Free Edge Lengths in Plane Graphs

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ABSTRACT

We study the impact of metric constraints on the realizability of planar graphs. Let G be a subgraph of a planar graph H (where H is the "host" of G). The graph G is *free* in H if for every choice of positive lengths for the edges of G, the host H has a planar straight-line embedding that realizes these lengths; and G is *extrinsically free* in H if all constraints on the edge lengths of G depend on G only, irrespective of additional edges of the host H.

We characterize all planar graphs G that are free in every host H, $G \subseteq H$, and all the planar graphs G that are extrinsically free in every host H, $G \subseteq H$. The case of cycles $G = C_k$ provides a new version of the celebrated carpenter's rule problem. Even though cycles C_k, $k \geq 4$, are not extrinsically free in all triangulations, it turns out that "nondegenerate" edge lengths are always realizable, where the edge lengths are considered degenerate if the cycle can be flattened (into a line) in two different ways.

Separating triangles, and separating cycles in general, play an important role in our arguments. We show that every star is free in a 4-connected triangulation (which has no separating triangle).

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General Terms

Theory

1. INTRODUCTION

Representing graphs in Euclidean space such that some or all of the edges have given lengths has a rich history. For example, the rigidity theory of bar-and-joint frameworks, motivated by applications in mechanics, studies edge lengths that guarantee a unique (or locally unique) representation of a graph. Our primary interest lies in simple *combinatorial* conditions that guarantee realizations for all possible edge lengths. We highlight two well-known results similar to ours. (1) Jackson and Jordán [4, 12] gave a combinatorial characterization of graphs that are generically globally rigid (i.e., admit unique realizations for *arbitrary* generic edge lengths). (2) Connelly et al. [5] showed that a cycle C_k, $k \geq 3$, embedded in the plane can be continuously unfolded into a convex polygon (i.e., the configuration space of the planar embeddings of C_k is connected), solving the so-called carpenter's rule problem.

We consider straight-line embeddings of planar graphs where some of the edges can have arbitrary lengths. A *straight-line embedding* (for short, *embedding*) of a planar graph is a realization in the plane where the vertices are mapped to distinct points, and the edges are mapped to line segments between the corresponding vertices such that any two edges can intersect only at a common endpoint. By Fáry's theorem [9], every planar graph admits a straight-line embedding with *some* edge lengths. However, it is NP-hard to decide whether a planar graph can be embedded with prescribed edge lengths [7], even for planar 3-connected graphs with unit edge lengths [3], but it is decidable in linear time for triangulations [6] and near-triangulations [3]. Finding a straight-line embedding of a graph with prescribed edge lengths involves a fine interplay between topological, metric, combinatorial, and algebraic constraints. Determining the impact of each of these constraints is a challenging task. In this paper, we characterize subgraphs for which the metric constraints on the straight-line embedding remain independent from any topological, combinatorial, and algebraic constraints. Such subgraphs admit arbitrary positive edge lengths in an appropriate embedding of the host graph. This motivates the following definition.

DEFINITION 1. *Let $G = (V, E)$ be a subgraph of a planar graph H (the* host *of G). We say that*

- *G is* **free** *in H when, for every length assignment $\ell \colon E \to \mathbb{R}^+$, there is a straight-line embedding of the host H in which every edge $e \in E$ has length $\ell(e)$;*
- *G is* **extrinsically free** *in H when, for every length assignment $\ell \colon E \to \mathbb{R}^+$, if G has a straight-line embedding with edge lengths $\ell(e)$, $e \in E$, then H also has a straight-line embedding in which every edge $e \in E$ has length $\ell(e)$.*

Intuitively, if G is free in H, then there is no restriction on the edge lengths of G; and if G is *extrinsically free* in H, then all constraints on the edge lengths depend on G alone, rather than the edges in $H \setminus G$. Clearly, if G is free in H, then it is also extrinsically free in H. But not all extrinsically free subgraphs are free. For example, K_3 is not free since the edge lengths have to satisfy the triangle inequality, but it is an extrinsically free subgraph in K_4. It is easily verified that every subgraph with exactly two edges is always free (every pair of lengths can be attained by an affine transformation); but a triangle K_3 is not free in any host (due to the triangle inequality).

Results. We characterize all graphs G that are free as a subgraph of *every* host H.

THEOREM 1. *A planar graph $G = (V, E)$ is* free *in* every *planar host H, $G \subseteq H$, if and only if G consists of isolated vertices and*

- *a matching, or*
- *a forest with at most 3 edges, or*
- *the disjoint union of two paths, each with 2 edges.*

Separating 3- and 4-cycles in triangulations play an important role in our argument. A *star* is a graph $G = (V, E)$, where $V = \{v, u_1, \ldots, u_k\}$ and $E = \{vu_1, \ldots, vu_k\}$. We obtain the following result for stars in 4-connected triangulations (the proof is available in the full version of the paper).

THEOREM 2. *Every star in a 4-connected triangulation is free.*

If a graph G is free in H, then it is extrinsically free, as well. We completely classify graphs G that are extrinsically free in every host H.

THEOREM 3. *Let $G = (V, E)$ be a planar graph. Then G is* extrinsically free *in* every *host H, $G \subseteq H$, if and only if G consists of isolated vertices and*

- *a forest as listed in Theorem 1 (a matching, a forest with at most 3 edges, the disjoint union of two paths, each with 2 edges), or*
- *a triangulation, or*
- *a triangle and one additional edge (either disjoint from or incident to the triangle).*

When $G = C_k$ is a cycle with prescribed edge lengths, the realizability of a host H, $C_k \subset H$, leads to a variant of the celebrated carpenter's rule problem. Even though cycles on four or more vertices are not extrinsically free, all nonrealizable length assignments are *degenerate* in the sense that the cycle C_k, $k \geq 4$, decomposes into four paths of lengths (a, b, a, b) for some $a, b \in \mathbb{R}^+$. Intuitively, a length assignment on a cycle C_k is degenerate if C_k has two noncongruent embeddings in the line (that is, in 1-dimensions) with prescribed edge lengths. We show that every host H, $C_k \subset H$, is realizable with prescribed edge lengths of C_k, i.e., H admits a straight-line embedding in which every edge of C_k has its prescribed length, if the length assignment of C_k is nondegenerate.

THEOREM 4. *Let H be a planar graph that contains a cycle $C = (V, E)$. Let $\ell : E \to \mathbb{R}^+$ be a length assignment such that C has a straight-line embedding with edge lengths $\ell(e)$, $e \in E$. If ℓ is nondegenerate, then H admits a straight-line embedding in which every edge $e \in E$ has length $\ell(e)$.*

Organization. Our negative results (i.e., a planar graph G is *not always* free) are confirmed by finding specific hosts H, $G \subseteq H$, and length assignments that cannot be realized (Section 2). We give a constructive proof that every matching is free in all planar graphs (Section 3). In fact, we prove a slightly stronger statement: the edge lengths of a matching G can be chosen arbitrarily in every *plane* graph H with a fixed combinatorial embedding (that is, the edge lengths *and* the outer face can be chosen arbitrarily). The key tools are edge contractions and vertex splits, reminiscent of the technique of Fáry [9]. Separating triangles pose technical difficulties: we should realize the host H even if one edge of a separating triangle has to be very short, and an edge in its interior has to be very long. Similar problems occur when two opposite sides of a separating 4-cycle are short. We use grid embeddings and affine transformations to recursively construct embeddings for all separating 3- and 4-cycles (Section 3.2). All other subgraphs listed in Theorem 1 have at most 4 edges. We show directly that they are free in every planar host (Section 4). In Section 5 we show that for cycles with prescribed edge lengths any host H is realizable if the length assignment is nondegenerate. We conclude with open problems in Section 7.

Related Problems. As noted above, the embeddability problem for planar graphs with given edge lengths is NP-hard [3, 7], but efficiently decidable for near-triangulations [3, 6]. Patrignani [16] also showed that it is NP-hard to decide whether a straight-line embedding of a *subgraph G* (i.e., a *partial embedding*) can be extended to an embedding of a host H, $G \subset H$. For *curvilinear* embeddings, this problem is known as planarity testing for partially embedded graphs (PEP), which is decidable in polynomial time [2]. Recently, Jelínek et al. [13] give a combinatorial characterization for PEP via a list of forbidden substructures. Sauer [17, 18] considers similar problems in the context of structural Ramsey theory of metric embeddings: For an edge labeled graph G and a set $\mathcal{R} \subset \mathbb{R}^+$ that contains the labels, he derived conditions that ensure the existence of a metric space M on $V(G)$ whose distances extend the labeling in G.

