On Erdős's eulerian trail game

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Dedicated to the memory of Paul Erdős

Abstract

Paul Erdős proposed the following graph game. Starting with the empty graph on n vertices, two players, Trailmaker and Breaker, draw edges alternatingly. Each edge drawn has to start at the endpoint of the previously drawn edge, so the sequence of edges defines a trail. The game ends when it is impossible to continue the trail, and Trailmaker wins if the trail is eulerian. For all values of n, we determine which player has a winning strategy.

1 Introduction

We investigate a graph game proposed by Erdős. Two players, Trailmaker and Breaker, begin playing on a board of n isolated vertices. One of them starts the game by drawing an arbitrary edge. At each step, the next player

[1]Research partially supported by NSF Grant CCR-9503430 and by the Alexander von Humboldt Foundation.

[2]Research partially supported by NSF Grant DMS-9304580.

must move by drawing an edge from the endpoint of the edge where the other player finished her move. Trailmaker wins the game, if together eventually they build an eulerian trail, that is, they draw all of the possible $\binom{n}{2}$ edges. If the trail arrives to a vertex which is already connected to all of the other $n-1$ vertices, but there are still edges of the complete graph which are not drawn yet, then Breaker wins.

Note that for even values $n \geq 4$, Trailmaker cannot win. In fact, Breaker wins for all but finitely many odd values of n as well.

Theorem 1 (*a*) *For $n = 2$ and $n = 3$, Trailmaker wins.*
(*b*) *For $n = 5$, the player who draws the first edge has a winning strategy.*
(*c*) *For all other values of n, Breaker has a winning strategy.*

We shall prove part (c) of Theorem 1 in a strong form. Namely, we show that Breaker can achieve that the trail contains only a linear number of edges.

Theorem 2 (*a*) *If Breaker starts then she can achieve that the trail contains $2n-3$ edges.*
(*b*) *If $n \geq 4$ and Trailmaker starts then Breaker can achieve that the trail contains at most $4n-10$ edges.*

Note that Theorem 2 implies Theorem 1(c) for $n \geq 6$. Of course, Trailmaker cannot win for $n = 4$ as well.

In September 1996, at the Minisemester on Combinatorics in Warsaw, the second author had the opportunity to describe Theorems 1 and 2 to Paul Erdős. Given how unbalanced the game is in favor of Breaker, Erdős asked what happens if Trailmaker is allowed to draw at least one but at most m edges at each of her turns. This modification has the honor of being one of the last problems proposed by Uncle Paul. It turns out that already for $m = 2$, the outcome of the game changes dramatically.

Theorem 3 *Suppose that Trailmaker is allowed to draw one or two edges when it is her turn to continue the trail. Then Trailmaker wins the game for all odd values of n, independently of the fact who starts the game.*

The game we consider here is one in the growing family of non-Ramsey type graph games. In these games, it does not matter which player chooses an edge; only the set of chosen edges is important. Some other examples of non-Ramsey type games can be found in [1], [2], [3], [4].

2 The original game

In this section, we prove Theorems 1 and 2. Throughout the section, we assume that the players adhere to the original rules, i.e., Trailmaker draws exactly one edge at each of her turns.

Let $V = \{v_1, v_2, ..., v_n\}$ denote the set of vertices and let $S \subseteq V$. We call two vertices a, b *equivalent outside of* S, if they have the same neighbours in $V \setminus S$. We say a is *equivalent* to b, if a is equivalent to b outside of $\{a, b\}$. Finally, a and b are *totally equivalent* if they are equivalent and adjacent.

In order to simplify the description of the game, we ask our players to adhere to the following convention. Suppose that at the moment a player has to continue the trail from the vertex v_x, and she intends to draw the edge $v_x v_y$. If there are other vertices equivalent to v_y, then we request the player to choose the *smallest* number z such that v_y, v_z are equivalent vertices, and draw the edge $v_x v_z$ instead of $v_x v_y$. So the game will start with the edge $v_1 v_2$ and continue with the edge $v_2 v_3$ (as 3 is the smallest index in the set of equivalent vertices $\{v_3, v_4, ..., v_n\}$ which still can be connected to v_2). Then the starter has two choices: either she draws $v_3 v_1$ or $v_3 v_4$ (as 4 is the smallest index in the set of equivalent vertices $\{v_4, v_5, ..., v_n\}$), etc. Clearly, this convention does not change the outcome of the game.

Now we begin the proof of Theorem 1. The cases $n = 2, 3$ are obvious. For $n = 5$, when Trailmaker starts, she can win by the following sequence of edges. Note that, according to the convention above, Breaker's moves are uniquely determined in this sequence.

$$v_1 v_2, v_2 v_3, v_3 v_1, v_1 v_4, v_4 v_2, v_2 v_5, v_5 v_3, v_3 v_4, v_4 v_5, v_5 v_1.$$

The subcase of the case $n = 5$, when Breaker makes the first move, is covered by Theorem 2(a). Hence it is enough to prove Theorem 2.

