Fast winning strategies in positional games

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Abstract

For the unbiased Maker-Breaker game, played on the hypergraph $\mathcal{H}$, let $\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win then set $\tau_M(\mathcal{H}) = \infty$). Similarly, for the unbiased Avoider-Enforcer game played on $\mathcal{H}$, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t moves (if the game is an Avoider's win then set $\tau_E(\mathcal{H}) = \infty$). In this paper, we investigate τ_M and τ_E and determine their value for various positional games.

1 Introduction

Let p and q be positive integers and let $\mathcal{H}$ be a hypergraph. In a $(p, q, \mathcal{H})$ Maker-Breaker game, two players, called Maker and Breaker, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Maker selects p vertices per move and Breaker selects q vertices per move. Maker wins if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Breaker wins. (Sometimes, when there is no risk of confusion, we will omit $\mathcal{H}$ in the notation above, calling a $(p, q, \mathcal{H})$-game simply a (p, q)-game.) For a $(1, 1, \mathcal{H})$ Maker-Breaker game, let

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$\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win, then set $\tau_M(\mathcal{H}) = \infty$).

Similarly, in a $(p, q, \mathcal{H})$ Avoider-Enforcer game two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Avoider selects p vertices per move and Enforcer selects q vertices per move. Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Enforcer loses. For a $(1, 1, \mathcal{H})$ Avoider-Enforcer game, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t rounds (if the game is an Avoider's win, then set $\tau_E(\mathcal{H}) = \infty$).

In this paper, our attention is restricted to games which are played on the edges of the complete graph on n vertices, that is, the vertex set of $\mathcal{H}$ will always be $E(K_n)$. For quite a few Maker-Breaker and Avoider-Enforcer games it is rather easy to determine the winner. For example, in the connectivity game played on the edges of the complete graph K_n on n vertices, Maker can easily construct a spanning tree by the end of the game. The Avoider-Enforcer planarity game, played on the edges of K_n for n sufficiently large, is an even more convincing example – Avoider creates a non-planar graph and thus loses the game in the end, irregardless of his strategy, the prosaic reason being that every graph on n vertices with more than $3n-6$ edges is non-planar. Thus, for games of this type, a more interesting question to ask is not who wins but rather how long it should take the winner to reach a winning position. This is the type of question we address in this paper.

We start with providing a brief overview of known and relevant results about fast wins in Maker-Breaker and Avoider-Enforcer games. As an immediate consequence of the result of Lehman [12], Maker has a fast winning strategy in the connectivity game. That is, $\tau_M(\mathcal{T}_n) = n - 1$, where $\mathcal{T}_n,\ n \geq 4$ is the hypergraph whose hyperedges are the (edge sets of the) spanning trees of K_n. This approach can be easily generalized to a fast winning strategy for Maker in the k-edge-connectivity game. Indeed, if K_n contains $2k$ pairwise edge disjoint spanning trees, then by partitioning them into k pairs and applying Lehman's strategy to each pair we get $\frac{1}{2}kn \leq \tau_M(\mathcal{T}_n^k) \leq k(n-1)$, where $\mathcal{T}_n^k,\ n \geq 4k$ is the hypergraph whose hyperedges are the spanning k-edge-connected subgraphs of K_n. The lower bound follows immediately since the minimum degree of a k-connected graph is at least k. In this paper we substantially reduce the gap between these two bounds. As another immediate consequence of Lehman's result, we get that Enforcer cannot win the Avoider-Enforcer cycle game faster than the trivial bound suggests, that is, $\tau_E(\mathcal{C}_n) = n$, where the hyperedges of $\mathcal{C}_n$ are all the cycles of K_n. A result of Bednarska [4] entails $\tau_M(\mathcal{TB}_n^k) = k - 1$, where the hyperedges of $\mathcal{TB}_n^k$ are all the copies of complete binary trees on k vertices in K_n, and $k = o(n)$. In [6], Chvátal and Erdős provide Maker with a fast winning strategy for the $(1, 1, \mathcal{H}_n)$ Hamilton cycle game, showing that $\tau_M(\mathcal{H}_n) \leq 2n$, where $\mathcal{H}_n$ is the hypergraph whose hyperedges are the Hamilton cycles of K_n. In this paper, we almost completely close the gap between this upper bound and the trivial lower bound of $n+1$. Maker can win the $(1, 1, \mathcal{K}_n^q)$ clique game in a constant (depending on q but not on n) number of moves, that is, $\tau_M(\mathcal{K}_n^q) = f(q)$, where the hyperedges of $\mathcal{K}_n^q$ are the q-cliques of K_n. The best upper bound, $f(q) = O((q-3)2^{q-1})$ is due to Pekeč (see [13]); Beck proved that the exponential dependency on q cannot be avoided, namely $f(q) = \Omega(\sqrt{2}^q)$ (see [3]). Note that Maker's strategy for the clique game provides him with a fast win in the non-planarity game and the non-r-colorability game by building a copy of K_5 and K_{r+1}, respectively (for background on these games, see [10]).

Some general sufficient conditions for winning Maker-Breaker games and Avoider-Enforcer games were proved in [2] and [9], respectively. Both are based on the "potential" method of Erdős and Selfridge [8]. These criteria, however, seem not to be very useful for winning quickly, as it is assumed that the game is played until every element of the board is claimed by some player. Nonetheless, using some "fake moves" trick (see [3]), they can be used to get certain, usually rather weak, results.

If Maker wins a $(1, q, \mathcal{H})$ Maker-Breaker game for some positive integer q, then $\tau_M(\mathcal{H}) \leq v(\mathcal{H})/(q+1)$, where $v(\mathcal{H})$ is the number of vertices in $\mathcal{H}$. Indeed, when playing the $(1, 1, \mathcal{H})$ game, Maker can use his winning strategy in the $(1, q, \mathcal{H})$ game. In every round, he imagines that additional $q - 1$ arbitrary unclaimed vertices were claimed by Breaker. Whenever Breaker claims a vertex which is already his in Maker's imagination, Maker imagines that another (arbitrary still unclaimed) vertex was claimed by Breaker. Clearly, after all vertices have been claimed (including the ones in Maker's imagination), Maker has already won, and the number of rounds played is $v(\mathcal{H})/(q + 1)$. Equivalently, this shows that if Breaker can keep from losing the $(1, 1, \mathcal{H})$ game within t rounds, then he can win the $(1, \frac{v(\mathcal{H})}{t} - 1, \mathcal{H})$ game. It was proved by Beck in [1] that Breaker, playing the $(1, 1, \mathcal{H})$ game on an almost disjoint n-uniform hypergraph $\mathcal{H}$, can keep from losing for at least $(2 - \varepsilon)^n$ moves, for any $\varepsilon > 0$. Hence, we can immediately deduce that Breaker can win the $(1, \frac{v(\mathcal{H})}{(2-\varepsilon)^n} - 1)$ game, on any almost disjoint n-uniform hypergraph $\mathcal{H}$ and for every $\varepsilon > 0$. Similarly, if Avoider wins the $(1, q, \mathcal{H})$ game for some positive integer q, then $\tau_E(\mathcal{H}) > v(\mathcal{H})/(q + 1)$. Indeed, when playing the $(1, 1, \mathcal{H})$ game, Avoider can use his winning strategy from the $(1, q, \mathcal{H})$ game. Equivalently, this shows that if Enforcer can win the game on $\mathcal{H}$ within t rounds, then he can also win the $(1, \frac{v(\mathcal{H})}{t} - 1, \mathcal{H})$ game.

To conclude, in order to say something non-trivial about the games we analyze, we will have to find winning strategies for Maker and Enforcer that are faster than the known strategies mentioned above (in case they exist).

1.1 Fast strategies for Maker and slow strategies for Breaker

We now turn back to the analysis of some concrete games. All games we consider here are played on the edges of the complete graph K_n.

Let $\mathcal{M}_n$ be the hypergraph whose hyperedges are all perfect matchings of K_n (or matchings that cover every vertex but one, if n is odd). Let $\mathcal{D}_n$ be the hypergraph whose hyperedges are all spanning subgraphs of K_n of positive minimum degree. We find the *exact* number of moves that Maker needs, in order to win the $(1, 1, \mathcal{M}_n)$ game and the $(1, 1, \mathcal{D}_n)$ game.

Obviously, Maker needs to make at least $\lfloor \frac{n}{2} \rfloor$ moves, as this is the size of a member of $\mathcal{M}_n$. We show that if n is odd, then he does not need more moves, whereas if n is even, then he needs just one more move. A similar result, showing the tightness of the obvious lower bound for the minimum degree game $\mathcal{D}_n$, easily follows.

Theorem 1.1 *(i)*

$$\tau_M(\mathcal{M}_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even} \end{cases}$$

(ii)

$$\tau_M(\mathcal{D}_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

As mentioned earlier, Chvátal and Erdős [6] proved that Maker can win the $(1,1)$ Hamilton cycle game on K_n within $2n$ rounds. Here we show that for sufficiently large n, Maker can win the $(1,1)$ Hamilton cycle game within $n+2$ rounds. This bound is now only 1 away from the obvious lower bound.

