Asymptotic random graph intuition for the biased connectivity game

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Abstract

We study biased Maker/Breaker games on the edges of the complete graph, as introduced by Chvátal and Erdős. We show that Maker, occupying one edge in each of his turns, can build a spanning tree, even if Breaker occupies $b \leq (1 - o(1)) \cdot \frac{n}{\ln n}$ edges in each turn. This improves a result of Beck, and is asymptotically best possible as witnessed by the Breaker-strategy of Chvátal and Erdős. We also give a strategy for Maker to occupy a graph with minimum degree c (where $c = c(n)$ is a slowly growing function of n) while playing against a Breaker who takes $b \leq (1 - o(1)) \cdot \frac{n}{\ln n}$ edges in each turn. This result improves earlier bounds by Krivelevich and Szabó. Both of our results support the surprising random graph intuition: the threshold bias is *asymptotically the same* for the game played by two "clever" players and the game played by two "random" players.

1 Introduction

In this paper we consider games played by two opponents on edges of the complete graph K_n on n vertices. The two players alternately take turns at claiming some number of unclaimed edges until all edges are claimed. One of the players, called Maker, aims to create a graph which possesses some fixed property P. The other player, called Breaker, tries to prevent Maker from achieving his goal: Breaker wins if, after all $\binom{n}{2}$ edges were claimed, Maker's graph does *not* posses P.

A classical graph game of this sort is the *Shannon switching game* analized completely by Lehman [8]. In the Shannon switching game both players take *one* edge per turn and Maker tries to occupy the edges of a spanning tree. Lehman proved that Maker wins this game "easily". Here by "easily" we mean that Maker does not need the full edge-set of K_n, he wins even if the game is played on the restricted board consisting of the edges of two edge-disjoint spanning trees.

Chvátal and Erdős [7] suggested to even out this advantage of Maker by allowing Breaker to occupy b edges in each round instead of just one. The integer $b = b(n) > 1$ is called the *bias* of Breaker. Chvátal and Erdős provided a strategy for Maker to occupy a spanning tree even if Breaker plays with bias $(\frac{1}{4} - o(1))\frac{n}{\ln n}$. They also showed that Breaker, playing with a bias $(1 + o(1))\frac{n}{\ln n}$ can

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occupy all edges incident to some vertex, thus of course making it impossible for Maker to build a spanning tree. Motivated partially by this problem, Beck [1] devised a general potential function method to deal with biased positional games under much more general circumstances. He then used his criterion to infer a strategy for Maker for occupying a spanning tree even if Breaker's bias is as large as $(\ln 2 - o(1))\frac{n}{\ln n} \approx 0.69\frac{n}{\ln n}$.

In one of the main results of the present paper we improve Beck's bound on Breaker's bias and establish the asymptotic optimality of the Breaker-strategy of Chvátal and Erdős.

Theorem 1.1 *Maker has a strategy to build a spanning tree while playing against a Breaker with bias* $b := (\ln n - \ln\ln n - 6)\frac{n}{\ln^2 n}$*, provided* n *is large enough.*

The constant 6 in the error term could be improved somewhat, but we do not know whether the second order term is best possible. Our proof is not based on the potential function technique of Beck, rather on the analysis of a quite natural strategy of Maker, involving a delicate inductive argument.

Random graph intuition For further discussion we introduce some notation. Let $\mathcal{F} \subseteq 2^{E(K_n)}$ be a monotone increasing family of subsets of edges of the complete graph. By $b_{\mathcal{F}}$ we denote the largest bias b of Breaker such that Maker, taking one edge in each turn, can occupy some member of $\mathcal{F}$ while Breaker takes b edges in each turn. The integer $b_{\mathcal{F}}$ is called the *threshold bias* of the game $\mathcal{F}$. Let $\mathcal{T} = \mathcal{T}(n)$ be the family of edge-sets of connected graphs on n vertices. Theorem 1.1, coupled with the Breaker-strategy of Chvátal and Erdős, establishes that $b_{\mathcal{T}} = \frac{n}{\ln n} + O\left(\frac{n\ln\ln n}{\ln^2 n}\right)$.