Breaker's strategy will be to force Trailmaker into a situation, where she has to move from one of a couple of totally equivalent vertices. Say a and b are totally equivalent and Trailmaker has to move from a. If a and b have k neighbours in $V \setminus \{a, b\}$ then Breaker can achieve that the trail ends after the drawing of $2(n - 2 - k)$ further edges. This is described in the following.

Finishing Strategy: Breaker always forces Trailmaker to move from $\{a, b\}$. She can do this, since a and b have the same neighbours, so whenever Trailmaker draws an edge ac (or bc) Breaker just moves back by cb (or ca). Since

a and b are connected, Trailmaker always must move to a vertex outside of $\{a, b\}$. Hence after $n-2-k$ repetions of this procedure, there are no more vertices not connected to a (or b) and the trail ends at one of the vertices a, b.

If Breaker starts the game then, after taking her first step, she is immediately in the situation described above. According to our convention, she starts by drawing the edge v_1v_2. Then v_1 and v_2 are totally equivalent and Trailmaker has to move from v_2. After drawing $2n-4$ further edges using the Finishing Strategy, the trail cannot be continued.

The solution is not this simple if Trailmaker starts the game. The first two edges of the trail are v_1v_2 and v_2v_3. Then Trailmaker has the option to go back to the set $\{v_1, v_2\}$ (by drawing v_3v_1) or not (at the moment, the only option is to draw v_3v_4). If she chooses v_3v_1 then Breaker has to draw v_1v_4, and again Trailmaker has to choose between going back to the set $\{v_1, v_2\}$ or not (since v_2 and v_3 are equivalent, she is not allowed to continue with v_4v_3, and so her only option is to draw v_4v_5).

We consider the stage of the game when Trailmaker first decides not to connect the last vertex of the trail to one of $\{v_1, v_2\}$. (If she always goes back to $\{v_1, v_2\}$ then the trail will have $2n-3$ edges, as in the case when Breaker started.) This means that, for some $k \geq 3$, the trail used $2k-4$ edges so far: v_1v_2, v_1v_j, v_2v_j for $3 \leq j \leq k-1$, and one of v_1v_k, v_2v_k (according to the parity of k). Since the vertices v_1, v_2 played a symmetric role so far, we assume that the last edge of the trail is v_2v_k. We distinguish three cases:

Case 1: $k = 3$, and Trailmaker continues with the edge v_3v_4.

Case 2: $k \geq 4$, and Trailmaker continues with the edge v_kv_{k+1}.

Case 3: $k \geq 5$, and Trailmaker continues with the edge v_kv_3.

Case 1: Breaker answers with v_4v_2. After that, her strategy is to keep v_2 and v_3 equivalent outside of the set $\{v_1, v_2, v_3, v_4\}$. This means that whenever Trailmaker draws the edge v_2v_j or v_3v_j for some $j \geq 5$, she answers with v_jv_3 or v_jv_2, respectively. While at the vertex v_2, Trailmaker has no other choice than to draw v_2v_j for the smallest still isolated vertex v_j. At the vertex v_3, she has the option to draw v_3v_1 (and when all isolated vertices are used, she will be forced to make this move). However, then Breaker answers with v_1v_4, making v_1 and v_4 totally equivalent, and then applies the Finishing Strategy. Since the set $V \setminus \{v_1, v_2, v_3, v_4\}$ remains independent during the game, the number of edges in the trail is at most $\binom{4}{2} + 4(n-4) = 4n-10$.

Case 2: Breaker answers with $v_{k+1}v_2$. After that, she keeps v_1 and v_2 equivalent outside of $\{v_1, v_2, v_3, v_k, v_{k+1}\}$. This means that whenever Trailmaker chooses to move outside of the set $\{v_1, v_2, v_3, v_k, v_{k+1}\}$ by drawing v_1v_j or v_2v_j, she answers with v_jv_2 or v_jv_1, respectively.