Theorem 1.2 *For sufficiently large n,*

$$n + 1 \leq \tau_M(\mathcal{H}_n) \leq n + 2.$$

A corollary of the proof of the previous theorem is that Maker can win the "Hamilton path" game within $n-1$ moves, which is clearly best possible.

Theorem 1.3 *For sufficiently large n,*

$$\tau_M(\mathcal{HP}_n) = n - 1,$$

where $\mathcal{HP}_n$ is the hypergraph whose hyperedges are all Hamilton paths of K_n.

Let $\mathcal{V}_n^k$ be the hypergraph whose hyperedges are all spanning k-vertex-connected subgraphs of K_n. The classical theorem of Lehman [12] asserts that Maker can build a 1-connected spanning graph in $n-1$ moves. From Theorem 1.2 it follows that Maker can build a 2-vertex-connected spanning graph for the price of spending just 3 more (that is, in $n+2$) moves.

In the following, we obtain a generalization of the latter fact for every $k \geq 3$. As every k-connected graph has minimum degree at least k, Maker needs at least $kn/2$ moves just to build a member of $\mathcal{V}_n^k$ (even if Breaker doesn't play at all). The next theorem shows that this trivial lower bound is asymptotically tight, that is, there is a strategy for Maker to build a k-vertex-connected graph in $kn/2 + o_k(n)$ moves.

Theorem 1.4 *For every fixed $k \geq 3$ and sufficiently large n,*

$$kn/2 \leq \tau_M(\mathcal{V}_n^k) \leq kn/2 + (k+4)(\sqrt{n} + 2n^{2/3}\log n).$$

An easy consequence of Theorems 1.1, 1.2 and 1.4, is that for every fixed $k \geq 1$ Maker can build a graph with minimum degree at least k within $(1+o(1))kn/2$ moves. This is clearly asymptotically optimal.

1.2 Slow strategies for Avoider and fast strategies for Enforcer

In the Avoider-Enforcer non-planarity game, Avoider loses the game as soon as his graph becomes non-planar. Clearly, Enforcer can win this game within $3n-5$ moves no matter how he plays; that is, $\tau_E(\mathcal{NP}_n) \leq 3n-5$, where $\mathcal{NP}_n$ is the hypergraph whose hyperedges are all non-planar subgraphs of K_n. On the other hand, Avoider can keep from losing for $\frac{3}{2}n - 3$ moves by simply fixing any triangulation and claiming its edges arbitrarily for as long as possible.

The following theorem asserts that the trivial upper bound is essentially tight, that is, Avoider can refrain from building a non-planar graph for at least $(3-o(1))n$ moves. More precisely,

Theorem 1.5

$$\tau_E(\mathcal{NP}_n) > 3n - 28\sqrt{n}.$$

In the Avoider-Enforcer non-k-coloring game $\mathcal{NC}_n^k$, Avoider loses the game as soon as his graph becomes non-k-colorable. Avoider can play for at least $(1-o(1))\frac{(k-1)n^2}{4k}$ moves without losing by simply fixing a copy of the k-partite Turán-graph and claiming half of its edges. On the other hand, it is not hard to see that the game is an Enforcer's win if it is played until the end (see [10]), so Avoider will lose after at most $\frac{1}{2}\binom{n}{2} \approx \frac{n^2}{4}$ moves. In our next theorem we essentially close the gap between the two bounds for the case $k=2$ (the "non-bipartite game"). We also improve the trivial lower bound and establish the order of magnitude of the second order term of $\tau_E(\mathcal{NC}_n^2)$.

Theorem 1.6

$$\frac{n^2}{8} + \frac{n-2}{12} \leq \tau_E(\mathcal{NC}_n^2) \leq \frac{n^2}{8} + \frac{n}{2} + 1.$$

Next, we look at two Avoider-Enforcer games that turn out to be of similar behavior. In the game $\mathcal{D}_n$ Enforcer wins as soon as the minimum degree in Avoider's graph becomes positive, and in the game $\mathcal{T}_n$ Enforcer wins as soon as Avoider's graph becomes connected and spanning. Enforcer wins both games (see [9]), entailing $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n}{2}$. On the other hand, Avoider can choose an arbitrary vertex v, and, for as long as possible, claim only edges which are not incident with v, implying $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) > \frac{1}{2}\binom{n-1}{2}$. This determines the first order term for both parameters. In the following theorem we determine the second order term and the order of magnitude of the third.

Theorem 1.7

$$\frac{1}{2}\binom{n-1}{2} + \left(\frac{1}{4} - o(1)\right)\log n < \tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1.$$

The rest of the paper is organized as follows: in Section 2 we prove Theorems 1.1, 1.2 and 1.4. In Section 3 we prove Theorems 1.5, 1.6 and 1.7. Finally, in Section 4 we present some open problems.

1.3 Preliminaries

For the sake of simplicity and clarity of presentation, we omit floor and ceiling signs whenever these are not crucial. Some of our results are asymptotic in nature and, whenever necessary, we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [7]. In particular, we use the following: for a graph G, denote its set of vertices by $V(G)$, and its set of edges by $E(G)$. Moreover, let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a graph $G = (V, E)$ and a set $A \subseteq V$ denote by $G[A]$ the subgraph of G induced by A. Let $N_G(A) = \{u \in V : \exists w \in A, (u, w) \in E\}$ be the neighborhood of A in G and let $\Gamma_G(A) = N_G(A) \setminus A$ be the external-neighborhood of A in G. Sometimes, when there is no risk of confusion, we abbreviate $N_G(A)$ to $N(A)$ and $\Gamma_G(A)$ to $\Gamma(A)$.

2 Maker-Breaker games

In our definition of Maker-Breaker games, Maker starts the game. In the following, whenever proving a result of the form $\tau_M(\mathcal{H}) \leq a$, we will assume that Breaker starts the game (thus proving a statement which is stronger than the one asserted in the corresponding theorem).

2.1 Building a perfect matching fast

Proof of Theorem 1.1.

Assume first that n is even. Obviously Maker needs at least $n/2$ edges to build a perfect matching. In fact he will need at least one more, as Breaker, seeing the first $n/2 - 1$ moves of Maker, can occupy the unique edge (if no such edge exists, then our claim immediately follows) which would extend Maker's graph into a perfect matching. Hence $\tau_M(\mathcal{M}_n) \geq \frac{n}{2} + 1$.

In the following we assume that Breaker starts the game and give a strategy for Maker to build his perfect matching in $\frac{n}{2} + 1$ moves. A *round* of the game consists of a move by Breaker and a counter move by Maker. A vertex is considered *bad*, if it is isolated in Maker's graph but not in Breaker's graph.

We will provide Maker with a strategy to ensure that for every $3 \leq r \leq \frac{n}{2}$, the following three properties hold after his rth move:

(a) Maker's edges form a forest consisting of $r - 1$ components: a path uvw of length two and $r - 2$ paths of length one;

(b) every isolated vertex of Maker's graph is adjacent to neither u nor w in Breaker's graph;

(c) there are at most two bad vertices.

First, let us see that, if these properties hold after Maker's $\frac{n}{2}$th move, then Maker wins the perfect matching game on his next move. Observe that by property (a) after the $\frac{n}{2}$th move of Maker there is exactly one isolated vertex z in Maker's graph, which, by property (b), is connected to neither u nor w in Breaker's graph. Hence, no matter which edge Breaker claims in his $(\frac{n}{2} + 1)$st move, Maker will be able to respond by claiming either (u, z) or (w, z). After that move Maker's graph is a spanning forest consisting of a path of length three and $\frac{n}{2} - 2$ paths of length one; obviously such a graph contains a perfect matching.

Next, we prove that for every $n \geq 6$, Maker can maintain properties (a) $-$ (c). First, it is easy to see that Maker can execute his first three moves such that these three properties hold.

We will prove that on his rth move, where $\frac{n}{2} \geq r > 3$, Maker can select two vertices that are isolated in his graph and connect them by an edge, while ensuring that, right after his move, properties (b) and (c) hold. Note that this strategy automatically ensures that property (a) holds as well.

Let I_r be the set of vertices which are isolated in Maker's graph after the rth round. Property (a) ensures that $|I_r| = n - (2r - 1)$ and property (c) implies that there are at most two vertices in I_r which are not isolated in Breaker's graph; in particular there is at most one edge in Breaker's graph spanned by I_r. Assume that the rth round, where $r \leq n/2 - 1$, has just ended, then $|I_r| \geq 3$.

In case Breaker claims an edge of the form (x, u) or (x, w) where $x \in I_r$, then Maker responds by claiming an edge (x, y) where $y \in I_r$. Such a vertex y for which the edge (x, y) was not previously claimed by Breaker always exists as only one of Breaker's edges is spanned by I_r, and there are at least three vertices in I_r. Since the vertex x will not be bad at the end of the $(r+1)$st round, the number of bad vertices does not increase and property (c) remains valid. Property (b) will also remain valid because the only new vertex which could dissatisfy it, x, is not isolated in Maker's graph anymore.

