE}\left(\frac{1}{2}+\lambda\right)$.

Now we condition further on $G-D$ having exactly $k \geq m/2$ edges. Note that then $G-D$ is distributed as $G(n-|D|, k)$. Under this condition then, by Lemma 3.23, $\mathcal{B}_D$ fails with probability at most $e^{-\Omega(m)}$. Since the events $|E(G-D)| = k$, $k \geq m/2$ partition the event $\mathcal{A}_D$, we obtain that $Pr\left[\overline{B_D}|\mathcal{A}_D\right] \leq e^{-\Omega(m)}$. Hence, in total we get that

$$Pr\left[\bigcup_{D\in\mathcal{D}} \overline{B_D}\right] \leq |\mathcal{D}|\left(e^{-\Omega(m)} + e^{-\Omega(m\ln n)}\right) = e^{-\Omega(m)}.$$

Thus, with probability at least $1 - e^{-\Omega(\ln n \ln\ln n)}$ both the event $\mathcal{E}$ and the events $\mathcal{B}_D$ hold for every $D \in \mathcal{D}$.

If $\mathcal{E}$ holds, then by Lemma 3.22, $G' - E(H) - D$ has property $\mathcal{RE}\left(\frac{1}{2}+\lambda\right)$ for every $D \in \mathcal{D}$ and $H \subseteq K_n$ with $\Delta(H) \leq \gamma\frac{m}{n}$.

If $\mathcal{B}_D$ holds, then $G-E(H)-D$ complements the $G'-E(H)-D$, because the latter has property $\mathcal{RE}\left(\frac{1}{2}+\lambda\right)$ and has at most $\frac{\beta m}{2}$ edges. Thus, by Proposition 3.21, $G-E(H)-D$ is Hamilton connected (recall that G' is a subgraph of G). $\square$

3.3.3 Proof of the Hamiltonicity threshold

We are now ready to return to the Hamiltonicity game.

Proof of upper bound in Theorem 1.5. Let $\epsilon > 0$ fixed and let $a = (1+\epsilon)\ln\ln n$. Furthermore let `CleverBreaker` play according to an arbitrary fixed strategy S.

Fix an $\alpha < \frac{1}{16e}\min\{\gamma_{(3.24)}, \frac{1}{32}\}$, such that inequality (2) holds as well. Recall that $t = \frac{\alpha}{2}n\ln n$. We show that `RandomMaker` builds a Hamilton cycle in the first t rounds of the $(a:1)$-biased Hamiltonicity game a.a.s.

By Proposition 2.1 we work in the setup where `RandomMaker` plays according to a random permutation $\sigma \in S_{E(K_n)}$ against `CleverBreaker`'s fixed strategy S. To use our random graph statements we generate σ in three steps. First we select the initial segment σ_1 of the first $\frac{at}{2}$ edges of σ uniformly at random. Then, independently, we select another sequence σ_2 of $\frac{at}{2}$ edges uniformly at random from *all* $\frac{\binom{n}{2}!}{\left(\binom{n}{2}-\frac{at}{2}\right)!}$ choices and append, in this order, those edges of σ_2 to σ_1 which do not appear in it already. Finally, we choose a uniformly random permutation σ_3 of the rest of the edges and append it, to obtain σ. We define the set $M_2 = M_2(\sigma_1, \sigma_2) \subseteq [at]$ to be the set of those coordinates where the edges of σ_2 appear in σ.

We thus refined the probability space to a triplet $(\sigma_1, \sigma_2, \sigma_3)$. But still, clearly the permutation σ created this way is a uniformly random permutation of the edges of K_n. Further, the graphs $G([at/2])$ and $G(M_2)$ as defined in Section 3.2.2 are independent and are drawn independently from the distribution of $G(n, at/2)$. We define five events. Let $\delta = \frac{\alpha}{4}(1+\epsilon)$.

First, let $\mathcal{A}$ be the event containing those triplets $(\sigma_1, \sigma_2, \sigma_3)$ for which $\Delta(H([at])) \le \frac{8e\alpha at}{n}$ and let $\mathcal{A}_1$ be the event containing those σ_1 for which $\Delta(H([at/2])) \le \frac{8e\alpha at}{n}$ (note that $H([at/2])$ depends only on σ_1). Observe that $\mathcal{A}$ implies $\mathcal{A}_1$ and by Lemma 3.11, $\mathcal{A}$ holds a.a.s.