Chvátal and Erdős [7] observed the following interesting paradigm, which we call the "random graph intuition". The threshold bias $b_{\mathcal{T}}$ of Breaker, which by definition involves two "clever" players, is of the same order of magnitude as the appropriately defined threshold bias of Breaker in a game where the players are mindless random edge generators! In this random game the player *RandomMaker* claims one random unclaimed edge per move, player *RandomBreaker* claims b random unclaimed edges per move. RandomMaker creates a random graph $G(n,m)$ with $m = \lceil \binom{n}{2}/(b+1) \rceil$ edges, so he wins the game $\mathcal{T}$ almost surely if and only if $m > \left(\frac{1}{2} + o(1)\right) n \ln n$, the sharp threshold edgenumber for the family $\mathcal{T}$. This implies that the threshold bias of RandomBreaker's win in the random game is almost surely $(1+o(1))\frac{n}{\ln n}$: just like in the "clever" game, as shown by Theorem 1.1.

Another classical game supporting the random graph intuition (at least in the order) is the hamiltonicity game, in which Maker's goal is to occupy a Hamilton cycle. Naturally, this game is harder for Maker to win, still the order of the threshold bias turned out to be the same as the one for the connectivity game: Beck [2] used his potential function method coupled with a nice ad-hoc argument to show that Maker can create a Hamilton cycle against a bias $(\frac{\ln 2}{27} - o(1))\frac{n}{\ln n}$ of Breaker. Recently this was improved to $(\ln 2 - o(1))\frac{n}{\ln n}$ by Krivelevich and Szabó [9], but the precise asymptotics still eludes us.

Further occurrences of the connection between random graphs and biased positional games were investigated in a series of papers by Beck [3, 4, 5] and Bednarska and Łuczak [6].

It is well-known from the theory of random graphs that several natural graph properties, like hamiltonicity, c-connectivity, or minimum degree at least c, for constant c, all have the same sharp threshold edge number $\frac{1}{2}n \ln n$. Let $\mathcal{H}$, $\mathcal{C}_c$, $\mathcal{D}_c$ be the corresponding families of edgesets. In [9] it was established that $(\ln 2 - o(1))\frac{n}{\ln n}$ is a lower bound for $b_{\mathcal{H}}, b_{\mathcal{C}_c}$ and $b_{\mathcal{D}_c}$, as well. Motivated by the extremely tight correlation between the appearence of the properties $\mathcal{T}$ and $\mathcal{D}_1$ in the random graph, it is conjectured that $b_{\mathcal{T}} = b_{\mathcal{D}_1}$ for all n large enough [9]. While this conjecture is still open, Theorem 1.1 does establish its asymptotic correctness.

In our second main theorem we improve the lower bound of [9] for the family $\mathcal{D}_c$, provided c is an arbitrary constant. We establish that $b_{\mathcal{D}_c} = (1 + o(1))\frac{n}{\ln n}$, which means that the random graph intuition is valid asymptotically for the minimum-degree-c game as well.

Theorem 1.2 *Let $c = c(n) < \frac{\ln \ln n}{3}$. Maker has a strategy to build a graph with minimum degree at least c while playing against Breaker with bias $b := (\ln n - \ln \ln n - (2c + 3))\frac{n}{\ln^2 n}$, provided n is large enough.*

As a third example of the surprising validity of the random graph intuition, for any constant $\varepsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that Maker is able to build a graph with minimum-degree at least $\delta \ln(n)$ while playing against a Breaker's bias $(1 - \varepsilon) \cdot \frac{n}{\ln(n)}$.

Theorem 1.3 *Let $\varepsilon > 0$ be a constant. Then Maker has a strategy to build a graph with minimum degree at least $\frac{\varepsilon}{3(1-\varepsilon)} \ln n$ while playing against a Breaker's bias of $(1 - \varepsilon) \cdot \frac{n}{\ln n}$.*

The order $\ln n$ for the largest achievable minimum degree against a bias of $(1 - \varepsilon)\frac{n}{\ln n}$ is obviously best possible: Maker has at most $\frac{\binom{n}{2}}{b+1}$ edges by the end, which allows a minimum degree at most $\frac{\ln n}{1-\varepsilon}$.

Finally, by merging the strategies of Maker for achieving a spanning tree and a graph of minimum degree c, respectively it can be proven that Maker has a strategy to accomplish both of these goals at the same time.

Theorem 1.4 *Let $c = c(n) < \frac{\ln \ln n}{3}$. Maker has a strategy to build a connected graph with minimum degree c while playing against Breaker with bias $b := (\ln n - \ln \ln n - (2c + 5))\frac{n}{\ln^2 n}$, provided n is large enough.*

As in the min-degree case we can show that Theorem 1.4 remains true if we let $\varepsilon > 0$ be a constant and replace c by $\delta \ln(n)$ with $\delta > 0$ being a constant depending on ε only.