Definitions. A *triangulation* is an edge-maximal planar graph with $n \geq 3$ vertices and $3n-6$ edges. Every triangulation has well-defined faces where all faces are triangles, since every triangulation is a 3-connected polyhedral graph for $n \geq 4$. A *near-triangulation* is a 3-connected planar graph in which all faces are triangles with at most one exception (which is typically the outer face). A 3-cycle t in a near-triangulation T is called a *separating triangle* if the vertices of t form a 3-cut in T. A triangulation T has no separating triangles iff T is 4-connected.

Tools from Graph Drawing. To show that a graph $G=(V,E)$ is free in every host H, $G \subseteq H$, we design algorithms that, for every length assignment $\ell\colon E \to \mathbb{R}^+$, construct a desired embedding of H. Our algorithms rely on several classical building blocks developed in the graph drawing community.

By Tutte's *barycenter embedding* method [20], every 3-connected planar graph admits a straight-line embedding in which the outer face is mapped to an arbitrarily prescribed convex polygon with the right number of vertices. Hong and Nagamochi [11] extended this result, and proved that every 3-connected planar graph admits a straight-line embedding in which the outer face is mapped to an arbitrarily prescribed star-shaped polygon with the right number of vertices.

A *grid-embedding* of a planar graph is an embedding in which the vertices are mapped to points in some small $h \times w$ section of the integer lattice $\mathbb{Z}^2$. For an n-vertex planar graph, the dimensions of the bounding box are $h, w \in O(n)$ [8, 19], which is the best possible [10]. The *angular resolution* of a straight-line embedding of a graph is the minimum angle subtended by any two adjacent edges. It is easy to see that the angular resolution of a grid embedding, where $h, w \in O(n)$, is $\Omega(n^{-2})$. By modifying an incremental algorithm by de Fraysseix et al. [8], Kurowski [14] constructed grid embeddings of n-vertex planar graphs on a $3n \times \frac{3}{2}n$ section of the integer lattice with angular resolution at least $\frac{\sqrt{2}}{3\sqrt{5}n} \in \Omega(1/n)$. Kurowski's algorithm embeds a n-vertex triangulation T with a given face (a,b,c) such that $a=(0,0)$, $b=(3n,0)$ and $c=(\lfloor 3n/2 \rfloor, \lfloor 3n/2 \rfloor)$. It has the following additional property used in our argument. When vertex c is deleted from the triangulation T, we are left with a 2-connected graph with an outer face $(a=u_1, u_2, \ldots, u_k=b)$. In Kurowski's embedding, the path $(a=u_1, u_2, \ldots, u_k=b)$ is x-monotone and the slope of every edge in this path is in the range $(-1,1)$.

2. SUBGRAPHS WITH CONSTRAINED EDGE LENGTHS

It is clear that a triangle is *not free*, since the edge lengths have to satisfy the triangle inequality in every embedding (they cannot be prescribed arbitrarily). This simple observation extends to all cycles.

OBSERVATION 1. *No cycle is free in any planar graph.*

Proof. Let C be a cycle with $k \geq 3$ edges in a planar graph H. If the first $k-1$ edges of C have unit length, then the length of the k-th edge is at most $k-1$ by repeated applications of the triangle inequality. □

OBSERVATION 2. *Let T be a triangulation with a separating triangle abc that separates edges e_1 and e_2. Then the subgraph G with edge set $E=\{ab, bc, e_1, e_2\}$ is not free in T. (See Fig. 1.)*

Proof. Since abc separates e_1 and e_2, in every embedding of T, one of e_1 and e_2 lies in the interior of abc. If ab and bc have unit length, then all edges of abc are shorter than 2 in every embedding (by the triangle inequality), and hence the length of e_1 or e_2 has to be less than 2. □

Based on Observations 1 and 2, we can show that most planar graphs G are not free in some appropriate triangulations T, $G \subseteq T$.

THEOREM 5. *Let $G=(V,E)$ be a forest with at least 4 edges, at least two of which are adjacent, such that G is not the disjoint union of two paths of length two. Then there is a triangulation T that contains G as a subgraph and G is not free in T.*

Proof. We shall augment G to a triangulation T such that Observation 2 is applicable. Specifically, we find four edges, $ab, bc, e_1, e_2 \in E$, such that either e_1 and e_2 are in distinct connected components of G or the (unique) path from e_1 to e_2 passes through a vertex in $\{a,b,c\}$. If we find four such edges, then G can be triangulated such that abc is a triangle (by adding edge ac), and it separates edges e_1 and e_2. See Fig. 1 for examples. We distinguish several cases based on the maximum degree $\Delta(G)$ of G.

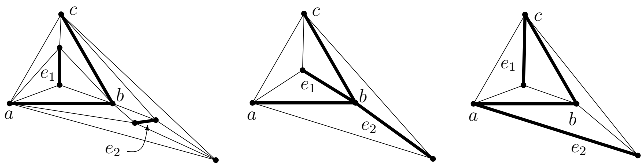

Figure 1: Triangulations containing a bold subgraph G with edges ab, bc, e_1 and e_2. In every embedding, one of e_1 and e_2 lies in the interior of triangle abc, and so $\min\{\ell(e_1), \ell(e_2)\} \leq \ell(ab)+\ell(bc)$. Left: G has four edges, two of which are adjacent. Middle: G is a star. Right: G is a path.

Case 1: $\Delta(G) \geq 4$. Let b be a vertex of degree at least 4 in G, with incident edges ab, bc, e_1 and e_2. Then e_1 and e_2 are in the same component of G, and the unique path between them contains b.

Case 2: $\Delta(G) = 3$. Let b be a vertex of degree 3, and let e_1 be an edge not incident to b. If e_1 and b are in the same connected component of G, then let ba be the first edge of the (unique) path from b to e_1; otherwise let ba be an arbitrary edge incident to b. Denote the other two edges incident to b by bc and e_2. This ensures that if e_1 and e_2 are in the same component of G, the unique path between them contains b.

Case 3: $\Delta(G) = 2$. If G contains a path with 4 edges, then let the edges of the path be (e_1, ab, bc, e_2). Now the (unique) path between e_1 and e_2 clearly contains a, b, and c, so we are done in this case. If a maximal path in G has 3 edges, then let these edges be (ab, bc, e_1), and pick e_2 arbitrarily from another component. Finally, if the maximal path in G has two edges, then let these edges be (ab, bc), and pick e_1 and e_2 from two distinct components (this is possible since G is not the edge-disjoint union of two paths of length two). □

3. EVERY MATCHING IS FREE

In this section, we show that every matching $M = (V, E)$ in every planar graph H is free. Given an arbitrary length assignment for a matching M of H, we embed H with the specified edge lengths on M. Our algorithm is based on a simple approach, which works well when M is "well-separated" (defined below). In this case, we contract the edges in M to obtain a triangulation $\widehat{H}$; embed $\widehat{H}$ on a grid $c\mathbb{Z}^2$ for a sufficiently large $c > 0$; and then expand the edges of M to the prescribed lengths. If $c > 0$ is large enough, then the last step is only a small "perturbation" of $\widehat{H}$, and we obtain a valid embedding of H with prescribed edge lengths. If, however, some edges in M appear in separating 3- or 4-cycles, then a significantly more involved machinery is necessary.

3.1 Edge Contraction and Vertex Splitting

A *near-triangulation* is a 3-connected planar graph in which all faces are triangles with at most one exception (which is typically considered to be the outer face). Let M be a matching in a planar graph H with a length assignment $\ell\colon M \to \mathbb{R}^+$. We may assume, by augmenting H if necessary, that H is a near-triangulation. Let D be an embedding of H where all the bounded faces are triangles. We will construct a new embedding of H with the *same* vertices on the outer face where every edge $e \in M$ has length $\ell(e)$.

Edge contraction is an operation for a graph $G = (V, E)$ and an edge $e = v_1v_2 \in E$: Delete v_1 and v_2 and all incident edges; add a new vertex $\hat{v}_e$; and for every vertex $u \in V \setminus \{v_1, v_2\}$ adjacent to v_1 or v_2, add a new edge $u\hat{v}_e$. Suppose G is a near triangulation and v_1v_2 does not belong to a separating triangle. Then v_1v_2 is incident to at most two triangle faces, say $v_1v_2w_1$ and $v_1v_2w_2$, and so there are at most two vertices adjacent to both v_1 and v_2. The cyclic sequence of neighbors of $\hat{v}_e$ is composed of the sequence of neighbors of v_1 from w_1 to w_2, and that of v_2 from w_2 to w_1 (in counterclockwise order). The inverse of an edge contraction is a *vertex split* operation that replaces a vertex $\hat{v}_e$ by an edge $e = v_1v_2$. See Fig. 2.