If Breaker does not claim an edge of the form (x, u) or (x, w), where $x \in I_r$, then Maker responds by claiming an edge with both endpoints in I_r such that property (c) remains valid. This can easily be done as there are at most two edges of Breaker with both endpoints in I_r, and $|I_r| \geq 3$. Property (b) was not affected by Breaker's move.

This concludes our description of Maker's strategy and the proof if n is even.

If n is odd, then Maker's strategy is essentially the same as his strategy for even n (in fact it is a little simpler). The main difference is that property (b) is redundant, property (a) is replaced with:

(a') After Maker's rth round, his graph is a matching with r edges,

and we don't need to consider separately, Maker's first three moves. We omit the straightforward details.

As for the positive minimum degree game, it is clear that $\tau_M(\mathcal{D}_n) \geq \lfloor n/2 \rfloor + 1$. Furthermore, if n is even, then by part (i) of Theorem 1.1 we get $\tau_M(\mathcal{D}_n) \leq \tau_M(\mathcal{M}_n) = n/2 + 1$. If n is odd, then Maker can build a matching that covers all vertices but one in $\lfloor n/2 \rfloor$ rounds, and then claim an arbitrary edge incident with the last remaining isolated vertex. Hence, we get $\tau_M(\mathcal{D}_n) = \lfloor n/2 \rfloor + 1$ as claimed.

□

2.2 Building a Hamilton cycle fast

Proof of Theorem 1.2.

In the proof, we use the method of Pósa rotations (see [14]). Let $P_0 = (v_1, v_2, \ldots, v_l)$ be a path of maximum length in a graph G. If $1 \leq i \leq l-2$ and (v_l, v_i) is an edge of G then $P' = (v_1, v_2, \ldots, v_i, v_l, v_{l-1}, \ldots, v_{i+1})$ is also of maximum length. It is called a *rotation* of P_0 with *fixed endpoint* v_1 and *pivot* v_i. The edge (v_i, v_{i+1}) is called the *broken* edge of the rotation. We can then, in general, rotate P' to get more maximum length paths.

We will assume that Breaker starts the game. A *round* consists of a move by Breaker and a counter move by Maker. Assume first that n is even. Maker's strategy is divided into three stages.

In the first stage, Maker builds a perfect matching with one additional edge, that is, he builds a path of length 3 and $(n-4)/2$ paths of length 1. From the proof of Theorem 1.1 we know that Maker can do this in $n/2+1$ moves.

In the second stage, which lasts exactly $n/2-2$ rounds, Maker connects endpoints of the paths in his graph. In each move he connects two paths to form one longer path. Hence, in each round he decreases the number of paths by one, and thus, by the end of the second stage he will have a Hamilton path.

For every $0 \leq i \leq n/2-3$, let B'_i be the subgraph of Breaker's graph, induced by the endpoints of Maker's paths, just after the $(i+1)$st move of Breaker in the second stage (recall that Breaker starts the second stage). Let B_i be the graph obtained from B'_i by removing all edges (x,y) such that x and y are endpoints of the same path of Maker. The unclaimed edges $(x,y) \in \binom{V(B_i)}{2}$, for which x and y are endpoints of different paths of Maker are called *available*.

The first move of Maker in this stage is somewhat artificial, thinking ahead about stage three. Let $w \in V(B_0)$ be a vertex of highest degree in Breaker's graph. On his first move of the second stage Maker claims an arbitrary available edge incident with w. Such an edge exists if n is large enough, since Breaker has $n/2+2$ edges, while there are $n-2$ endpoints in $V(B_0)$. Note that for any two vertices $z', z'' \in V(B_1)$, the sum of the degrees of z' and z'' in Breaker's graph is at most $n/3+4$ (we will use this observation only in stage three).

Maker's goal is now the following: he will make sure that $e(B_i) \leq v(B_i)-1$ for every $1 \leq i \leq n/2-3$. This easily holds for $i=1$ provided n is large enough. Assume that the statement holds for some $1 \leq i \leq n/2-4$ and let us prove that Maker can claim an available edge while ensuring that $e(B_{i+1}) \leq v(B_{i+1})-1$.

Case 1.j. (for every $0 \leq j \leq 3$). $e(B_i) \leq v(B_i)-1-j$ and there is an available edge incident with at least $3-j$ edges of B_i. Maker claims this edge entailing $e(B_{i+1}) \leq e(B_i)-(3-j)+1 \leq v(B_i)-3 = v(B_{i+1})-1$.

Case 2. There is a vertex v of degree at least 3 in B_i. Hence by Case 1.0 we can assume that there is no available edge incident with v, that is, the degree of v in B_i is exactly $v(B_i)-2$ (recall that there are no edges in B_i between the endpoints of the same path of Maker). Note that by the induction hypothesis there is at most one edge in B_i which is not incident with v. Since $i \leq n/2-4$, $v(B_i) \geq 6$, and so v has at least four neighbors in B_i.

Assume first that every edge of B_i is incident with v, entailing $e(B_i) = v(B_i)-2$. Among the four neighbors of v there has to be at least one available edge. This edge is incident with two edges of Breaker and so Case 1.1 applies.

Suppose now that there is an edge of B_i which is not incident with v. One of its endpoints z is a neighbor of v. Hence, since $v(B_i) \geq 6$, there must exist an available edge between z and another neighbor of v; thus Case 1.0 applies.

Case 3. The maximum degree of B_i is at most 2. Hence every connected component of

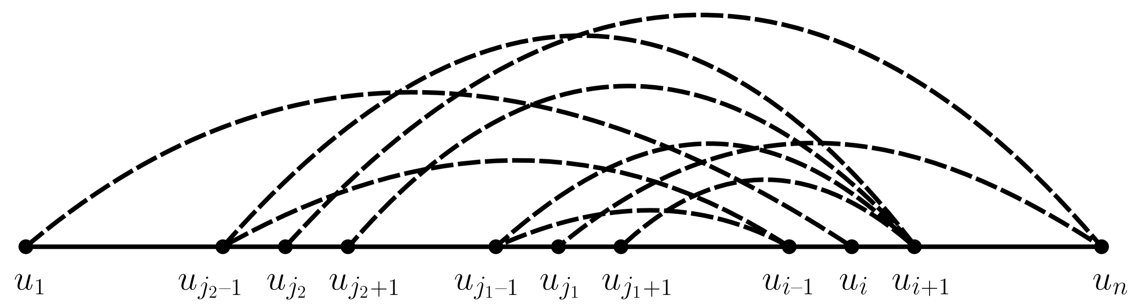

Figure 1: Dashed edges are unclaimed by Breaker.

B_i is either a path or a cycle. By Case 1.3 we can assume that $e(B_i) > v(B_i) - 4$. If $e(B_i) = v(B_i) - 3$, then by Case 1.2 Maker can claim any available edge which is incident with some edge of Breaker. If $e(B_i) = v(B_i) - 2$, then there is a vertex x of degree 2, since $v(B_i) \geq 6$. By Case 1.1 Maker can claim any available edge which is incident with x. Finally, if $e(B_i) = v(B_i) - 1$, then again there is a vertex x of degree 2. Moreover, there is an available edge incident with x whose other endpoint y is a non-isolated vertex in B_i (such a non-isolated vertex exists, since $v(B_i) \geq 6$ and $e(B_i) = v(B_i) - 1$). Maker claims the edge (x, y) and Case 1.0 applies.

This means that after $n/2 - 3$ moves in the second stage Maker has successfully built a spanning forest consisting of two paths such that Breaker's graph $B_{n/2-3}$ on the four endpoints of these two paths satisfies $e(B_{n/2-3}) \leq v(B_{n/2-3}) - 1$. Hence, there exists at least one available edge in $B_{n/2-3}$. Maker claims this edge, thus creating his Hamilton path.

In the third stage, Maker uses Pósa rotations to close his Hamilton path $u_1, u_2, \ldots, u_n$ to a Hamilton cycle. Let u_i, u_{j_1}, u_{j_2} be three vertices on this path such that $i-1 > j_1+1 > j_2+1$ and, just before Maker's first move in this stage, none of the edges (u_1, u_i), (u_{j_1}, u_n), (u_{j_2}, u_n), (u_{i+1}, u_{j_1-1}), (u_{i-1}, u_{j_1-1}), (u_{i+1}, u_{j_1+1}), (u_{i+1}, u_{j_2-1}), (u_{i-1}, u_{j_2-1}), (u_{i+1}, u_{j_2+1}) were previously claimed by Breaker (see Figure 1). In his first move of the third stage, Maker claims the edge (u_1, u_i). In his next move, Breaker cannot claim both (u_{j_1}, u_n) and (u_{j_2}, u_n). Assume without loss of generality that he does not claim (u_{j_1}, u_n). In his next move Maker claims (u_{j_1}, u_n), and then he claims either (u_{i+1}, u_{j_1-1}) or (u_{i-1}, u_{j_1-1}) or (u_{i+1}, u_{j_1+1}) (Breaker cannot neutralize these three simultaneous threats with only two edges). This yields a Hamilton cycle. Note that stage three lasts exactly 3 rounds.