Furthermore, let $\mathcal{B}_1$ be the event containing those σ_1 for which the uniform random graph $G([at/2])$, with $\frac{at}{2} = m = \delta n \ln n \ln\ln n$ edges, has the property that for any subset $B \subseteq V$, $|B| \le \ln n$, any sequence of at most $k \le \ln n$ pairs of vertices $a_1, \dots a_k, b_1, \dots b_k \in V \setminus B$, and any graph $H \subseteq K_n$ with maximum degree $\Delta(H) \le \frac{m}{32n}$ there exist $k-1$ pairwise disjoint paths $P_i \subseteq G([at/2]) - E(H) - B - \{a_1, b_k\}$, $i = 1, \dots k-1$, of length $\le \ln n$ each, connecting b_i to a_{i+1}. By Corollary 3.18, we have that $\mathcal{B}_1$ holds a.a.s.

Let $\sigma_1 \in \mathcal{A}_1 \cap \mathcal{B}_1$. For a set $B \in \binom{V}{\le \ln n}$ and for a sequence $a_1, \dots a_k, b_1, \dots, b_k \in V \setminus B$ of at most $2\ln n$ distinct vertices let us denote by $D^*(B, a_1, \dots a_k, b_1, \dots, b_k) \subseteq V$ the union of $B \cup \{a_2, \dots, a_k, b_1, \dots, b_{k-1}\}$ with the union of the vertex sets of the $k-1$ pairwise disjoint paths of length $\le \ln n$ connecting b_i to a_{i+1}, for $i = 1, \dots k-1$, in $G([at/2]) - E(H([at/2])) - B - \{a_1, b_k\}$. Note that these $k-1$ paths do exist since the maximum degree $\Delta(H([at/2])) \le \frac{8e\alpha at}{n} \le \frac{m}{32n}$ by $\sigma_1 \in \mathcal{A}_1$ and hence the property from $\mathcal{B}_1$ can be applied. (In case the choice of the family of paths is not unique then it is selected according to an arbitrary, but fixed preference order.)

Let us denote by $\mathcal{D}^*(\sigma_1) = \mathcal{D}^*$ the family containing $D^*(B, a_1, \dots a_k, b_1, \dots, b_k)$ for all choices of $B \in \binom{V}{\le \ln n}$, and $a_1, \dots, a_k$, $b_1, \dots, b_k \in V \setminus B$. Clearly, $|\mathcal{D}^*| < n^{3\ln n}$. Furthermore note that every $D^* \in \mathcal{D}^*$ has at most $\ln n + (\ln n - 1)\ln n = \ln^2 n$ elements.

Let $\mathcal{B}_2$ be the event containing the pairs (σ_1, σ_2) for which $\sigma_1 \in \mathcal{A}_1 \cap \mathcal{B}_1$ and for which the uniform random graph $G(M_2) \sim G(n, \frac{at}{2})$, has the property that for every $D \in \mathcal{D}^*(\sigma_1)$ and any graph H with maximum degree $\Delta(H) \le \gamma_{(3.24)} \frac{m}{n}$ the graph $G(M_2) - E(H) - D$ is Hamilton connected. Note that by Corollary 3.24, the event $\mathcal{B}_2$ conditioned on any $\sigma_1 \in \mathcal{A}_1 \cap \mathcal{B}_1$ holds with probability at least $1 - e^{-\Omega(\ln n \ln\ln n)}$. Here it is crucial that, although M_2 depends on both σ_1 and σ_2, the graph $G(M_2)$ is independent of σ_1 by construction.

Finally, we let $\mathcal{S}$ be the event containing those triplets $(\sigma_1, \sigma_2, \sigma_3)$ for which after t rounds there are disjoint paths $Q_1, \dots, Q_k$, $k \le \ln n$, covering the set B of α-bad vertices and having their endpoints, and only those, among the α-good vertices. Note that by Corollary 3.9, $\mathcal{S}$ holds a.a.s.