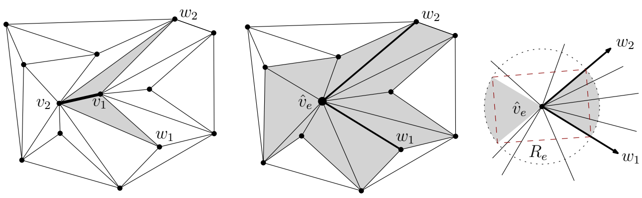

Figure 2: Left: An edge $e = v_1v_2$ of a near triangulation incident to the shaded triangles $v_1v_2w_1$ and $v_1v_2w_2$. Middle: e is contracted to a vertex $\hat{v}_e$. The triangular faces incident to $\hat{v}_e$ form a star-shaped polygon. Right: We position edge e such that it contains $\hat{v}_e$, and lies in the shaded double wedge, and in the kernel of the star-shaped polygon centered at $\hat{v}_e$. For simplicity, we consider only part of the double wedge, lying in a rectangle R_e of diameter 2ε.

Suppose that we are given an embedding of a triangulation, and we would like to split an interior vertex $\hat{v}_e$ into an edge $e = v_1v_2$ such that (1) all other vertices remain at the same location; and (2) the common neighbors of v_1 and v_2 are w_1 and w_2 (which are neighbors of $\hat{v}_e$). Note that the bounded triangles incident to $\hat{v}_e$ form a star-shaped polygon, whose kernel contains $\hat{v}_e$ in the interior. We position $e = v_1v_2$ in the kernel of this star-shaped polygon such that the line segment e contains the point $\hat{v}_e$, and vertices w_1 and w_2 are on opposite sides of the supporting line of e. Therefore, e must lie in the double wedge between the supporting lines of $\hat{v}_ew_1$ and $\hat{v}_ew_2$ (Fig. 2, right). In Subsection 3.2, we position $e = v_1v_2$ such that its midpoint is $\hat{v}_e$; and in Section 4, we place either v_1 or v_2 at $\hat{v}_e$, and place the other vertex in the appropriate wedge incident to $\hat{v}_e$.

3.2 Proof of Theorem 7

We now recursively prove that every matching in every planar graph is free. In one step of the recursion, we construct an embedding of a subgraph in the interior of a separating triangle (resp., a separating 4-cycle), where the length of one edge is given (resp., the lengths of two edges are given). The work done for a separating triangle or 4-cycle is summarized in the following lemma.

Lemma 6. *Let $H = (V, E)$ be a near-triangulation and let $M \subset E$ be a matching with a length assignment $\ell\colon M \to \mathbb{R}^+$.*

(1) Suppose that a 3-cycle (v_1, v_2, v_3), where $v_1v_2 \in M$, is a face of H. There is an $L > 0$ such that for every triangle abc with side length $|ab| = \ell(v_1v_2)$, $|bc| > L$ and $|ca| > L$, there is an embedding of H with prescribed edge lengths where the outer face is abc and v_1, v_2 and v_3 are mapped to a, b and c, respectively.

(2) Suppose that a 4-cycle (v_1, v_2, v_3, v_4), where $v_1v_2 \in M$ and $v_3v_4 \in M$, is a face of H. There is an $L > 0$ such that for every convex quadrilateral $abcd$ with side lengths $|ab| = \ell(v_1v_2)$, $|cd| = \ell(v_3v_4)$, $|ac| > L$, there is an embedding of H with prescribed edge lengths where the outer face is $abcd$ and v_1, v_2, v_3 and v_4 are mapped to a, b, c and d, respectively.

Proof. We proceed by induction on the size of the matching M. We may assume, by applying an appropriate scaling, that $\min\{\ell(e) : e \in M\} = 1$.

(1) Consider an embedding D of H where $v_1v_2 \in M$ is an edge of the outer face, and let $M' = M \setminus \{v_1v_2\}$. Let $C_1, \ldots, C_k$ be the maximal separating triangles that include some edge from M', and the chordless separating 4-cycles that include two edges from M' (more precisely, we consider all such separating triangles and separating chordless 4-cycles and among them we choose those that are not contained in the interior of any other such separating triangle or chordless 4-cycle). Let H_0 be the subgraph of H obtained by deleting all vertices and incident edges lying in the interiors of the cycles $C_1, \ldots, C_k$. Let $M_0 \subseteq M'$ denote the subset of edges of M' contained in H_0. Let

$$\lambda_0 = \max\{\ell(e) : e \in M_0\}. \tag{1}$$

For $i = 1, \ldots, k$, let H_i denote the subgraph of H that consists of the cycle C_i and all vertices and edges that lie in C_i in the embedding D; and let $M_i \subset M'$ be the subset of edges of M' in H_i. Applying induction for H_i and M_i, there is an $L_i > 0$ such that H_i can be embedded with the prescribed lengths for the edges of M_i in every triangle (resp., convex quadrilateral) with two edges of lengths at least L_i. Let $L' = \max\{L_i : i = 1, \ldots, k\}$.

By construction, M_0 is a well-separated matching in H_0 (recall that v_1v_2 is not in M_0). Successively contract every edge $e = uv \in M_0$ to a vertex $\hat{v}_e$. We obtain a planar graph $\widehat{H}_0 = (\widehat{V}_0, \widehat{E}_0)$ on at most n (and at least 3) vertices.

Let $\widehat{D}_0$ be a grid embedding of $\widehat{H}_0$ constructed by the algorithm of Kurowski [14], where the outer face is a triangle with vertices $(0,0)$, $(3n-7,0)$, and $(\lfloor\frac{3n-7}{2}\rfloor, \lfloor\frac{3n-7}{2}\rfloor)$; the only horizontal edge is the base of the outer triangle; and the angular resolution of $\widehat{D}_0$ is $\varrho \geq \frac{\sqrt{2}}{3\sqrt{5}n} \in \Omega(1/n)$. The minimum edge length is 1, since all vertices have integer coordinates. There is an $\varepsilon \in \Omega(1/n)$ such that if we move each vertex of $\widehat{D}_0$ by at most ε, then the directions of the edges change by an angle less than $\varrho/2$, and thus we retain an embedding. We could split each vertex $\hat{v}_e$, $e \in M$, into an edge e that lies in the ε-disk centered at $\hat{v}_e$, and in the double wedge determined by the edges between $\hat{v}_e$ and the common neighbors of the endpoints of e (Fig. 2, right). However, we shall split the vertices $\hat{v}_e$, $e \in M$, only after applying the affine transformation α that maps the outer triangle of $\widehat{D}_0$ to a triangle abc such that $\alpha(v_1) = a$, $\alpha(v_2) = b$ and $\alpha(v_3) = c$. (The affine transformation α would distort the prescribed edge lengths if we split the vertices now.)

In the grid embedding $\widehat{D}_0$, the central angle of such a double wedge is at least $\varrho \in \Omega(1/n)$, i.e., the angular resolution of $\widehat{D}_0$. The boundary of the double wedge intersects the boundary of the ε-disk in four vertices of a rectangle that we denote by R_e. Note the center of R_e is $\hat{v}_e$, and its diameter is $2\varepsilon \in \Omega(1/n)$. Hence, the aspect ratio of each R_e, $e \in M_0$, is at least $\tan(\varrho/2) \in \Omega(1/n)$, and the width of R_e is $\Omega(1/n^2)$.

We show that if $L = \max\{10n(L' + 2\lambda_0 + |ab|), \xi n^3\lambda_0\}$, for some constant $\xi > 0$, then the affine transformation α satisfies the following two conditions. The first condition allows splitting the vertices $\hat{v}_e$, $e \in M$, into edges of desired lengths, and the second one ensures that the existing edges remain sufficiently long after the vertex splits.

(i) every rectangle R_e, $e \in M_0$, is mapped to a parallelogram $\alpha(R_e)$ of diameter at least λ_0 (defined in (1));

(ii) every nonhorizontal edge in $\widehat{D}_0$ is mapped to a segment of length at least $L' + 2\lambda_0$.