It remains to prove that the three vertices u_i, u_{j_1}, u_{j_2} with the desired properties exist. Recall that, by Maker's first move in the second stage, we have $deg_{B_1}(u_1)+deg_{B_1}(u_n) \leq n/3+4$. In the second and third stages Breaker adds $n/2$ more edges, entailing $deg_{B_{n/2-3}}(u_1) + deg_{B_{n/2-3}}(u_n) \leq 5n/6 + 4$. Hence, for sufficiently large n, there are at least $n/7$ vertices u_k such that neither (u_1, u_k) nor (u_k, u_n) was claimed by Breaker. Thus there are at least $n^2/200$ pairs of vertices u_i, u_j such that $i - 1 > j + 1$ and both (u_1, u_i) and (u_j, u_n) were not claimed by Breaker. Moreover, Breaker has only $O(n)$ edges and every edge (u_p, u_q) he claims affects at most four of the pairs (u_i, u_j), namely (u_{p-1}, u_{q-1}), (u_{p-1}, u_{q+1}), (u_{p+1}, u_{q-1})

and (u_{p+1}, u_{q+1}). Hence, there exist two such pairs u_i, u_{j_1} and u_i, u_{j_2}.

If n is odd, then the proof is essentially the same, with just a few small technical changes:

1. The first stage lasts $\lfloor n/2 \rfloor + 1$ rounds and, when it ends, Maker has one path of length 2 and $(n-3)/2$ paths of length 1.

2. The second stage lasts exactly $\lceil n/2 \rceil - 2$ rounds.

3. In B_0 there are $n-1$ vertices and at most $\lfloor n/2 \rfloor + 2$ edges.

□

2.3 Building a k-connected graph fast

Proof of Theorem 1.4. Let $K_n = (V, E)$ where $V = \{1, 2, \ldots, n\}$. Assume first that n is even and let $m = kn/2$. We will present a random strategy for Maker, which enables him to build a k-vertex-connected graph within $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ rounds, with positive probability. This, however, will imply the existence of a deterministic strategy for Maker with the same outcome.

Before we start with a detailed description of Maker's strategy, we give an short overview of his actions. The game consists of two stages (it is possible that the second stage will not take place). In the first stage most of Maker's moves are used for building a graph which is "not far" from being a random k-regular graph. The motivation for this approach is that random k-regular graphs are known to be k-vertex-connected a.s. (for more on random regular graphs, the reader is referred to [5], [11] and [15]). In this stage Maker also has to watch out for Breaker's maximum degree growing too large; he will handle this by momentarily abandoning the creation of the pseudo-random graph in order to occupy some edges incident with the "dangerous vertex" (that is, a vertex of high degree in Breaker's graph). In the second stage, Maker occupies some more edges to neutralize possible damage to his pseudo-random graph, caused by Breaker during the first stage.

Before the beginning of the game, Maker does the following. With every $1 \leq i \leq n$, he associates a set $W_i = \{i_1, i_2, \ldots, i_k\}$ of "copies" of i, the sets being pairwise disjoint. Maker then draws uniformly at random a perfect matching P of the $2m$ elements of $W = \bigcup_{i=1}^{n} W_i$. Let $S = ((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m))$ be an arbitrary ordering of the matched pairs. Note that the selection of the perfect matching P, can be done equivalently by choosing the pairs one at a time. That is, Maker repeatedly draws a pair randomly, uniformly on all unmatched elements of W. Sometimes this point of view is more convenient for our analysis. If $a_r \in W_i$ and $b_r \in W_j$, then we say that the pair (a_r, b_r) corresponds to the edge (i, j). Clearly, different pairs can correspond to the same edge, and so it is possible to get parallel edges. Furthermore, it is possible that $\{a_r, b_r\} \subseteq W_i$ and so the pair (a_r, b_r) corresponds to the loop (i, i). Thus the pairing P corresponds to a k-regular multi-graph. We will discard loops and parallel edges and thus obtain a simple graph of maximum degree at most k.

A vertex $i \in V$ will be called *dangerous* if its degree in Breaker's graph is at least $k\sqrt{n}$. As soon as such a vertex appears, Maker "treats" it immediately (this process will be described in the following paragraph). Throughout the game, let D denote the set of all dangerous vertices which were already "treated". Before the game starts we set $D = \emptyset$.

Stage 1: During this stage, if there are no dangerous vertices outside D, then Maker claims edges of K_n according to the ordering S (note that the matching P and its ordering S are not known to Breaker). That is, let r be the smallest positive integer such that the pair (a_r, b_r) was not considered by Maker before. Maker then claims the edge (i, j), where $(a_r, b_r) = (i_p, j_q)$ for some $1 \leq i, j \leq n$ and $1 \leq p, q \leq k$. If $i = j$ or the edge (i, j) was previously claimed, either by him or by Breaker, then Maker skips his turn (that is, he claims an arbitrary edge which will not be considered in the analysis) and the pair (a_r, b_r) is marked a *failure*. As soon as some $u \in V$ becomes dangerous (if there are several dangerous vertices, then Maker picks one arbitrarily), Maker suspends the above mentioned strategy and plays as follows. He arbitrarily picks $2k + 8$ vertices $w_1, w_2, \ldots, w_{2k+8} \notin D$ such that the edges (u, w_j) are unclaimed for every $1 \leq j \leq 2k + 8$ and, at that point, no w_j is adjacent in Maker's graph to any vertex in D. This is always possible since the first stage lasts less than $kn/2$ moves, so there can be at most $\sqrt{n}$ dangerous vertices. Handling each such vertex takes $k + 4$ moves, so any dangerous vertex, when handled, has degree at most $k\sqrt{n} + (k + 4)\sqrt{n}$ in Breaker's graph, and every vertex which is not in D has degree at most $k + 1$ in Maker's graph. During his next $k + 4$ moves, Maker claims some $k + 4$ edges from the set $\{(u, w_1), (u, w_2), \ldots, (u, w_{2k+8})\}$. He then labels u treated, adds it to D and returns to his usual strategy. The first stage ends as soon as every dangerous vertex is treated and all but $kn^{2/3}$ pairs of S are considered by Maker. The last $kn^{2/3}$ pairs of S are also considered to be failures.

Lemma 2.1 *During the first stage there are at most $n^{2/3} \log n$ failures almost surely.*

Proof of Lemma 2.1: It is well-known that for every fixed k, an n-vertex k-regular multi-graph that corresponds to a random pairing, almost surely contains at most $n^{2/3}$ loops and parallel edges (see e.g. [11]). Hence, it suffices to bound from above the number of failures that correspond to edges that were previously claimed by Breaker. Throughout the first stage, there are at most $\sqrt{n}$ vertices in D. Hence, after considering at most $kn/2 - kn^{2/3}$ pairs of S, there are at least $n^{2/3} < 2n^{2/3} - \sqrt{n} - (k + 4)\sqrt{n}$ vertices of degree strictly smaller than k in Maker's graph. It follows that at any point during the first stage there are at least $\binom{n^{2/3}}{2} - kn/2$ edges available for Maker to continue his configuration (following S). Since Breaker has claimed at most $kn/2$ edges to this point, the probability that any specific pair (a_i, b_i) corresponds to an edge that was previously claimed by Breaker (here we view S as if it was built sequentially) is at most

$$\frac{kn/2}{\binom{n^{2/3}}{2} - kn/2} \leq \frac{2k}{n^{1/3}}.$$

Let F be the random variable that counts the number of the first $kn/2 - kn^{2/3}$ pairs of S, that correspond to edges that were previously claimed by Breaker. Then

$$\mathbb{E}(F) \leq \frac{kn}{2} \cdot \frac{2k}{n^{1/3}} \leq k^2 n^{2/3}.$$

Using Markov's inequality we obtain

$$Pr(F \geq n^{2/3}(\log n - k - 1)) = o(1).$$

It follows that almost surely throughout Stage 1 there are at most $n^{2/3} \log n$ failures ($n^{2/3}(\log n - k - 1)$ for hitting Breaker's edges, $n^{2/3}$ for loops and parallel edges and $kn^{2/3}$ for the last $kn^{2/3}$ pairs of S), which proves the statement of the lemma. □

Let $G_1 = (V, E)$ denote the graph that Maker has built in the first stage, following his random strategy. Let X be the set of all vertices of $V \setminus D$ that are incident with at least one edge, that corresponds to a failure pair, and let $V = V_1 \cup V_2$ be a partition of V, where $V_1 = D \cup X$. Observe that each vertex of V_2 is incident with k random edges of the random graph defined by P. We can thus derive expansion properties of subsets of V_2 from those of the random k-regular graph. This is done in the following claim.