Theorem 1.5 *Let $\varepsilon > 0$ be a constant. Then Maker has a strategy to build a connected graph with minimum degree $\frac{\varepsilon}{3(1-\varepsilon)} \ln n$ while playing against a Breaker's bias of $(1 - \varepsilon) \cdot \frac{n}{\ln n}$.*

Notation Ceiling and floor signs are routinely omitted whenever they are not crucial for clarity. We denote by H_s the sth harmonic number $\sum_{j=1}^{s} \frac{1}{j}$ and often use the well-known fact that for every positive integer s

$$\ln s \leq H_s \leq \ln s + 1. \tag{1}$$

2 Building a spanning tree

Proof of Theorem 1.1: An important message of Beck's potential function argument was that instead of concentrating on "making" a spanning tree, one concentrates on "breaking" into every cut – the succes of which would then imply connectivity. In our proof we abandon this dual approach and plainly focus on the original goal: building a spanning tree.

We assume that Breaker starts the game. Otherwise Maker can start with an arbitrary first move, then follow his strategy. If his strategy calls for something he occupied before he takes an arbitrary edge; no extra move is disadvantegous for him.

In the following proof by a *component* we always mean a connected component of Maker's graph. For a vertex v, we denote by $C(v)$ the component containing v. We call a component *dangerous* if it contains at most $2b$ vertices. By the *degree of a vertex* v (or $\deg(v)$ in short) we always mean the ordinary degree of v in Breaker's graph.

We define a *danger function* on the vertex set. Let

$$\mathsf{dang}(v) = \begin{cases} \deg(v) & \text{if } C(v) \text{ is dangerous} \\ -1 & \text{otherwise} \end{cases}$$

Maker's Strategy At the beginning every vertex is *active.* For his ith move, Maker identifies a vertex v_i with the largest danger value among active vertices (ties are broken arbitrarily) and he occupies one arbitrary free edge connecting $C(v_i)$ to another component. He deletes v_i from the set of active vertices and calls v_i *deactivated.*

This strategy of Maker will be denoted by S_M. Maker can always make a move according to S_M unless his graph is a tree (thus he won) or Breaker occupied a cut (and Breaker won). Hence during our analysis Maker's graph is always a forest.

The following observation follows easily from S_M by induction on the number of rounds.

Observation 1 Every component contains exactly one active vertex.

Proof of Maker's win Suppose, for a contradiction, that Breaker has a strategy S_B to win the $(1 : b)$ connectivity game against Maker. Let B_i and M_i denote the i'th move of Breaker and Maker, respectively, in the game where they play against each other using their respective strategies S_B and S_M. Let g be the length of this game, i.e., g is the smallest integer, that Breaker finished occupying all edges in a cut $(K, V \setminus K)$ in move B_g. We call this the end of the game. Note that $g \leq n - 1$, as Maker's strategy does not allow him to occupy a cycle.

Let $|K| \leq |V \setminus K|$. Observe that $|K| \leq 2b$, since otherwise Breaker would have had to occupy at least $2b(n-2b) > gb$ edges in $g < n$ rounds, a contradiction for large n. This implies that during the game there is always at least one dangerous component. Since Maker's strategy prefers to deactivate active vertices in dangerous components we also have the following.

Observation 2 Vertex v_i is in a dangerous component at and before its deactivation.

In his last move Breaker takes b edges to completely occupy all edges between K and $V \setminus K$. In order to be able to do that, directly before Breaker's last move all vertices of K must have degree at least $n - 2b - b$. Let $v_g \in K$ be an arbitrary active vertex; by Observation 1 there is one in each component inside K.

Recall that $v_1, \ldots, v_{g-1}$ were defined during the game. For $0 \leq i \leq g-1$, let $I_i = \{v_{g-i}, \ldots v_g\}$. For a subset $I \subseteq V$, let $\overline{\mathsf{dang}}_{B_i}(I) = \frac{\sum_{v \in I} \mathsf{dang}(v)}{|I|}$ denote the average danger value of the vertices in I directly before move B_i of Breaker. Analogously, $\overline{\mathsf{dang}}_{M_i}(I)$ denotes the average danger value before M_i.

The following lemma is a consequence of Maker's strategy. It considers the change of danger during Maker's move.