For (i), note that α maps a grid triangle of diameter $3n - 7 < 3n$ into triangle abc of diameter L. Hence, it stretches every vector parallel to the preimage of the diameter of abc by a factor of at least $L/(3n)$. Since the width of a rectangle R_e, $e \in M_0$, is $\Omega(1/n^2)$, the diameter of $\alpha(R_e)$ is at least $\Omega(L/n^3)$. If $L \in \Omega(n^3\lambda_0)$ is sufficiently large, then the diameter of every $\alpha(R_e)$ is at least λ_0.

For (ii), we may assume w.l.o.g. that the triangle abc is positioned such that $a = (0,0)$ is the origin, $b = (|ab|, 0)$ is on the positive x-axis, and c is above the x-axis (i.e., it has a positive y-coordinate). Then, the affine transformation α is a linear transformation with an upper triangular matrix:

$$\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cy \end{bmatrix},$$

where $A, C > 0$, and by symmetry we may assume $B \geq 0$. We show that if $L \geq 10n(L' + 2\lambda_0 + |ab|)$, then α maps every nonhorizontal edge of $\widehat{D}_0$ to a segment of length at least $L' + 2\lambda_0$.

A nonhorizontal edge in the grid embedding $\widehat{D}_0$, directed upward, is an integer vector (x, y) with $x \in [-3n+7, 3n-7]$ and $y \in [1, \frac{3n-7}{2}]$. It is enough to show that $(Ax + By)^2 + (Cy)^2 > (L' + 2\lambda_0)^2$ for $x \in [-3n, 3n]$ and $y \in [1, \frac{3}{2}n]$. Since α maps the right corner of the outer grid triangle $(3n-7, 0)$ to $b = (|ab|, 0)$, we have $A = |ab|/(3n-7)$. Since $|ac| > L$, where $a = (0,0)$ and $c = \alpha\left((\lfloor\frac{3n-7}{2}\rfloor, \lfloor\frac{3n-7}{2}\rfloor)\right)$, we have

$$\left(A \cdot \frac{3n-7}{2} + B \cdot \frac{3n-7}{2}\right)^2 + \left(C \cdot \frac{3n-7}{2}\right)^2 = $$
$$= |ac|^2 > L^2 \geq 100n^2(L' + 2\lambda_0 + |ab|)^2. \qquad (2)$$

We distinguish two cases based on which term is dominant in the left hand side of (2).

Case 1: $(C \cdot \frac{3n-7}{2})^2 \geq 50n^2(L' + 2\lambda_0 + |ab|)^2$. In this case, we have $C^2 > (L' + 2\lambda_0)^2$, and so $(Cy)^2 > (L' + 2\lambda_0)^2$ since $y \geq 1$.

Case 2: $(A \cdot \frac{3n-7}{2} + B \cdot \frac{3n-7}{2})^2 > 50n^2(L' + 2\lambda_0 + |ab|)^2$. In this case, we have $A \cdot \frac{3n-7}{2} + B \cdot \frac{3n-7}{2} > 7n(L' + 2\lambda_0 + |ab|)$. Combined with $A = |ab|/(3n-7)$, this gives $B > 4(L' + 2\lambda_0 + |ab|)$. It follows that $(Ax + By)^2 > (L' + 2\lambda_0)^2$, as claimed, since $|Ax| \leq |ab|$ and $y \geq 1$.

We can now reverse the edge contraction operations, that is, split each vertex $\hat{v}_e$, $e \in M_0$, into an edge e of length $\ell(e)$ within the parallelogram $\alpha(R_e)$. By (i), we obtain a embedding of H_0. Each cycle C_i, $i = 1, \ldots, k$, is a triangle (resp., quadrilateral) where the edges of M_0 have prescribed lengths, and any other edge has length at least $L' = \max\{L_i : i = 1, \ldots, k\}$ by (ii). By induction, we can insert an embedding of H_i with prescribed lengths on the matching M_i into the embedding of the cycle C_i, for $i = 1, \ldots, k$. We obtain the required embedding of H.

(2) The proof for the case when the outer face of H is a 4-cycle follows the same strategy as for (1), with some additional twists.

Suppose we are given a convex quadrilateral $abcd$ as described in the statement of the lemma. Denote by q the intersection of its diagonals. We show that ($|aq|$ and $|bq|$ are both at least $L/3$) or ($|cq|$ and $|dq|$ are both at least $L/3$) if $L > 9\max(|ab|, |cd|)$. Indeed, we have $|ac| > |bc| - |ab| > \frac{8}{9}L$ from the triangle inequality for abc. Since $|ac| = |aq| + |cq|$, we have $|aq| > \frac{4}{9}L$ or $|cq| > \frac{4}{9}L$. If $|aq| > \frac{4}{9}L$, then $|bq| > |aq| - |ab| > \frac{1}{3}L$ from the triangle inequality for abq; otherwise $|dq| > |cq| - |cd| > \frac{1}{3}L$. Assume w.l.o.g. that $|aq| > L/3$ and $|bq| > L/3$. In the remainder of the proof, we embed H such that almost all vertices lie in the triangle abq, and the vertices v_1, v_2, v_3, and v_4 are mapped to a, b, c, and d, respectively.

Similarly to (1), we define H_0 as the graph obtained by deleting all vertices and incident edges lying in the interior of maximal separating triangles or chordless 4-cycles, containing an edge from $M \setminus \{v_1v_2\}$. Define L' as before, by using the inductive hypothesis in the separating cycles. Contract successively all remaining edges of $M \setminus \{v_1v_2\}$ that are in H_0 (including edge v_3v_4) to obtain a graph $\hat{H}_0$. Denote by $\hat{v}_3$ the vertex of $\widehat{H}_0$ corresponding to $v_3v_4 \in M$, and consider an embedding of $\overline{H}$ with the outer face $v_1v_2\hat{v}_3$.

We again use the embedding $\widehat{D}_0$ of Kurowski [14], such that v_1, v_2 and $\hat{v}_3$ are mapped to $(0,0)$, $(3n-7, 0)$, and $(\lfloor\frac{3n-7}{2}\rfloor, \lfloor\frac{3n-7}{2}\rfloor)$, respectively. We first split vertex $\hat{v}_3$ into two vertices v_3 and v_4, exploiting the fact that $\hat{v}_3$ is a boundary vertex in $\widehat{D}_0$ and some special properties of the embedding in [14] (described below); and then split all other contracted vertices of $\hat{H}_0$ similarly to (1).

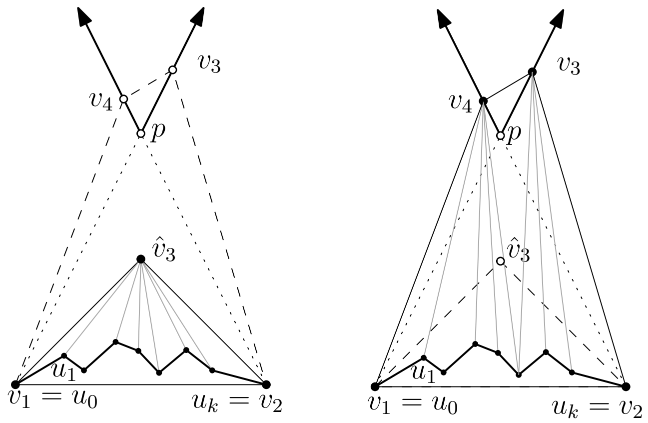

Figure 3: Left: The embedding $\widehat{D}_0$ into a triangle $v_1 v_2 \hat{v}_3$, and the x-monotone path $v_1 = u_0, u_1, \ldots, u_k$ formed by the neighbors of $\hat{V}_3$. A point p lies above $\hat{v}_3$, and the rays emitted by p in directions $(1, 2)$ and $(-1, 2)$. Right: Vertex $\hat{v}_3$ is split into v_3 and v_4 on the two rays emitted by p.

Denote the neighbors of $\hat{v}_3$ in $\widehat{D}_0$ in counterclockwise order by $v_1 = u_0, u_1, \ldots, u_k = v_2$ (Fig. 3, left). The grid embedding in [14] has the following property (mentioned in Section 1): the path $u_0, \ldots, u_k$ is x-monotone and the slope of every edge is in the range $(-1, 1)$. Let $p = (\lfloor \frac{3n-7}{2} \rfloor, 2\lfloor \frac{3n-7}{2} \rfloor)$, and note that the slope of every line between p and $u_1, \ldots, u_k$, is outside of the range $(-2, 2)$. Similarly, if we place the points v_3 (resp., v_4) on the ray emitted by p in direction $(1, 2)$ (resp., $(-1, 2)$), then the slope of every line between v_3 (resp., v_4) and $u_1, \ldots, u_k$ is outside of $(-2, 2)$.