Claim 2.2 *The following holds almost surely. There exists a constant $c > 0$ such that if $A \subseteq V_2$ and $|A| < c \log n$, then $|\Gamma(A)| \geq (k-2)|A|$, and if $A \subseteq V_2$, $B \subseteq V \setminus A$, where $c \log n \leq |A| \leq |B|$ and $|B| \geq n - k - |A|$, then there is an edge between a vertex of A and a vertex of B. Moreover, if $|A| = 1$, then $|\Gamma(A)| \geq k$, and if $|A| = 2$, then $|\Gamma(A)| \geq 2k - 3$.*

The proof of Claim 2.2 is essentially the same as the proof of Theorem 7.32 from [5]. We omit the straightforward details.

As we already mentioned, since we are looking at a finite, perfect information game with no chance moves, it follows that Maker has a deterministic strategy to build $G_1 = (D \cup X \cup V_2, E)$ within $kn/2 + (k+4)\sqrt{n}$ moves, such that $|D| \leq \sqrt{n}$, $|X| \leq 2n^{2/3} \log n$, and V_2 satisfies the properties described in Claim 2.2.

Stage 2: For every $u \in X$, Maker arbitrarily picks $2k+8$ vertices $w_1^u, w_2^u, \ldots, w_{2k+8}^u \in V \setminus N(D)$, such that the edges (u, w_j^u) are unclaimed for every $1 \leq j \leq 2k+8$ and $\{w_1^u, w_2^u, \ldots, w_{2k+8}^u\} \cap \{w_1^v, w_2^v, \ldots, w_{2k+8}^v\} = \emptyset$ for every $u \neq v \in X$. This is possible as $|X| \leq 2n^{2/3} \log n$, $|D| \leq \sqrt{n}$, and each vertex in X has $n - o(n)$ unclaimed edges incident with it, as $X \cap D = \emptyset$. Using an obvious pairing strategy, Maker claims $k+4$ of the edges (u, w_j^u) for every $u \in X$.

Let G_M denote the graph built by Maker during the entire game. We claim that it is k-vertex-connected. Assume for the sake of contradiction, that a small set separates G_M, that is, $V = A \cup S \cup B$, where $1 \leq a = |A| \leq |B|$, $|S| = s < k$ and there are no edges between A and B in G_M. If $a \leq 5$ and $x \in A \cap V_1$, then by Maker's strategy $|(\Gamma(A) \cup A) \setminus \{x\}| \geq |\Gamma(x)| \geq k+4 > |(A \cup S) \setminus \{x\}|$ which is a contradiction as $(\Gamma(A) \cup A) \setminus \{x\} \subseteq (A \cup S) \setminus \{x\}$. On the other hand, if $A \cap V_1 = \emptyset$, then $|\Gamma(A)| \geq k$ by Claim 2.2 (recall that $k \geq 3$). Hence, from now on we assume that $6 \leq a < c \log n$. If $|A \cap V_1| \geq a/4$, then by Maker's strategy $|N(A \cap V_1)| \geq (k+4)a/4 > a + k \geq |A \cup S|$ which is a contradiction as $N(A \cap V_1) \subseteq A \cup S$. Otherwise, $|A \cap V_1| < a/4$ and so by Claim 2.2 we have $|\Gamma(A \cap V_2)| \geq (k-2)3a/4 \geq a/4 + k > |(A \cap V_1) \cup S|$, where the second inequality follows since $a \geq 6$ and $k \geq 3$. Again, this is a contradiction.

If n is odd, then Maker plays as follows. He arbitrarily picks some vertex u and then plays two disjoint games in parallel. One is on the board $\{(u, v) : v \in V \setminus \{u\}\}$, which is played until he claims exactly k of its elements, and the other is on $K_n[V \setminus \{u\}] \cong K_{n-1}$, where Maker plays according to the above strategy. It is easy to see that the resulting graph is k-vertex-connected (adding a vertex to a k-connected graph and then connecting it to k arbitrary vertices of the graph produces a k-connected graph).

Finally, note that by Maker's strategy and by Lemma 2.1, in both stages Maker plays at most $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ moves. □

3 Avoider-Enforcer games

3.1 Keeping the graph planar for long

Proof of Theorem 1.5 We begin by introducing some terminology. Let v be a vertex, and let S be a set of vertices. Let $N_{\mathcal{A}}(v, S)$ denote the set of neighbors of v in Avoider's graph, belonging to S. Similarly, let $N_{\mathcal{E}}(v, S)$ denote the set of neighbors of v in Enforcer's graph, belonging to S.

We will provide Avoider with a strategy for keeping his graph planar for at least $3n - 28\sqrt{n}$ rounds. The strategy consists of three stages.

Before the game starts, we partition the vertex set

$$V(K_n) = \{v_1\} \dot\cup \{v_2\} \dot\cup A \dot\cup N_{1,1} \dot\cup N_{1,2} \dot\cup N_{2,1} \dot\cup N_{2,2},$$

such that $|N_{1,1}| = |N_{1,2}| = |N_{2,1}| = |N_{2,2}| = \sqrt{n} - 1$ and $|A| = n - 4\sqrt{n} + 2$.

In the first stage, Avoider claims edges according to a simple pairing strategy. For every vertex $a \in A$, we pair up the edges (a, v_1) and (a, v_2). Whenever Enforcer claims one of the paired edges, Avoider immediately claims the other edge of that pair. If Enforcer claims an edge which does not belong to any pair, then Avoider claims the edge (a, v_1), for some $a \in A$, for which neither (a, v_1) nor (a, v_2) were previously claimed. He then removes the pair $(a, v_1), (a, v_2)$ from the set of considered edge pairs.

The first stage ends as soon as Avoider connects every $a \in A$ to either v_1 or v_2. Note that, at that point, Avoider's graph consists of two vertex-disjoint stars centered at v_1 and v_2, and the isolated vertices in $N_{1,1} \cup N_{1,2} \cup N_{2,1} \cup N_{2,2}$. Hence, during the first stage, Avoider has claimed exactly $n - 4\sqrt{n} + 2$ edges. Define $A_1 := N_{\mathcal{A}}(v_1, A)$, and $A_2 := N_{\mathcal{A}}(v_2, A)$.

Before the second stage starts, we pick four vertices $n_{1,1} \in N_{1,1}$, $n_{1,2} \in N_{1,2}$, $n_{2,1} \in N_{2,1}$ and $n_{2,2} \in N_{2,2}$, such that $|N_{\mathcal{E}}(n_{i,j}, A)| \leq \sqrt{n}$ for every $i, j \in \{1, 2\}$. Clearly, such a choice of vertices is possible as the total number of edges Enforcer has claimed during the first stage is $n - 4\sqrt{n} + 2 < \sqrt{n} \cdot (\sqrt{n} - 1)$. Define $G_1 := N_{\mathcal{E}}(n_{1,1}, A_1) \cup N_{\mathcal{E}}(n_{1,2}, A_1)$, and $G_2 := N_{\mathcal{E}}(n_{2,1}, A_2) \cup N_{\mathcal{E}}(n_{2,2}, A_2)$. Note that $|G_1| \leq 2\sqrt{n}$, $|G_2| \leq 2\sqrt{n}$, and $|N_{\mathcal{E}}(n_{1,1}, A_1 \setminus G_1)| = |N_{\mathcal{E}}(n_{1,2}, A_1 \setminus G_1)| = |N_{\mathcal{E}}(n_{2,1}, A_2 \setminus G_2)| = |N_{\mathcal{E}}(n_{2,2}, A_2 \setminus G_2)| = 0$.

Using a pairing strategy similar to the one used in the first stage, Avoider connects each vertex of $A_1 \setminus G_1$ to either $n_{1,1}$ or $n_{1,2}$, and each vertex of $A_2 \setminus G_2$ to either $n_{2,1}$ or $n_{2,2}$. More precisely, for every $a \in A_1 \setminus G_1$ we pair up the edges $(a, n_{1,1})$ and $(a, n_{1,2})$, and for every $a \in A_2 \setminus G_2$ we pair up edges $(a, n_{2,1})$ and $(a, n_{2,2})$. Avoider then proceeds as in the first stage.

The second stage ends as soon as Avoider connects every $a \in A_1 \setminus G_1$ to either $n_{1,1}$ or $n_{1,2}$, and every $a \in A_2 \setminus G_2$ to either $n_{2,1}$ or $n_{2,2}$. We define $A_{1,1} := N_{\mathcal{A}}(n_{1,1}, A_1)$, $A_{1,2} := N_{\mathcal{A}}(n_{1,2}, A_1)$, $A_{2,1} := N_{\mathcal{A}}(n_{2,1}, A_2)$ and $A_{2,2} := N_{\mathcal{A}}(n_{2,2}, A_2)$. Since $|A_{1,1}| + |A_{1,2}| = |A_1| - |G_1|$, $|A_{2,1}| + |A_{2,2}| = |A_2| - |G_2|$ and $|A_1| + |A_2| = |A|$, we infer that the number of edges Avoider has claimed in the second stage is at least $n - 8\sqrt{n}$. Note that during the first two stages Avoider did not claim any edge with both endpoints in one of the sets $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, $A_{2,2}$.