Then, formally, we have that

$$
\begin{aligned}
&Pr\left[\overline{\mathcal{A}}\right] + Pr\left[\overline{\mathcal{B}_1}\right] + Pr\left[\overline{\mathcal{B}_2}\right] + Pr\left[\overline{\mathcal{S}}\right] = \\
&= Pr\left[\overline{\mathcal{A}}\right] + Pr\left[\overline{\mathcal{B}_1}\right] + \sum_{\sigma_1 \in \mathcal{A}_1 \cap \mathcal{B}_1} Pr\left[\overline{\mathcal{B}_2} \mid \sigma_1\right] Pr\left[\sigma_1\right] + Pr\left[\overline{\mathcal{S}}\right] \\
&= o(1) + o(1) + e^{-\Omega(\ln n \ln\ln n)} \sum_{\sigma_1 \in \mathcal{A}_1 \cap \mathcal{B}_1} Pr\left[\sigma_1\right] + o(1) = o(1).
\end{aligned}
$$

It remains to show that for any triplet $(\sigma_1, \sigma_2, \sigma_3)$ such that $(\sigma_1, \sigma_2, \sigma_3) \in \mathcal{A} \cap \mathcal{S}$, $\sigma_1 \in \mathcal{B}_1$ and $(\sigma_1, \sigma_2) \in \mathcal{B}_2$ hold, `RandomMaker` following the permutation strategy according to the σ induced by the triplet $(\sigma_1, \sigma_2, \sigma_3)$ builds a Hamilton cycle against CleverBreaker playing with his fixed strategy S (by the end of round t).

First we show that RandomMaker's graph after t rounds contains a single path of length at most $\ln^2 n$ covering the set B of all α-bad vertices. Indeed, $\mathcal{S}$ guarantees paths $Q_1, \ldots, Q_k$, $k \le \ln n$, partitioning B, and having their endpoints $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$, and only those among the α-good vertices. Recall that $|B| \le \ln n$. Since $\sigma_1 \in \mathcal{B}_1 \cap \mathcal{A}_1$, there are $k-1$ pairwise disjoint paths $P_i \subseteq G([at/2]) - E(H([at/2])) - B - \{a_1, b_k\}$, $i = 1, \ldots k-1$, of length at most $\ln n$ connecting b_i to a_{i+1}. Since only good vertices are involved in these paths, by Lemma 3.10 the paths are indeed in RandomMaker's graph. The concatenation of the paths P_i and Q_j gives a single a_1, b_k-path P of length at most $\ln^2 n$ covering all bad vertices.

Since $(\sigma_1, \sigma_2, \sigma_3) \in \mathcal{A}$ and $(\sigma_1, \sigma_2) \in \mathcal{B}_2$, the graph $G(M_2) - H([at]) - (V(P) \setminus \{a_1, b_k\})$ is Hamilton connected, and thus contains a Hamilton path Q connecting a_1 and b_k. Note that removing $V(P) \setminus \{a_1, b_k\}$ removes all bad vertices and thus, again by Lemma 3.10, Q is contained in RandomMaker's graph.

The concatenation of Q and P gives a Hamilton cycle. Here we used that even though $G([at/2])$ and $G(M_2)$ might have common edges, for Q we used only those edges of $G(M_2)$ that are left after deleting the internal vertices of P. $\square$

4 Remarks and Open Problems

In this paper we determined the sharp threshold bias of the minimum-degree-k, connectivity and Hamiltonicity games in the half-random RandomMaker vs CleverBreaker scenario. To prove that the sharp threshold bias of the half-random k-connectivity game is also $\ln \ln n$, we can proceed as we did when deriving connectivity and minimum-degree-k. Suppose there is a vertex cut S of size at most $k-1$ in RandomMaker's graph. Note that for every bad vertex there exists k vertex disjoint paths to good vertices, so the deletion of $k-1$ vertices will not disconnect all of these paths: any bad vertex will still be connected to a good vertex after the deletion of S. So it is enough to show that the graph of RandomMaker induced by the good vertices is not disconnected with the removal of S. This can be done similarly as we show the 1-connectedness of the graph: the only difference is that for any X, $|X| \le n/2$ we show that the number of edges going to $\overline{X} \setminus S$ is at least $\Omega(|X| \ln n \ln \ln n) > 0$. For this one needs to also subtract from the calculations there the number of edges incident to S, which is negligeble.