We can now split vertex $\hat{v}_3$ as follows. Refer to Fig. 3. Let α be the affine transformation that maps the triangle $v_1 v_2 p$ to abq such that $\alpha(v_1) = a$, $\alpha(v_2) = b$ and $\alpha(p) = q$. Since the diagonals ac and ce intersect at q, the segments $v_1\alpha^{-1}(c)$ and $v_2\alpha^{-1}(d)$ intersect at p. We split vertex $\hat{v}_3$ into $v_3 = \alpha^{-1}(c)$ and $v_4 = \alpha^{-1}(d)$. By the above observation, the edges incident to v_3 and v_4 remain above the x-monotone path $u_0, \ldots, u_k$. (Note, however, that the angles between edges incident with v_3 or v_4 may be arbitrarily small.)

With a very similar computation as for (1), we conclude that for a large enough $L \in \Omega(L' + \lambda_0)$ we can guarantee the same two properties we needed in (1), that is, α maps every small rectangle R_e to a parallelogram $\alpha(R_e)$ whose diameter is at least λ_0, and every nonhorizontal edge to a segment of length at least $L' + 2\lambda_0$. Hence, every remaining contracted vertex v_e in $\widehat{D}_0$ can be split within the parallelogram $\alpha(R_e)$ as in (1). To finish the construction, it remains to apply the inductive hypothesis to fill in the missing parts in the maximal separating triangles or 4-cycles. $\square$

We are now ready to prove the main result of this section.

THEOREM 7. *Every matching in a planar graph is free.*

Proof. Let $H = (V, E)$ be a planar graph, and let $M \subseteq E$ be a matching with a length assignment $\ell \colon M \to \mathbb{R}^+$. We may assume, by augmenting H with new edges if necessary, that H is a triangulation. Consider an embedding of H such that an edge $e \in M$ is on the outer face. Now Lemma 6 completes the proof. $\square$

4. GRAPHS WITH 3 OR 4 EDGES

By Theorems 5 and 7, a graph G with at least five edges is free in every host H if and only if G is a matching. For graphs with four edges, the situation is also clear except for the case of the disjoint union of two paths of two edges each. In this section we show that every forest with three edges, as well as the disjoint union of two paths of length two are always free.

We show (Lemma 9) that it is enough to consider hosts H in which G is a spanning subgraph, that is, $V(G) = V(H)$. For a planar graph $G = (V, E)$, the *triangulation of* G is an edge-maximal planar graph T, $G \subset T$, on the vertex set V.

LEMMA 8. *If G is a subgraph of a triangulation H with $0 < |V(G)| < |V(H)|$, then there is an edge in H between a vertex in $V(H)$ and a vertex in $V(H) - V(G)$ that does not belong to any separating triangle of H.*

Proof. Let $V = V(G)$ denote the vertex set of G and $U = V(H) \setminus V$. Let $E(U, V)$ be the set of edges in H between U and V. Since H is connected, $E(U, V)$ is nonempty. Consider an arbitrary embedding of H (with arbitrary edge lengths). For every edge $uv \in E(U, V)$, let $k(uv)$ denote the maximum number of vertices of H that lie in the interior of a triangle (u, v, w) of H. Let $uv \in E(U, V)$ be an edge that minimizes $k(uv)$. If $k(uv) = 0$, then uv does not belong to any separating triangle, as claimed. For the sake of contradiction, suppose $k(uv) > 0$, and let (u, v, w) be a triangle in H that contains exactly $k(uv)$ vertices of H. Since H is a triangulation, there is a path between u and v via the interior of (u, v, w). Since $u \in U$ and $v \in V$, one edge of this path must be in $E(U, V)$, say $u'v' \in E(U, V)$. Note that any triangle (u', v', w') of H lies inside the triangle (u, v, w), and hence contains strictly fewer vertices than (u, v, w). Hence $k(u'v') < k(u, v)$ contradicting the choice of edge uv. $\square$

LEMMA 9. *If a planar graph G is free (or extrinsically free) in every triangulation of G, then G is free (or extrinsically free, respectively) in every planar host H, $G \subseteq H$.*

Proof. Let $G = (V, E)$ be a planar graph with a length assignment $\ell : E \to \mathbb{R}^+$. It is enough to prove that G is (extrinsically) free in every triangulation H, $G \subset H$. We proceed by induction on $n' = |V(H)| - |V(G)|$, the number of extra vertices in the host H. If $n' = 0$, then H is a triangulation *of* G, and G is free in H by assumption. Consider a triangulation H, $G \subset H$, and assume that the claim holds for all smaller triangulations H', $G \subset H'$.

By Lemma 8, there is an edge $e = uv$ in H between $v \in V(G)$ and $u \in V(H) - V(G)$ that does not belong to any separating triangle. Contract e into a vertex $\hat{v}_e$ to obtain a triangulation H', $G \subset H'$. By induction, H' admits a straight-line embedding in which the edges of G have prescribed lengths. Since e is not part of a separating triangle of H', we can split vertex $\hat{v}_e$ into u and v such that v is located at point $\hat{v}_e$, and v lies in a sufficiently small neighborhood of $\hat{v}_e$ (refer to Fig. 2). Thus, we obtained a straight line embedding of H in which edges of G have prescribed lengths. $\square$

The next theorem finishes the characterization of free graphs.

THEOREM 10. *Let G be a subgraph of a planar graph H, such that G is*

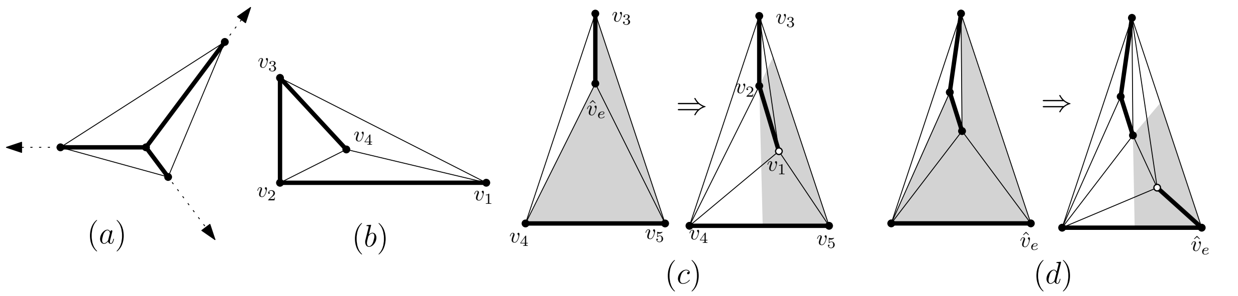

Figure 4: (a) Embedding of a star with three leaves. (b) Embedding of a path of three edges. (c) Graph H' and regions that are used for splitting $\hat{v}_e$. (d) Graph H' and its spanning subgraph G whose edges have prescribed lengths.

(1) the star with three edges; or

(2) the path with three edges; or

(3) the disjoint union of a path with two edges and a path with one edge; or

(4) the disjoint union of two paths with two edges each.

Then G is free in H.