In the third stage, Avoider claims only edges with both endpoints contained in the sets $A_{i,j}$, for some $i, j \in \{1, 2\}$. His goal in this stage is to build a "large" linear forest in $A_{1,1}$. (A *linear forest* is a vertex-disjoint union of paths.) In the beginning of the third stage, Avoider's graph induced on the vertices of $A_{1,1}$ is empty, that is, it consists of $|A_{1,1}|$ paths of length 0 each. For as long as possible, Avoider claims edges that connect endpoints of two of his paths in $A_{1,1}$, creating a longer path. When this is no longer possible, every edge that connects endpoints of two different paths must have been previously claimed by Enforcer. Since the total number of edges that Enforcer has claimed so far is at most $3n$, the number of paths of Avoider in $A_{1,1}$ is at most $2\sqrt{n}$. Hence, Avoider has claimed at least $|A_{1,1}| - 2\sqrt{n}$ edges to this point of the third stage.

Similarly, Avoider builds a "large" linear forest in $A_{1,2}$, $A_{2,1}$, and finally $A_{2,2}$, all in the

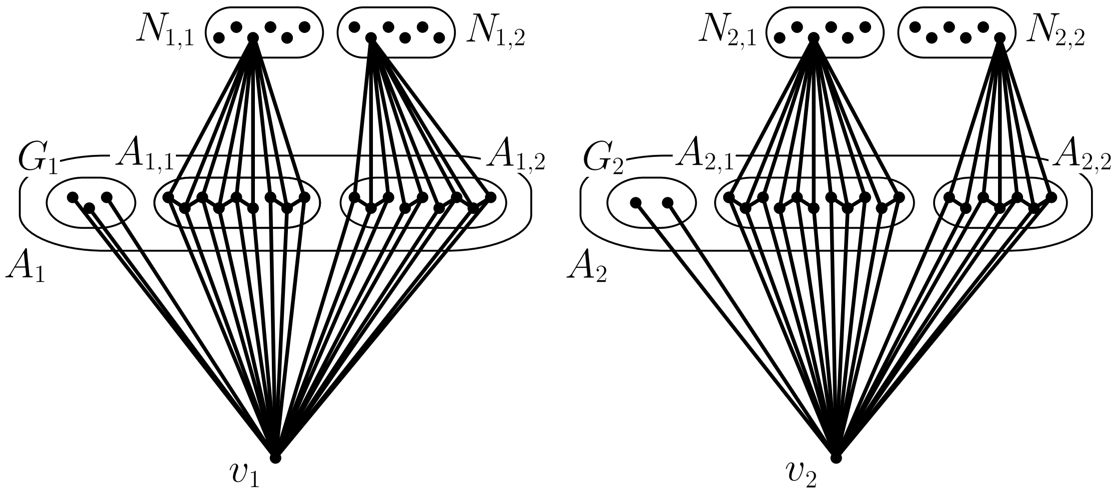

Figure 2: Avoider's graph.

same way. Thus, the total number of edges he claims during the third stage is at least

$$\begin{aligned}\sum_{i,j\in\{1,2\}} (|A_{i,j}| - 2\sqrt{n}) &\geq |A_1| - |G_1| + |A_2| - |G_2| - 8\sqrt{n} \\ &\geq |A| - 12\sqrt{n} \\ &\geq n - 16\sqrt{n}.\end{aligned}$$

The total number of edges claimed by Avoider during the entire game is therefore at least $(n-4\sqrt{n})+(n-8\sqrt{n})+(n-16\sqrt{n}) = 3n-28\sqrt{n}$. Moreover, at the end of the third stage (which is also the end of the game), Avoider's graph is the pairwise edge disjoint union of two stars, four other graphs - each being a subgraph of a union of K_{2,n_i} and a linear forest which is restricted to one side of the bipartition (see Figure 2). Clearly, such a graph is planar.

□

3.2 Avoiding an odd cycle for long

Proof of Theorem 1.6

Forcing an odd cycle fast. First, we provide Enforcer with a strategy that will force Avoider to claim the edges of an odd cycle during the first $\frac{n^2}{8} + \frac{n}{2} + 1$ moves. In every stage of the game, each connected component of Avoider's graph is a bipartite graph with a unique bipartition of the vertices (we stop the game as soon as Avoider is forced to close an odd cycle). In every move, Enforcer's primary goal is to claim an edge which connects two opposite sides of the bipartition of one of the connected components of Avoider's graph. If no such edge is available, then Enforcer claims an arbitrary edge, and that edge is marked as "possibly bad". Clearly, in his following move Avoider cannot play inside any of the connected components of his graph either, and so he is forced to merge two of his connected components (that is, he has to claim an edge (x, y) such that x and y are in different connected components of his graph). As the game starts with n connected components, this situation can occur at most $n - 1$ times.

Therefore, when Avoider is not able to claim any edge without creating an odd cycle, his graph is bipartite, and all of Enforcer's edges, except some of the "possibly bad" ones, are compatible with the bipartition of Avoider's graph. The total number of edges that were claimed by both players to this point is at most $\frac{n^2}{4} + n - 1$, and so the total number of moves Avoider has played in the entire game is at most $\frac{n^2}{8} + \frac{n}{2} + 1$.

Avoiding an odd cycle for long. Next, we provide Avoider with a strategy for keeping his graph bipartite for at least $\frac{n^2}{8} + \frac{n-2}{12}$ rounds. For technical reasons we assume that n is even; however, a similar statement holds for odd n as well. During the game Avoider will maintain a family of ordered pairs (V_1, V_2), where $V_1, V_2 \subseteq V(K_n)$, $V_1 \cap V_2 = \emptyset$ and $|V_1| = |V_2|$, which he calls *bi-bunch*es. We say that two bi-bunches (V_1, V_2) and (V_3, V_4) are disjoint if $(V_1 \cup V_2) \cap (V_3 \cup V_4) = \emptyset$. At any point of the game, Avoider calls a vertex *untouched* if it does not belong to any bi-bunch and all the edges incident with it are unclaimed. During the entire game, we will maintain a partition of the vertex set $V(K_n)$ into a number of pairwise disjoint bi-bunches, and a set of untouched vertices.

Avoider starts the game with n untouched vertices and no bi-bunches. In every move, his primary goal is to claim an edge *across* some existing bi-bunch, that is, an edge (x, y) where $x \in V_1$ and $y \in V_2$ for some bi-bunch (V_1, V_2). If no such edge is available, then he claims an edge joining two untouched vertices x and y, introducing a new bi-bunch $(\{x\}, \{y\})$. If he is unable to do that either, then he claims an edge connecting two bi-bunches, that is, an edge (x, y) such that there exist two bi-bunches (V_1, V_2) and (V_3, V_4) with $x \in V_1$ and $y \in V_3$. He then replaces these two bi-bunches with a single new one $(V_1 \cup V_4, V_2 \cup V_3)$.

Whenever Enforcer claims an edge (x, y) such that neither x nor y belong to any bi-bunch, we introduce a new bi-bunch $(\{x, y\}, \{u, v\})$, where u and v are arbitrary untouched vertices. If at that point of the game there are no untouched vertices (clearly this can happen at most once), then the new bi-bunch is just $(\{x\}, \{y\})$. If Enforcer claims an edge (x, y) such that there is a bi-bunch (V_1, V_2) with $x \in V_1$ and y is untouched, then the bi-bunch (V_1, V_2) is replaced with $(V_1 \cup \{y\}, V_2 \cup \{u\})$, where u is an arbitrary untouched vertex. Finally, if Enforcer claims an edge (x, y) such that there are bi-bunches (V_1, V_2) and (V_3, V_4) with $x \in V_1$ and $y \in V_3$, than these two bi-bunches are replaced with a single one $(V_1 \cup V_3, V_2 \cup V_4)$. Note that by following his strategy, and updating the bi-bunch partition as described, Avoider's graph will not contain an edge with both endpoints in the same side of a bi-bunch at any point of the game.

Observe that the afore-mentioned bi-bunch maintenance rules imply the following. If Enforcer claims an edge (x, y), such that before that move x was an untouched vertex, then the edge (x, y) will be contained in the same side of some bi-bunch, that is, after that move there will be a bi-bunch (V_1, V_2) with $x, y \in V_1$ (unless x and y were the last two isolated vertices).

Assume that in some move Avoider claims an edge (x, y), such that before that move x was an untouched vertex. It follows from Avoider's strategy that y was untouched as well, and there were no unclaimed edges across a bi-bunch at that point. Thus, in his next move, Enforcer will also be unable to claim an edge across a bi-bunch and so, by the bi-bunch maintenance rules for Enforcer's moves, the edge he will claim in that move will have both its endpoints in the same side of some bi-bunch.