Sooner or later Trailmaker is forced to draw an edge inside $\{v_1, v_2, v_3, v_k, v_{k+1}\}$. This can happen only when the trail currently ends at v_1, as v_2 is already connected to the other four vertices. Since v_k and v_{k+1} are equivalent, Trailmaker is forced to draw v_1v_k. Then Breaker answers with v_kv_3, and after that her strategy is to keep v_3 and v_k equivalent outside of $\{v_1, v_2, v_3, v_k, v_{k+1}\}$ the usual way, always drawing an edge back to the set $\{v_3, v_k\}$. Again, Trailmaker will be forced to draw an edge inside $\{v_1, v_2, v_3, v_k, v_{k+1}\}$. This edge can be only v_3v_{k+1}, and then Breaker answers with $v_{k+1}v_1$. This makes v_1 and v_2 totally equivalent, and Breaker completes the trail according to the Finishing Strategy. The set $V \setminus \{v_1, v_2, v_3, v_k\}$ remains independent during the game, so the trail contains at most $4n - 10$ edges.

Case 3: Breaker answers with v_3v_4, and after that keeps v_3 and v_4 equivalent outside of $\{v_1, v_2, v_3, v_4, v_k\}$ the usual way. When Trailmaker is forced to draw an edge inside the set $\{v_1, v_2, v_3, v_4, v_k\}$, it must be v_4v_k; then Breaker answers with v_kv_1. This makes v_1 and v_2 totally equivalent, and Breaker can apply the Finishing Strategy. Since the set $V \setminus \{v_1, v_2, v_3, v_4\}$ remains independent during the game, the trail contains at most $4n - 10$ edges.

This finishes the proof of Theorem 2, and also the proof of Theorem 1.

3 The modified game

The purpose of this section is to prove Theorem 3. Throughout the section, we consider the variant of the game when Trailmaker is allowed to choose whether she continues the trail by drawing one or two edges.

First, we consider the case when Breaker starts the game. Assume that the size of the board is $n = 2k + 1$. We prove the existence of a winning strategy for Trailmaker by induction on k. Actually, we prove a slightly more general statement: there exists a winning strategy for Trailmaker, such that all the isolated vertices, except v_3, are visited first by Breaker.

The above statement is obvious for $k = 1$. Let us assume that there is an appropriate winning strategy on a board of size $2k - 1$, and let us consider the case $n = 2k + 1$.

According to the conventions about choosing between equivalent vertices, Breaker starts the game with v_1v_2. Trailmaker should answer with the two edges v_2v_3, v_3v_1. After this, Trailmaker considers the set $S = V \setminus \{v_2, v_3\}$ as a board of $2k-1$ vertices and applies her strategy there. Next, Breaker must draw the edge v_1v_4, and by this step she starts a game on S. From now on, if Breaker draws an edge to a vertex of S which is not isolated, then Trailmaker follows her existing strategy for a board of size $2k-1$. Whenever Breaker draws an edge to an isolated vertex $a \in S$, Trailmaker moves av_2 and keeps v_2, v_3 equivalent outside of $\{v_2, v_3, a\}$. Sooner or later Breaker is forced to draw v_3a. In the meantime, always Breaker's edges ended at previously isolated vertices, since Trailmaker always drew edges ending in the set $\{v_2, v_3\}$.

After Breaker draws v_3a, Trailmaker has to move from a, so she just continues her strategy on the board S. If Breaker draws an edge to another isolated vertex b, Trailmaker connects it with v_2 right away and repeats the above process. Finally Breaker must connect back from v_3 to b, and then Trailmaker continues according her strategy on S.

Breaker never gets the chance to draw an edge ending in the set $\{v_2, v_3\}$, because all those edges drawn by Trailmaker which end in S reach non-isolated vertices; however, the non-isolated vertices are already connected to v_2 and v_3.

This way, the strategy for the board S can be applied, with the interruptions when edges incident to $\{v_2, v_3\}$ are drawn. After these interruptions, the game returns into S at the vertex of exit with the appropriate person's turn to move, so the strategy on S can be continued.

With the exception of v_3, Trailmaker never visited an isolated vertex. Within S, with the exception of v_5, this is true because of the inductive hypothesis. For v_5, which is the equivalent of the "exceptional" vertex v_3 on the smaller board S, we show that it was first visited by Breaker. Since v_4 was isolated before Breaker drew v_1v_4, Trailmaker will move v_4v_2 and in the next step Breaker is forced to draw v_2v_5. This means that v_5 will not be isolated when Trailmaker first draws an edge ending at v_5, following the strategy on S. (This will occur in the doublestep v_4v_5, v_5v_1.) There are only two vertices outside S, namely v_2 and v_3, and the isolated vertex v_2 was first visited by Breaker.

Thus we constructed an appropriate strategy for the board of size $2k+1$ and proved Theorem 3 for the case when Breaker starts.

Let us note that with this strategy Trailmaker used its extra power, the doublestep, only k times, which is linear as a function of the number of vertices.

When Trailmaker starts, she can reduce the game easily to a situation considered above, by drawing v_3v_1 as her second move (after the steps v_1v_2 by Trailmaker and v_2v_3 by Breaker). From here, she just continues with the strategy described above.

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