Proof. By Lemma 9 it is enough to prove the theorem in the case when G is a spanning subgraph of H. We can also assume that H is a triangulation.
(1) If G is the star with three edges, then H is K_4. Embed the center of the star at the origin. Place the three leaves lie on three rotationally symmetric rays emitted by the origin, at prescribed lengths from the origin (Fig. 4(a)). The remaining three edges are embedded as straight line segments on the convex hull of the three leaves.
(2) Let G be the path (v_1, v_2, v_3, v_4) with $\ell(v_1v_2) \geq \ell(v_3v_4)$. Embed v_2 at the origin, place v_1 and v_3 on the positive x- and y-axis respectively, at prescribed distance from v_2. Note that $\Delta = \text{conv}(v_1, v_2, v_3)$ is a right triangle whose diameter (hypotenuse) is larger than the other two sides (Fig. 4(a)). Thus we can embed v_4 at a point in the interior of Δ at distance $\ell(v_3v_4)$ from v_3. Since the four vertices have a triangular convex hull, $H = K_4$ embeds as a straight-line graph.
(3) Suppose that $G = (V, E)$ is the disjoint union of path (v_1, v_2, v_3) and (v_4, v_5). Since H has five vertices there exists at most one separating triangle in H. Thus, the path of G with two edges contains an edge, say $e = v_1v_2$, that does not belong to any separating triangle. Contract edge e to a vertex $\hat{v}_e$, obtaining a triangulation $H' = K_4$ on four vertices, and a perfect matching $G' \subset H'$. Let us embed the two edges of G' with prescribed lengths such that one lies on the x-axis, the other lies on the orthogonal bisector of the first edge at distance $\ell(e)$ from the x-axis. This defines a straight-line embedding of H', as well. We obtain a desired embedding of H by splitting vertex $\hat{v}_e$ into edge e such that v_2 is embedded at point $\hat{v}_e$ and v_1 is mapped to a point in the kernel of the appropriate star-shaped polygon (c.f. Fig. 2). By the choice of our embedding of H', the diameter of this kernel is more than $\ell(e)$, and we can split $\hat{v}_e$ without introducing any edge crossing (Fig.4(c))
(4) Assume that G is the disjoint union of two paths P_1 and P_2, each with two edges. Since G is a spanning subgraph of H, neither path can span a separating triangle. Moreover, as there exist at most two separating triangles in H. One of the paths, say P_1, contains an edge e that is not part of a separating triangle. Contract edge e to $\hat{v}_e$, obtaining a triangulation H' and a subgraph G'. Similarly to the case (3), embed H' respecting the lengths of all the edges of G' such that all edges between the two components of G' have length at least $\ell(e)$. By the choice of our drawing of H', the kernel of the appropriate star-shaped polygon has diameter at least $\ell(e)$. Therefore, we can split e into two vertices such that the middle vertex of P_1 remains at $\hat{v}_e$, and the endpoint of P_1 is embedded at distance $\ell(e)$ from $\hat{v}_e$ (Fig. 4(d)). □

5. EMBEDDING A CYCLE WITH NONDEGENERATE LENGTHS

We say that a length assignment $\ell : E \to \mathbb{R}^+$ for a cycle $C = (V, E)$ is *feasible* if C admits a straight-line embedding with edge length $\ell(e)$ for all $e \in E$. Lenhart and Whitesides [15] showed that ℓ is feasible for C iff no edge is supposed to be longer than the semiperimeter $s = \frac{1}{2}\sum_{e \in E} \ell(e)$. Recall that three positive reals, a, b and c, satisfy the triangle inequality iff each of them is less than $\frac{1}{2}(a + b + c)$.

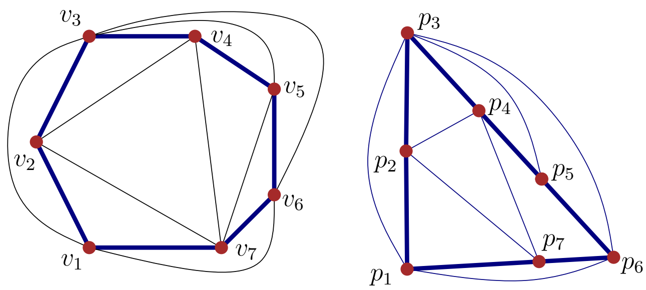

Figure 5: Left: A planar graph H with a Hamilton cycle C (think lines). Right: The graph H has a 3-cycle (1,3,6) such that C admits a straight-line embedding with the same edge lengths as in the left and all edges of C are along the edges of triangle (1,3,6).

By Lemma 9, it is enough to prove Theorem 4 in the case when C is a Hamilton cycle in H. Consider a Hamilton cycle C in a triangulation H. We construct a straight-line embedding of H with given nondegenerate edge lengths using the following two-step strategy. We first embed C on the boundary of a triangle T such that each edge of $H - C$ is either an internal diagonal of C or a line segment along one of the sides of the triangle T (Lemma 11). If any edge of $H - C$ overlaps with edges C, then this is not a proper embedding of H yet. In a second step, we perturb the embedding of C to accommodate all edges of H (see Section 5.1).

LEMMA 11. *Let H be a triangulation with a Hamilton cycle $C = (V, E)$ and a feasible nondegenerate length assignment $\ell : E \to \mathbb{R}^+$. Then, there is a 3-cycle (v_i, v_j, v_k) in H such that the prescribed arc lengths of C between these vertices, i.e., the three sums of lengths of edges corresponding to these three arcs, satisfy the triangle inequality.*

Proof. Consider an arbitrary embedding of H (with arbitrary lengths). The edges of H are partitioned into three subsets: edges E of the cycle C, interior chords E_{int} and exterior chords E_{ext}. Each chord $v_iv_j \in E_{\text{int}} \cup E_{\text{ext}}$ decomposes C into two paths. If the length assignment ℓ is nondegenerate, then there is at most one chord $v_iv_j \in E_{\text{int}} \cup E_{\text{ext}}$ that decomposes C into two paths of equal length. Assume, by exchanging interior and exterior chords if necessary, that no edge in E_{ext} decomposes C into two paths of equal length.

Denote by $\delta_{ij} > 0$ the absolute value of the difference between the sums of the prescribed lengths on the two paths that an exterior chord v_iv_j produces. Let $v_iv_j \in E_{\text{ext}}$ be an exterior chord that minimizes δ_{ij}. The chord v_iv_j is adjacent to two triangles, say $(v_iv_jv_k)$ and $(v_iv_jv_{k'})$, where v_k and $v_{k'}$ are vertices of two different paths determined by v_iv_j. Assume, without loss of generality, that v_k is part of the longer path (measured by the prescribed length). The path length between v_i and v_k (resp., v_j and v_k) cannot be less than δ_{ij} otherwise $\delta_{kj} < \delta_{ij}$ (resp., $\delta_{ik} < \delta_{ij}$). Therefore the three arcs between v_i, v_j, and v_k satisfy the triangle inequality. $\square$

5.1 A Hamilton Path with Given Edge Lengths

Our main tool to "perturb" a straight-line drawing with collinear edges is the following lemma.

LEMMA 12. *Let H be a planar graph with $n \geq 3$ vertices and a fixed combinatorial embedding; let $P = (V, E)$ be a Hamilton path in H with both of its endpoints incident to the outer face of H; and let $\ell : E \to \mathbb{R}^+$ be a length assignment with $L = \sum_{e \in E} \ell(e)$ and $\ell_{\min} = \min_{e \in E} \ell(e)$.*

For every $\varepsilon \in (0, \ell_{\min})$, H admits a straight-line embedding such that the two endpoints of P are at points origin $(0, 0)$ and $(0, L - \varepsilon)$ on the x-axis, and every edge $e \in E$ has length $\ell(e)$.

Proof. We proceed by induction on $n = |V|$, the number of vertices of H. The base case is $n = 3$, where P consists of a single edge, $P = H$, and we can place the two endpoints of P at $(0, 0)$ and $(L - \varepsilon, 0)$. Assume now that $n > 3$ and the claim holds for all instances where H has fewer than n vertices.

We may assume, by adding dummy edges if necessary, that H is a triangulation. Denote the vertices of the path P by $(v_1, v_2, \ldots, v_n)$. By assumption, the endpoints v_1 and v_n are incident to the outer face (i.e., outer triangle). Denote by v_k, $1 < k < n$, the third vertex of the outer triangle. Let $P_1 = (v_1, \ldots, v_k)$ and $P_2 = (v_k, \ldots, v_n)$ be two subpaths of P, with total lengths $L_1 = \sum_{i=1}^{k-1} \ell(v_iv_{i+1})$ and $L_2 = \sum_{i=k}^{n-1} \ell(v_iv_{i+1})$. We may assume without loss of generality that $L_1 \leq L_2$. We may assume, by applying a reflection if necessary, that the triple (v_1, v_k, v_n) is clockwise in the given embedding of H. Let H_1 (resp., H_2) be the subgraph of H induced by the vertices of P_1 (resp., P_2); and let $E_{1,2}$ denote the set of edges of H between $\{v_1, \ldots, v_{k-1}\}$ and $\{v_{k+1}, \ldots, v_n\}$. In the remainder of the proof, we embed P_1 and P_2 by induction, after choosing appropriate parameters ε_1 and ε_2.

We first choose "preliminary" points p_i for each vertex v_i as follows. Let (p_1, p_k, p_n) be a triangle with clockwise orientation, where $p_1 = (0, 0)$, $p_n = (L - \varepsilon, 0)$, and the edges p_1p_k and p_kp_n have length L_1 and L_2 respectively. (See Fig. 6.) Place the points $p_2, \ldots, p_{k-1}$ on segment p_1p_k, and the points $p_{k+1}, \ldots, p_{n-1}$ on segment p_kp_n such that the distance between consecutive points is $|p_ip_{i+1}| = \ell(v_iv_{i+1})$ for $i = 1, \ldots, n - 1$.