Lemma 2.1 *For every i, $1 \leq i \leq g-1$, directly before M_{g-i}* $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) \geq \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})$

Proof: All the vertices $v_{g-i+1}, \ldots, v_g$ constituting I_{i-1} are in a dangerous components directly before B_{g-i+1}, so their danger value does not change during M_{g-i}. Hence $\overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1}) = \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})$. Maker deactivated v_{g-i} in M_{g-i}, because its danger was maximum among active vertices. All vertices of I_{i-1} were still active before M_{g-i}, thus $\mathsf{dang}(v_{g-i}) \geq \max\{\mathsf{dang}(v_{g-i+1}), \ldots, \mathsf{dang}(v_g)\}$ implying $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) \geq \overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1})$ and the lemma follows. □

The next lemma bounds the change of the danger value during Breaker's moves. The first estimate is used during the main game. It will guarantee the existence of many vertices with large average degree, which eventually leads to a contradiction. For the rounds closer to the end we need a stronger inductive statement, which is provided by the second estimate of the lemma.

Lemma 2.2 *Let i be an integer, $1 \leq i \leq g-1$.*

(i) $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \leq \frac{2b}{i+1}$

(ii) $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \leq \frac{b+i+a(i-1)-a(i)}{i+1}$*, where $a(i)$ denotes the number of edges spanned by I_i which Breaker took in the first $g-i-1$ rounds.*

Proof All the components $C(v_{g-i}), \ldots, C(v_g)$ are dangerous before M_{g-i}. Since components do not change during Breaker's move the danger value of the vertices of I_i depend solely on their degrees. In B_{g-i} Breaker claims b edges, so the increase of the sum of degrees of $v_{g-i}, \ldots, v_g$ during B_{g-i} is at most $2b$. Hence $\overline{\mathsf{dang}}(I_i)$ increases by at most $\frac{2b}{i+1}$, which proves (i).

For (ii), we will be more careful. Let e_{double} denote the number of edges taken in B_{g-i} whose both endpoints are in I_i. Then the increase of $\sum_{j=0}^{i} \deg(v_{g-j})$ during B_{g-i} is at most $b + e_{double}$. Hence

$\overline{\mathsf{dang}}(I_i)$ increases by at most $\frac{e_{\text{double}}+b}{i+1}$. We now bound e_{double}. By definition, Breaker occupied $a(i)$ edges spanned by I_i in his first $g-i-1$ moves. So, all in all, Breaker occupied $a(i)+e_{\text{double}}$ edges spanned by I_i in his first $g-i$ moves. On the other hand, we know that among these edges exactly $a(i-1)$ are spanned by $I_{i-1}=I_i\setminus\{v_{g-i}\}$ and there are at most i edges in I_i incident to v_{g-i}. Hence $a(i)+e_{\text{double}}\le a(i-1)+i$, giving us $e_{\text{double}}\le i+a(i-1)-a(i)$. □

Using Lemmas 2.1 and 2.2 we derive that before B_1, $\overline{\mathsf{dang}}(I_{g-1})>0$. This is of course in contradiction with the fact that at the beginning of the game every vertex has danger value 0.

Let $k:=\lfloor\frac{n}{\ln n}\rfloor$. For the analysis, we split the game into two parts: The main game, and the end game consisting of the last k rounds.

Recall that the danger value of v_g directly before B_g is at least $n-3b$.

Assume first that $k>g$.

$$
\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0)+\\
&\quad \sum_{i=1}^{g-1}\Big(\overline{\mathsf{dang}}_{M_{g-i}}(I_i)-\overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})\Big)-\sum_{i=1}^{g-1}\Big(\overline{\mathsf{dang}}_{M_{g-i}}(I_i)-\overline{\mathsf{dang}}_{B_{g-i}}(I_i)\Big)\\
&\ge n-3b+\sum_{i=1}^{g-1}0-\sum_{i=1}^{g-1}\frac{b+i+a(i-1)-a(i)}{i+1}\\
&\ge n-3b-b(H_g-1)-(g-1)-\frac{a(0)}{2}+\sum_{i=1}^{g-2}\frac{a(i)}{(i+2)(i+1)}+\frac{a(g-1)}{g}\\
&\ge n-b(H_g+2)-g \qquad [\text{since } a(0)=0 \text{ and } a(i)\ge 0]\\
&\ge n-b(\ln k+3)-k \qquad [\text{since } g\le k]\\
&\ge n-\frac{n}{\ln n}(\ln n-\ln\ln n+3)-\frac{n}{\ln n}\\
&\ge \frac{n\ln\ln n}{\ln n}-O\left(\frac{n}{\ln n}\right)\\
&> 0.
\end{aligned}
$$