It would also be interesting to study other natural half-random games, for example non-planarity, non-k-colorability, and k-minor games, as well as their half-random Avoider-Enforcer and Waiter-Client variants. Hefetz, Krivelevich, Stojaković and Szabó [14] proved that the threshold biases for the clever Maker-Breaker version of all these games are linear in n. The half-random RandomMaker-CleverBreaker threshold of all these games will also be larger than $n^{1-\epsilon}$

for every $\epsilon > 0$. This is because for each of these properties $\mathcal{P}$ there exists a graph $H = H(\mathcal{P}, \epsilon)$ possessing $\mathcal{P}$, which is of constant size (bounded by some function $f(1/\epsilon)$) and sparse (in the sense that $1/m_2(H) > 1 - \epsilon$). Then `RandomMaker` will occupy this H against `CleverBreaker` a.a.s. by the results of [5]. It would be interesting to decide whether for these games `RandomMaker` performs roughly the same as `CleverMaker` (like for $\mathcal{K}_H$) or significantly worse (i.e., in the order of magnitude). Note that for the games of the present paper, with spanning graph properties, `RandomMaker` performs significantly worse than `CleverMaker`, because the isolation of just one vertex already can win the game for `CleverBreaker`.

Finally, it is well-known that for a fixed graph H the threshold bias $n^{1/m(H)}$ of the random H-building game is coarse. It is unclear however whether the `RandomMaker` vs `CleverBreaker` half-random H-creation game cannot have a sharp threshold bias, we tend to think it does. Note that by [5] we know the order of magnitude of the threshold, it is $n^{1/m_2(H)}$, where $m_2(H)$ is the usual maximum 2-density of H.

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5 Appendix

Proof of Lemma 3.12. Fix a set $X \subset [n]$ of size $|X| \le n/2$. Applying Theorem 3.14 to the edge set $\left(X, \overline{X}\right)$ between X and its complement, with $F = \binom{n}{2}$ and $l = m$, the probability that less than $\frac{m|X|(n-|X|)}{2\binom{n}{2}} \ge |X|\frac{m}{2n}$ edges are present in G between X and $\overline{X}$ is at most $e^{-|X|\frac{m}{8n}}$. Taking the union bound over all subsets X, we obtain that the failure probability is $\sum_{k=1}^{n/2}\binom{n}{k}e^{-k\frac{m}{8n}} = \sum_{k=1}^{n/2} e^{k(O(\ln n)-\Omega(\ln n \ln\ln n))} = o(1)$. $\square$

Proof of Lemma 3.15. The following claim directly implies part (a) of the lemma.

Claim: With probability at least $1-e^{-\Omega(\ln n \ln\ln n)}$ for every vertex set X of size at most $|X| \le \frac{n^2}{m}$ there is a sequence of vertices $v_1, \dots, v_{|X|/2} \in X$ and disjoint sets $N_1, \dots, N_{|X|/2} \subset [n] \setminus X$ of size $\frac{m}{4n}$ each, such that for all $i = 1, \dots, \frac{|X|}{2}$ we have $N_i \subseteq N(v_i)$.

Proof. To prove the claim, let us first fix X and write $X = \left\{s_1, \dots, s_{|X|}\right\}$. We inductively define sets M_i: if there exist at least $\frac{m}{4n}$ neighbors of s_1 in $\overline{X}$, then let M_1 be the set containing the $\frac{m}{4n}$ neighbours with the lowest index. Otherwise let $M_1 = \emptyset$. Similarly, let then M_i be the set of the $\frac{m}{4n}$ neighbors of s_i in $\overline{X} \setminus \left(\bigcup_{j<i} M_j\right)$ with the lowest index provided there are at least $\frac{m}{4n}$ such neighbors, and the empty set otherwise.