Note that segment p_1p_k has a positive slope, say $\overline{s}$; and p_kp_n has negative slope, $\underline{s}$. The slope of every segment p_ip_j, for $v_iv_j \in E_{1,2}$, is in the open interval $(\underline{s}, \overline{s})$. Let $[\underline{r}, \overline{r}]$ be the smallest closed interval that contains the slopes of all segments p_ip_j for $v_iv_j \in E_{1,2}$. Let $\overline{t} \in (\overline{r}, \overline{s})$ and $\underline{t} \in (\underline{s}, \underline{r})$ be two arbitrary reals that "separate" the sets of slopes. We shall perturb the vertices $p_2, \ldots, p_{n-1}$ such that the directions of the edges of H_2, $E_{1,2}$, and H_1 remain in pairwise disjoint intervals $(2\underline{s}, \underline{t})$, $(\underline{t}, \overline{t})$, and $(\overline{t}, 2\overline{s})$, respectively.

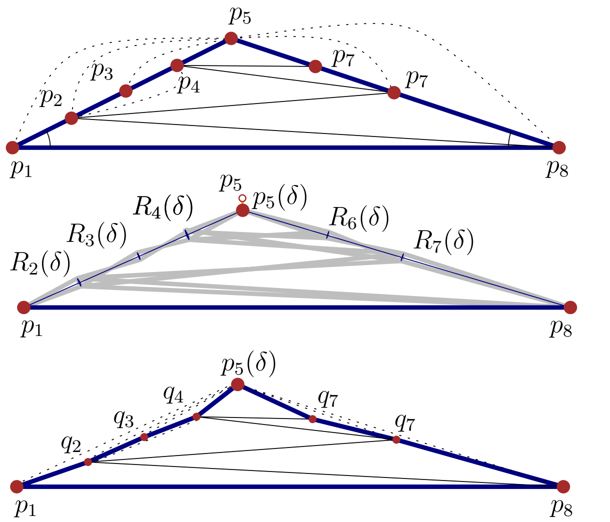

Figure 6: Top: A path $P = (p_1, \ldots p_8)$ embedded on the boundary of a triangle (p_1, p_5, p_8) with prescribed edge lengths. The edges of $H - P$ between different sides of the triangle are solid thin lines, the edges of $H - P$ between vertices of the same side of the triangle are dotted. Middle: When point p_5 is shifted down to $p_5(\delta)$, then in any embedding of C with prescribed edge lengths, vertex p_i is located in a region $R_i(\delta)$ for $i = 2, 3, 4, 6, 7$. Bottom: A straight-line embedding of H is obtained by embedding the subgraphs induced by $P_1 = (p_1, \ldots, p_5)$ and $P_2 = (p_5, \ldots, p_8)$ by induction.

Suppose that we move point p_k to position $p_k(\delta) = p_k + (0, -\delta)$. In any straight-line embedding of P_1 with $v_1 = p_1$ and $v_k = p_k(\delta)$, each vertex v_i, $i = 2, \ldots, k-1$, must lie in a region $R_i(\delta)$, which is the intersection of two disks centered at p_1 and $p_k(\delta)$ of radius $|p_1p_i|$ and $|p_ip_k|$, respectively (Figure). Similarly, in any straight-line embedding of P_2 with

$v_k = p_k(\delta)$ and $v_n = p_n$, each vertex v_i, $i = k+1, \ldots, n-1$, must lie in a region $R_i(\delta)$, which is the intersection of two disks centered at $p_k(\delta)$ and p_n of radius $|p_k p_i|$ and $p_i p_n|$, respectively. We also define one-point regions $R_1(\delta) = \{p_1\}$, $R_k(\delta) = \{p_k(\delta\}$, and $R_n(\delta) = \{p_n\}$. Choose a sufficiently small $\delta > 0$ such that the slope of any line intersecting $R_i(\delta)$ and $R_j(\delta)$ is in the interval

- $(\bar{t}, 2\bar{s})$ if $1 \leq i < j \leq k$;
- $(\underline{t}, \bar{t})$ if $v_i v_j \in E_{1,2}$;
- $(2\underline{s}, \underline{t})$ if $k \leq i < j \leq n$.

Embed vertices v_1, v_k and v_n at points p_1, $p_k(\delta)$ and p_n, respectively. If H_1 (resp., H_2) has three or more vertices, embed it by induction such that the endpoints of path P_1 are p_1 and $p_k(\delta)$ (resp., the endpoints of P_2 are $p_k(\delta)$ and p_n). Each vertex v_i is embedded in a point in the region R_i, for $i = 1, \ldots n$. By the choice of δ, the slopes of the edges of H_1 and H_2 are in the intervals $(\bar{t}, 2\bar{s})$ and $(2\underline{s}, \underline{t})$, respectively, while the slopes of the edges in $E_{1,2}$ are in a disjoint interval $(\underline{t}, \bar{t})$. Thus, these edges are pairwise noncrossing, and we obtain a proper embedding of graph H. □

6. CONSTRUCTIONS WITH DEGENERATE LENGTHS

In this section, we show that cycles on more than three vertices are not extrinsically free. For an integer $k \geq 4$, we define the graph H_n on the vertex set $\{v_1, v_2, \ldots, v_n\}$ as a union of a Hamilton cycle $C_n = (v_1, v_2, \ldots, v_n)$, and two spanning stars centered at v_1 and v_n respectively. Note that H_n is planar: the two stars can be embedded in the interior and the exterior of an arbitrary embedding of C_n. Fig. 7 (left) depicts a straight-line embedding of H_6.

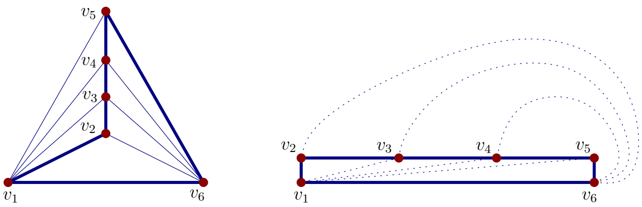

Figure 7: Left: A straight-line embedding of H_6. Right: A straight-line embedding of C_6 with prescribed edge lengths.

We show that C_n is not extrinsically free in the host H_n. Consider the following length assignment on the edges of C_n: let $\ell(v_1 v_2) = \ell(v_{n-1} v_n) = \frac{1}{4}$, $\ell(v_i, v_{i+1}) = 1$ for $i = 2, 3, \ldots, n-2$, and $\ell(v_1 v_n) = n-3$. Fig. 7 shows a straight-line embedding of C_n with the prescribed edge lengths. In the full version of the paper, we prove that H_n admits no straight-line embedding that realizes the prescribed lengths on the edges of the cycle C_n.

6.1 Proof of Theorem 4

By Lemma 9, it is enough to prove Theorem 4 in the case when C is a Hamilton cycle in H.

THEOREM 13. *Let H be a planar graph that contains a cycle $C = (V, E)$. Let $\ell : E \to \mathbb{R}^+$ be a feasible nondegenerate length assignment. Then H admits a straight-line embedding in which each $e \in E$ has length $\ell(e)$.*

Proof. We may assume that H is an edge-maximal planar graph, that is, H is a triangulation. By Lemma 11, H contains a 3-cycle (v_a, v_b, v_c) such that the prescribed arc lengths of C between these vertices, i.e., the three sums of lengths of edges corresponding to these three arcs, satisfy the triangle inequality.

Let P_1, P_2, and P_3 denote the paths along C between the vertex pairs (v_a, v_b), (v_b, v_c), and (v_c, v_a); and let their prescribed edge lengths be L_1, L_2, and L_3, respectively. For $j = 1, 2, 3$, let H_j be the subgraphs of H induced by the vertices of the path P_j. Denote by $E_{1,2,3}$ the set of edges of H between an interior vertex of P_1, P_2, or P_3, and a vertex not on the same path. Consider a combinatorial embedding of H (with arbitrary edge lengths) such that (v_a, v_b, v_c) is triangle in the *exterior* of C. In this embedding, all edges in $E_{1,2,3}$ are *interior* chords of C.