By the previous paragraphs, we conclude that after every round in which at least one of the players claims an edge which is incident with an untouched vertex (which is not the next to last untouched vertex), the edge Enforcer claims in this round will be contained in the same side of some bi-bunch. By the bi-bunch maintenance rules, during every round the number of untouched vertices is decreased by at most 6. Hence, by the time all but two vertices are not untouched at least $(n-2)/6$ edges of Enforcer will be contained in the same side of a bi-bunch. Therefore, when Avoider can no longer claim an edge without creating an odd cycle, both players have claimed together all the edges of a balanced bipartite graph which is in compliance with the bi-bunch bipartition, and at least another $(n-2)/6$ edges. This gives a total of at least $\frac{n}{2} \cdot \frac{n}{2} + (n-2)/6$ edges claimed, which means that at least $\frac{n^2}{8} + \frac{n-2}{12}$ rounds were played to that point. □

3.3 Keeping an isolated vertex for long

Proof of Theorem 1.7. Clearly $\tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n)$ and so it suffices to prove that $\tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ and that, $\tau_E(\mathcal{D}_n) > \frac{1}{2}\binom{n-1}{2} + (1/4 - \varepsilon)\log n$ for every $\varepsilon > 0$ and sufficiently large n.

Forcing a spanning tree fast. Starting with the former inequality, we provide Enforcer with a strategy to force Avoider to build a connected spanning graph within $\frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ rounds. At any point of the game, we call an edge that was not claimed by Avoider *safe*, if both its endpoints belong to the same connected component of Avoider's graph. An edge which is not safe and was not claimed by Avoider is called *dangerous*. Denote by G_D the graph consisting of dangerous edges claimed by Enforcer. We will provide Enforcer with a strategy to make sure that, throughout the game, the maximum degree of the graph G_D does not exceed $4k$, where $k = \log_2 n$.

Assuming the existence of such a strategy, the assertion of the theorem readily follows. Indeed, assume for the sake of contradiction that after $\frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ rounds have been played (where Enforcer follows the afore-mentioned strategy), Avoider's graph is disconnected. Let $C_1, \ldots, C_r$, where $r \geq 2$ and $|C_1| \leq \ldots \leq |C_r|$, be the connected components in Avoider's graph at that point. By Enforcer's strategy, the maximum degree of the graph G_D does not exceed $4k$. Hence, the number of edges claimed by both players to this point does not exceed

$$\sum_{i=1}^{r} \binom{|C_i|}{2} + 4k \sum_{i=1}^{r-1} |C_i|.$$

Assuming that $r \geq 2$ and n is sufficiently large, this sum above attains its maximum for $r = 2$, $|C_1| = 1$ and $|C_2| = n-1$; that is, the sum is bounded from above by $\binom{n-1}{2} + 4\log_2 n$ - a contradiction.

Now we provide Enforcer with a strategy for making sure that, throughout the game, the maximum degree of the graph G_D does not exceed $4k$. In every move, if there exists an unclaimed safe edge, Enforcer claims it (if there are several such edges, Enforcer claims one arbitrarily). Hence, whenever Enforcer claims a dangerous edge, Avoider has to merge two connected components of his graph in the following move, and the number of Avoider's connected components is decreased by one. We will use this fact to estimate the number of dangerous edges at different points of the game.

When all edges within each of the connected components of Avoider's graph are claimed, Enforcer has to claim a dangerous edge. His strategy for choosing dangerous edges is divided into two phases. The first phase is divided into k stages. In the ith stage Enforcer will make sure that the maximum degree of the graph G_D is at most $2i$; other than that, he claims dangerous edges arbitrarily. He proceeds to the following stage only when it is not possible to play in compliance with this condition. Let c_i, $i = 1, \ldots, k$, denote the number of connected components in Avoider's graph after the ith stage. Let $c_0 = n$, be the number of components at the beginning of the first stage. During the ith stage, a vertex v is called *saturated*, if $d_{G_D}(v) = 2i$. Note that at the beginning of the first stage the maximum degree of G_D is $2 \cdot 0 = 0$.

We will prove by induction that $c_i \leq n2^{-i} + 2i$, for all $i = 0, 1, \ldots, k$. The statement trivially holds for $i = 0$.

Next, assume that $c_j \leq n2^{-j} + 2j$, for some $0 \leq j < k$. At the beginning of the $(j+1)$st stage Avoider's graph has exactly c_j connected components, and at the end of this stage it has exactly c_{j+1} components. It follows that during this stage Avoider merged two components of his graph $c_j - c_{j+1}$ times. Hence, Enforcer has not claimed more than $c_j - c_{j+1}$ dangerous edges during the $(j+1)$st stage. As the maximum degree of the graph G_D before this stage was $2j$, the number of saturated vertices at the end of the $(j+1)$st stage is at most $c_j - c_{j+1}$. It follows that there are at least $n - (c_j - c_{j+1})$ non-saturated vertices at this point. The non-saturated vertices must be covered by at most $2(j+1)$ connected components of Avoider's graph. Indeed, assume for the sake of contradiction that there are non-saturated vertices

$u_1, u_2, \ldots, u_{2j+3}$ and connected components $U_1, U_2, \ldots, U_{2j+3}$, such that $u_p \in U_p$ for every $1 \leq p \leq 2j+3$. Since $deg_{G_D}(u_p) \leq 2j+1$ for every $1 \leq p \leq 2j+3$, it follows that there must exist an unclaimed edge (u_r, u_s) for some $1 \leq r < s \leq 2j+3$, contradicting the fact that the $(j+1)$st stage is over. Therefore, there are at least $c_{j+1} - 2(j+1)$ connected components in Avoider's graph that do not contain any non-saturated vertex. Clearly every such component has size at least one, entailing $(c_{j+1} - 2j - 2) + (n - c_j + c_{j+1}) \leq n$. Applying the inductive hypothesis we get $c_{j+1} \leq c_j/2 + j + 1 \leq n2^{-(j+1)} + 2(j+1)$. This completes the induction step.

It follows, that at the end of the first phase, after the kth stage, the number of connected components in Avoider's graph, is at most $c_k \leq n2^{-k} + 2k \leq 2k+1$.

In the second phase, whenever Enforcer is forced to claim a dangerous edge, he claims one arbitrarily. Since at the beginning of the second phase, there are at most $2k+1$ connected components in Avoider's graph, Enforcer will claim at most $2k$ dangerous edges during this phase.

It follows that at the end of the game, the maximum degree in G_D will be at most $4k$, as claimed.

Keeping an isolated vertex for long. Fix $\varepsilon > 0$ and set $l := \frac{1-4\varepsilon}{2} \log n$. We provide Avoider with a strategy to keep an isolated vertex in his graph for at least $\frac{1}{2}\binom{n-1}{2} + \frac{l}{2}$ rounds.

Throughout the game, Avoider's graph will consist of one connected component, which we denote by C, and $n - |C|$ isolated vertices. A vertex $v \in V(K_n) \setminus C$ is called *bad*, if there is an even number of unclaimed edges between v and C; otherwise, v is called *good*.

For every vertex $v \in V(K_n)$ let $d_{\mathcal{E}}(v)$ denote the degree of v in Enforcer's graph. If at any point of the game there exists a vertex $v \in V(K_n) \setminus C$ such that $d_{\mathcal{E}}(v) \geq l$, then Avoider simply proceeds by arbitrarily claiming edges which are not incident with v, for as long as possible. The total number of rounds that will be played in that case is at least $\frac{1}{2}\binom{n-1}{2} + \frac{l}{2}$, which proves the theorem. We will show that Avoider can make sure that such a vertex $v \in V(K_n) \setminus C$, with $d_{\mathcal{E}}(v) \geq l$, will appear before the order of his component C reaches $n - l\varepsilon^{-1} - 1$. Hence, from now on, we assume that $|C| \leq n - l\varepsilon^{-1} - 2$.

Whenever possible, Avoider will claim an edge with both endpoints in C. If this is not possible, he will join a new vertex to the component, that is, he will connect it by an edge to an arbitrary vertex of C. Note that this is always possible. Indeed, assume that every edge between C and $V(K_n) \setminus C$ was already claimed by Enforcer. If $|C| \geq l$ then there exists a vertex $v \in V(K_n)$ such that $d_{\mathcal{E}}(v) \geq l$ and so we are done by the previous paragraph. Otherwise, $|C| < l$ and thus, until this point, Enforcer has claimed at most $l^2 < l(n-l)$ edges. As for the way he chooses this new vertex, we consider three cases. Let $\overline{\overline{d}}$ denote the average degree in Enforcer's graph, taken over all the vertices of $V(K_n) \setminus C$, that is,

$$\overline{\overline{d}} := \frac{\sum_{v \in V(K_n) \setminus C} d_{\mathcal{E}}(v)}{n - |C|}.$$

Throughout the case analysis, C and $\overline{\overline{d}}$ represent the values as they are just before Avoider makes his selection.