Further, call each s_i a *success*, if $M_i \neq \emptyset$ or $\sum_{j=1}^{i-1} d(s_j) \ge \frac{m}{2}$. Note that if there are less than $\frac{|X|}{2}$ failures, then either (1) the claim holds for X, or (2) X has at least $\frac{m}{4}$ incident edges. However, the number of edges incident to X has the hypergeometric distribution with parameters $F = \binom{n}{2}$, $f = |X|\,(n-|X|) + \binom{|X|}{2}$ and $l = m$. For the expectation we have $\frac{|X|m}{n} \le \frac{fl}{F} \le \frac{|X|m}{2(n-1)} \le \frac{m}{8}$, hence by Theorem 3.14 the probability of (2) is at most $e^{-m|X|/3n}$. We show now that with high probability there are less than $\frac{|X|}{2}$ failures.

We go through the s_i in increasing order, and determine the probability of a failure, conditioned under the exact sets of neighbors of the $s_1, \dots, s_{i-1}$. So, fix an $i \le |X|$ and condition on the event that for $j < i$, the neighborhood of s_j is $N_G(s_j) = B_j$ for some fixed sets B_j.
Now if $\sum_{j<i} |B_j| \ge \frac{m}{2}$, then s_i is a guaranteed success. Otherwise, there are $l \ge \frac{m}{2}$ edges left to place in G, and $F \le \binom{n}{2}$ potential edges to choose from. The "good" edges are all the edges from s_i to $\overline{X} \setminus \bigcup_{j<i} M_j$, there are at least $f \ge n - |X| - (i-1)\frac{m}{4n} \ge \frac{n}{2}$ of them. For the expectation we have $\frac{fl}{F} \ge \frac{m}{2n}$, so by Theorem 3.14 the probability that s_i is a failure, that is, that less than $\frac{m}{4n}$ of the good edges are realized, is at most $e^{-m/16n}$.

Thus, by the union bound the probability that there are $\frac{|X|}{2}$ elements of X that are failures is at most

$$\binom{|X|}{|X|/2}\left(e^{-m/16n}\right)^{|X|/2} = e^{-\Omega(m|X|/n)}.$$

This means that the probability that the claim does not hold for X is at most $e^{-\Omega(m|X|/n)} + e^{-m|X|/3n}$.

Now taking the union bound over all X, $|X| \le \frac{n^2}{m}$, the probability that the event of the claim does not hold is at most

$$\sum_{x=1}^{n/\ln^2 n} e^{x \ln n} e^{-\Omega(mx/n)} = e^{-\Omega(\ln n \ln\ln n)}.$$

□

We prove that part (b) holds with probability at least $1 - e^{-n/10}$. Let us fix sets X and N. The number of edges between X and $[n] \setminus (X \cup N)$ has the hypergeometric distribution with parameters $F = \binom{n}{2}$, $f = |X| \, |[n] \setminus (X \cup N)| \geq |X| \, \frac{n}{4}$ and $l = m$. By Theorem 3.14, we have that the probability that there are less than $|X| \, \frac{m}{8n} \leq \frac{fl}{2F}$ edges between X and $[n] \setminus (X \cup N)$ is at most $e^{-fl/8F} \leq e^{-|X| \frac{m}{32n}} \leq e^{-2n}$. Since there are at most $2^n 2^n$ pairs of sets X and N, the probability that the statement fails is at most $e^{(2 \ln 2 - 2)n} \leq e^{-n/10}$.

Finally we show that part (c) holds with probability at least $1 - e^{-m/129}$. Let us fix disjoint vertex sets $X, Y \subset [n]$ of size at least $|X|, |Y| \geq \frac{n}{4}$. The number of edges between X and Y follows the hypergeometric distribution with parameters $F = \binom{n}{2}$, $f = |X| \, |Y| \geq \frac{|X| n}{4}$ and $l = m$. By Theorem 3.14, we have that the probability that there are less than $|X| \, \frac{m}{8n} \leq \frac{fl}{2F}$ edges between X and Y is at most $e^{-fl/8F} \leq e^{-|X| m/32n} \leq e^{-m/128}$. Since there are at most $2^n 2^n \leq e^{o(m)}$ pairs of sets X and Y, the claim follows by taking the union bound over all such X and Y. $\square$