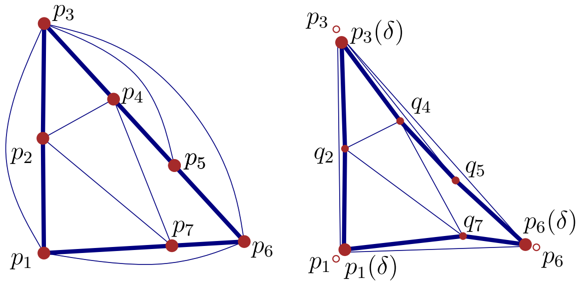

Figure 8: Left: A cycle $C = (p_1, \ldots p_8)$ embedded on the boundary of a triangle (p_1, p_3, p_6) with prescribed edge lengths. Right: When the vertices of the triangle are translated by δ towards the center of the triangle, we can embed the subgraphs induced by (p_1, p_2, p_3), (p_3, p_4, p_5, p_6) and (p_6, p_7, p_1) by straight-line edges so that they do not cross any of the diagonals between different sides of the triangle.

Similarly to the proof of Lemma 12, we start with a "preliminary" embedding, where the vertices v_i are embedded as follows. Let (p_a, p_b, p_c) be a triangle with edge lengths $|p_a p_b| = L_1$, $|p_b p_c| = L_2$, and $|p_c p_a| = L_3$. Place all other points p_i on the boundary of the triangle such that the distance between consecutive points is $|p_i p_{i+1}| = \ell(v_i v_{i+1})$ for $i = 1, \ldots, n-1$. Suppose, without loss of generality, that no two points have the same x-coordinate. Note that the slope of every line segment $p_i p_j$, for $v_i v_j \in E_{1,2,3}$ is *different* from the slopes of the sides of the triangle that contains p_i and p_j. Let η be the minimum difference between the slopes of two segments $p_i p_j$, with with $v_i v_j \in E_{1,2,3}$.

Move points p_a, p_b, and p_c toward the center of triangle (p_a, p_b, p_c) by a vector of length $\delta > 0$ to positions $p_a(\delta)$, $p_b(\delta)$, and $p_c(\delta)$. In any straight-line embedding of C with $v_a = p_a(\delta)$, $v_b = p_b(\delta)$ and $v_c = p_c(\delta)$, each vertex v_i, $i = 2, \ldots, n$, must lie in a region $R_i(\delta)$, which is the intersection of two disks centered at two vertices of the triangle $(p_a(\delta), p_b(\delta), p_c(\delta))$. Choose a sufficiently small $\delta > 0$ such that the slopes of a line intersecting $R_i(\delta)$ and $R_j(\delta)$ with $v_i v_j \in H$ is within $\eta/2$ from the slope of the segment $p_i p_j$.

Embed vertices v_i, v_j and v_k at points $p_i(\delta)$, $p_j(\delta)$ and $p_k(\delta)$, respectively. If H_1 (resp., H_2 and H_3) has three or more vertices, embed it using Lemma 12 such that the endpoints of the path P_1 are $p_i(\delta)$ and $p_k(\delta)$ (resp., $p_j(\delta), p_k(\delta)$ and $p_k(\delta), p_i(\delta)$). Each vertex v_i is embedded in a point in the region R_i, for $i = 1, \dots n$. By the choice of δ, the slopes of the edges of H_1, H_2, and H_3 are in three small pairwise disjoint intervals, and these intervals are disjoint from the slopes of any edge $v_iv_j \in E_{1,2,3}$. Therefore, the edges of H are pairwise noncrossing, and we obtain a proper embedding of H. $\square$

7. CONCLUSION

We have characterized the planar graphs G that are free subgraphs in every host H, $G \subseteq H$. In Section 3, we showed that every triangulation T has a straight-line embedding in which a matching $M \subset T$ has arbitrarily prescribed edge lengths, and the outer face is fixed. Several related questions remain unanswered.

1. Given a length assignment $\ell\colon M \to [1, \lambda]$ for a matching M in an n-vertex planar graph G, what is the minimum Euclidean *diameter* (resp., *area*) of an embedding of G with prescribed edge lengths?

2. Is there a polynomial time algorithm for deciding whether a subgraph G of a planar graph H is free or extrinsically free in H?

3. Is there a polynomial time algorithm for deciding whether a planar graph H is realizable such that the edges of a cycle $C = (V, E)$ have given (possibly degenerate) lengths?

4. Which planar graphs G are free in every *4-connected* triangulation H, $G \subseteq H$? We know that stars are, but we do not have a complete characterization.

Recently, Alamdari et al. [1] proved that given any two topologically equivalent embeddings of a planar graph, one can continuously morph one embedding into the other in $O(n^2)$ successive linear morphs (where each vertex moves with constant speed). Combined with our Theorem 1, this implies that if we are given *two* length assignments $\ell_1\colon M \to \mathbb{R}^+$ and $\ell_2\colon M \to \mathbb{R}^+$ for a matching M in an n-vertex triangulation T, one can continuously morph an embedding with one length assignment into an embedding with the other assignment in $O(n^2)$ linear morphs. It remains an open problem whether fewer linear morphs suffice between the embeddings that admit two different length assignments of M.

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8. REFERENCES

[1] S. Alamdari, P. Angelini, T. M. Chan, G. Di Battista, F. Frati, A. Lubiw, M. Patrignani, V. Roselli, S. Singla, and B. T. Wilkinson, Morphing planar graph drawings with a polynomial number of steps, in *24th SODA*, ACM-SIAM, 2013, pp. 1656–1667.

[2] P. Angelini, G. Di Battista, F. Frati, V. Jelínek, J. Kratochvíl, M. Patrignani, and I. Rutter, Testing planarity of partially embedded graphs, in *21st SODA*, ACM-SIAM, 2010, pp. 202–221.

[3] S. Cabello, E. D. Demaine, and G. Rote, Planar embeddings of graphs with specified edge lengths, *J. Graph Alg. Appl.* **11(1)** (2007), 259–276.

[4] R. Connelly, Generic global rigidity, *Discrete Comput. Geom.* **33(4)** (2005), 549–563.

[5] R. Connelly, E. D. Demaine, and G. Rote, Straightening polygonal arcs and convexifying polygonal cycles. *Discrete Comput. Geom.* **30(2)** (2003), 205–239.

[6] G. Di Battista and L. Vismara, Angles of planar triangular graphs, *SIAM J. Discrete Math.* **9(3)** (1996), 349–359.

[7] P. Eades and N. C. Wormald, Fixed edge-length graph drawing is NP-hard, *Discrete Applied Mathematics* **28** (1990), 111–134.

[8] H. de Fraysseix, J. Pach, and R. Pollack, How to draw a planar graph on a grid, *Combinatorica* **10(1)** (1990), 41–51.

[9] I. Fáry, On straight line representation of plane graphs, *Acta. Sci. Math. Szeged* **11** (1948), 229–233.

[10] F. Frati and M. Patrignani, A note on minimum-area straight-line drawings of planar graphs, in *15th Graph Drawing*, LNCS 4875, Springer, 2008, pp. 339–344.

[11] S.-H. Hong and H. Nagamochi, Convex drawings of graphs with non-convex boundary constraints *Discrete Appl. Math.* **156** (2008), 2368–2380.

[12] B. Jackson and T. Jordán, Connected rigidity matroids and unique realizations of graphs, *J. Combin. Theory Ser. B* **94(1)** (2005), 1–29.

[13] V. Jelínek, J. Kratochvíl, and I. Rutter, A Kuratowski-type theorem for planarity of partially embedded graphs, *Comput. Geom. Theory Appl.* **46(4)** (2013), 466–492.

[14] M. Kurowski, Planar straight-line drawing in an $O(n) \times O(n)$ grid with angular resolution $\Omega(1/n)$, in *SOFSEM*, LNCS 3381, 2005, Springer, pp. 250–258.

[15] W. J. Lenhart and S. H. Whitesides, Reconfiguring closed polygonal chains in euclidean d-space, Discrete Comput. Geom. **13** (1995), 123–140.

[16] M. Patrignani, On extending a partial straight-line drawing, *Found. Comput. Sci.* **17(5)** (2006), 1061–1069.

[17] N. W. Sauer, Distance sets of Urysohn metric spaces, *Canad. J. Math.* **65** (2013), 222–240.

[18] N. W. Sauer, Edge labelled graphs and metric spaces, in *Abstracts of Erdős Centennial*, 2013, p. 81.

[19] W. Schnyder, Embedding planar graphs in the grid, in *1st SODA*, 1990, ACM-SIAM, pp. 138–147.

[20] W.T. Tutte, How to draw a graph, *Proc. London Math. Soc.*, 1963, **3-13(1)**, 743–767