1. There exists a vertex $v \in V(K_n) \setminus C$, such that $d_{\mathcal{E}}(v) \leq \overline{\overline{d}} - 1$.

   Avoider joins v to his component C. Then $|C|$ increases by one, and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - (\overline{\overline{d}} - 1)}{n - |C| - 1} \quad = \quad \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

2. Every vertex $v \in V(K_n) \setminus C$ satisfies $d_{\mathcal{E}}(v) > \overline{\overline{d}} - 1$, and $\overline{\overline{d}} < \lfloor \overline{\overline{d}} \rfloor + 1 - \varepsilon$.

   Let D denote the set of vertices $u \in V(K_n) \setminus C$ such that $d_{\mathcal{E}}(u) = \lfloor \overline{\overline{d}} \rfloor$. Note that there must be at least $\varepsilon(n - |C|)$ vertices in D. We distinguish between the following two subcases.

   (a) There is a good vertex in D. Avoider joins it to his component C (if there are several good vertices, then he picks one arbitrarily). Since v was a good vertex, Enforcer must claim at least one edge (x, y) such that $x \notin C \cup \{v\}$, before Avoider is forced again to join another vertex to his component. After this move of Enforcer $|C|$ is (still) increased by (just) one, and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - \lfloor \overline{\overline{d}} \rfloor + 1}{n - |C| - 1} \quad \geq \quad \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

   (b) All vertices in D are bad. Knowing that $d_{\mathcal{E}}(v) \leq l - 1$ for all vertices $v \in V(K_n) \setminus C$, and $|C| \leq n - l\varepsilon^{-1} - 2$, we have

$$\max_{v \in D} d_{\mathcal{E}}(v) = \lfloor \overline{\overline{d}} \rfloor < l - 1 + 2\varepsilon \leq \varepsilon(n - |C|) - 1 \leq |D| - 1$$

   and hence there have to be two vertices $u, w \in D$ such that (u, w) is unclaimed. Avoider joins u to his component C, and thus w becomes good. If Enforcer, in his next move, claims an edge (w, v) for some $v \in C$, then $|C|$ is increased by one and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - \lfloor \overline{\overline{d}} \rfloor + 1}{n - |C| - 1} \quad \geq \quad \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

Otherwise, in his next move Avoider joins w to C. Since w was good, then, as in the previous subcase, Enforcer will be forced to claim an edge (x, y) such that $x \notin C \cup \{w\}$. After that move of Enforcer, we will have that $|C|$ is still increased just by two and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - \lfloor\overline{\overline{d}}\rfloor - \lfloor\overline{\overline{d}}\rfloor + 1}{n-|C|-2} \geq \overline{\overline{d}} + \frac{1}{n-|C|-2}.$$

3. Every vertex $v \in V(K_n) \setminus C$ satisfies $d_{\mathcal{E}}(v) > \overline{\overline{d}} - 1$, and $\overline{\overline{d}} \geq \lfloor\overline{\overline{d}}\rfloor + 1 - \varepsilon$.

   Let D denote the set of vertices in $V(K_n) \setminus C$ with degree either $\lfloor\overline{\overline{d}}\rfloor$ or $\lfloor\overline{\overline{d}}\rfloor + 1$. Clearly, $|D| \geq \frac{1}{2}(n - |C|)$. We distinguish between the following two subcases.

   (a) There is a good vertex in D. Similarly to subcase 2(a), Avoider joins that vertex to his component C, and after Enforcer claims some edge with at least one endpoint outside C, we have that $|C|$ is increased by one and the new value of $\overline{\overline{d}}$ is at least

   $$\frac{(n-|C|)\overline{\overline{d}} - (\overline{\overline{d}} + \varepsilon) + 1}{n-|C|-1} = \overline{\overline{d}} + \frac{1-\varepsilon}{n-|C|-1}.$$

   (b) All vertices in D are bad. Similarly to subcase 2(b), Avoider can find two vertices in D such that the edge between them is unclaimed. He joins them to his component C, one after the other. After Enforcer claims some edge with at least one endpoint outside C, we have that $|C|$ increased by two and the new value of $\overline{\overline{d}}$ is at least

   $$\frac{(n-|C|)\overline{\overline{d}} - (\overline{\overline{d}} + \varepsilon) - (\overline{\overline{d}} + \varepsilon) + 1}{n-|C|-2} = \overline{\overline{d}} + \frac{1-2\varepsilon}{n-|C|-2}.$$

It follows that in all cases the value of $\overline{\overline{d}}$ grows by at least $\frac{1-2\varepsilon}{n-|C|-1}$, whenever $|C|$ grows by at most 2. Hence, when the size of C reaches $n - l\varepsilon^{-1} - 2$, we have

$$\begin{aligned}
\overline{\overline{d}} &\geq \sum_{i=2}^{n/2-\frac{1}{2\varepsilon}l-1} \frac{1-2\varepsilon}{n-2i-1} \\
&\geq \frac{1-2\varepsilon}{2} \sum_{i=4}^{n-l\varepsilon^{-1}-2} \frac{1}{n-i-1} \\
&\geq \frac{1-2\varepsilon}{2} \left(\sum_{i=1}^{n-5} \frac{1}{i} - \sum_{i=1}^{l\varepsilon^{-1}} \frac{1}{i} \right) \\
&\geq \frac{1-3\varepsilon}{2} \left(\log n - \log(l\varepsilon^{-1}) \right) \\
&\geq l,
\end{aligned}$$

which concludes the proof of the theorem.

□

4 Concluding remarks and open problems

- It was proved in Theorem 1.4 that Maker can win the $(1,1)$ k-vertex-connectivity game on K_n within $kn/2+o(n)$ moves. It would be interesting to decide whether the $o(n)$ term can be replaced with some function of k, if not for this game, then for the k-edge-connectivity game or the minimum-degree-k game.

- It was proved in Theorem 1.6 that $\tau_E(\mathcal{NC}_n^2) \leq \frac{n^2}{8}+\Theta(n)$. For $k \geq 3$, we know just the trivial bounds $\frac{(k-1)n^2}{4k} \leq \tau_E(\mathcal{NC}_n^k) \leq \frac{1}{2}\binom{n}{2}$. It would be interesting to close, or at least reduce, the gap between these bounds. It seems reasonable that, as in the case $k=2$, the truth is closer to the trivial lower bound, and maybe $\tau_E(\mathcal{NC}_n^k) \leq (1+o(1))\frac{(k-1)n^2}{4k}$ for every $k \geq 3$.

- It was proved in Theorem 1.7 that $\tau_E(\mathcal{T}_n)$ and $\tau_E(\mathcal{D}_n)$ are "almost the same". This is reminiscent of the well-known property of random graphs, that the hitting time of being connected and the hitting time of having minimum positive degree are a.s. the same, and it motivates us to raise the following conjecture.

  Conjecture 4.1 *$\tau_E(\mathcal{D}_n) = \tau_E(\mathcal{T}_n)$.*

- It would be interesting to obtain good estimates on $\tau_E(\mathcal{M}_n)$ and $\tau_E(\mathcal{H}_n)$.

References

[1] J. Beck, On positional games, J. of Combinatorial Theory, Ser. A 30 (1981) 117-133.

[2] J. Beck, Remarks on positional games, Acta Math. Acad. Sci. Hungar. 40 (1-2) (1982) 65-71.

[3] J. Beck, **Tic-Tac-Toe Theory**, Cambridge University Press, 2006.

[4] M. Bednarska, On biased positional games, Combin. Probab. Comput. 7 (1998) 339-351.

[5] B. Bollobás, **Random graphs**, 2^{nd} Ed., Cambridge University Press, 2001.

[6] V. Chvátal and P. Erdős, Biased positional games, Annals of Discrete Math. 2 (1978) 221-228.

[7] R. Diestel, **Graph Theory**, 2^{nd} ed., Springer, 1999.

[8] P. Erdős and J. Selfridge, On a combinatorial game, J. of Combinatorial Theory, Ser. A 14 (1973) 298-301.

[9] D. Hefetz, M. Krivelevich and T. Szabó, Avoider-Enforcer games, J. of Combinatorial Theory, Ser. A, to appear.

[10] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Planarity, colorability and minor games, submitted.

[11] S. Janson, T. Łuczak and A. Ruciński, **Random graphs**, Wiley, New York, 2000.

[12] A. Lehman, A solution of the Shannon switching game, J. Soc. Indust. Appl. Math. 12 (1964) 687-725.

[13] A. Pekeč, A winning strategy for the Ramsey graph game, Combinat. Probab. Comput. 5 (1996), 267-276.

[14] L. Pósa, Hamilton circuits in random graphs, Discrete Math. 14 (1976) 359-364.

[15] N.C. Wormald, *Models of random regular graphs*, in: Surveys in Combinatorics, 1999, J.D. Lamb and D.A. Preece eds., pp. 239–298.