Assume now that $k \le g$. We then have

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1}\Big(\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})\Big) \\
&\quad - \sum_{i=1}^{k-1}\Big(\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i)\Big) - \sum_{i=k}^{g-1}\Big(\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i)\Big) \\
&\ge n - 3b + \sum_{i=1}^{g-1} 0 - \sum_{i=1}^{k-1} \frac{b + i + a(i-1) - a(i)}{i+1} - \sum_{i=k}^{g-1} \frac{2b}{i+1} \\
&\ge n - 3b - b(H_k - 1) - (k-1) - \frac{a(0)}{2} + \sum_{i=1}^{k-2} \frac{a(i)}{(i+2)(i+1)} + \frac{a(k-1)}{k} - 2b(H_g - H_k) \\
&\ge n - b(2H_g - H_k + 2) - k \\
&\ge n - b(2\ln n - \ln k + 4) - k \\
&\ge n - \left(\frac{n}{\ln n} - \frac{n \ln\ln n}{\ln^2 n} - 6\frac{n}{\ln^2 n}\right)(\ln n + \ln\ln n + 5) - \frac{n}{\ln n} \\
&\ge \frac{n(\ln\ln n)^2}{\ln^2 n} \\
&> 0.
\end{aligned}$$

□

3 Achieving large minimum degree

Proof of Theorem 1.2: As in the previous proof we assume that Breaker starts the game. We say that the game ends when either all vertices have degree at least c in Maker's graph (and Maker won) or one vertex has degree at least $n-c$ in Breaker's graph (and Breaker won). With $\deg_M(v)$ and $\deg_B(v)$ we denote the degree of a vertex v in Maker's graph and in Breaker's graph, respectively. A vertex v is called *dangerous* if $\deg_M(v) \le c-1$. To establish Maker's strategy we define the *danger value* of a vertex v as $\mathsf{dang}(v) := \deg_B(v) - 2b \cdot \deg_M(v)$.

Maker's Strategy S_M Before his ith move Maker identifies a dangerous vertex v_i with the largest danger value, ties are broken arbitrarily. Then, as his ith move Maker claims an edge incident to v_i. We refer to this step as "easing v_i".

Observation 3 Maker can always make a move according to his strategy unless no vertex is dangerous (thus he won) or Breaker occupied at least $n-c$ edges incident to a vertex (and Breaker won).

Observation 4 Vertex v_i is dangerous any time before Maker's ith move.

Suppose, for a contradiction, that Breaker, playing with a bias b, has a strategy S_B to win the min-degree-c game against Maker who plays with bias 1. Let B_i and M_i denote the ith move of Breaker and Maker, respectively, in the game where they play against each other using their respective strategies S_B and S_M. Let g be the length of this game, i.e., the maximum degree of Breaker's graph becomes larger than $n-1-c$ in move B_g. We call this the end of the game.

For a set $I \subseteq V$ of vertices we let $\overline{\mathsf{dang}}(I)$ denote the average danger value $\frac{\sum_{v\in I}\mathsf{dang}(v)}{|I|}$ of the vertices of I. When there is risk of confusion we add an index and write $\mathsf{dang}_{B_i}(v)$ or $\mathsf{dang}_{M_i}(v)$ to emphasize that we mean the danger-value of v *directly before* B_i or M_i, respectively.

In his last move Breaker takes b edges to increase the maximum Breaker-degree of his graph to at least $n-c$. In order to be able to do that, directly before Breaker's last move B_g there must be a dangerous vertex v_g whose Breaker-degree is at least $n-c-b$. Thus $\mathsf{dang}_{B_g}(v_g) \geq n-c-b-2b(c-1)$.

Recall that $v_1,\dots,v_{g-1}$ were defined during the game. For $0 \leq i \leq g-1$, let $I_i = \{v_{g-i},\dots v_g\}$.

The following lemma estimates the change in the average danger during Maker's move.

Lemma 3.1 *Let i, $1 \leq i \leq g-1$,*

(i) if $I_i \neq I_{i-1}$, then $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0.$

(ii) if $I_i = I_{i-1}$, then $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq \frac{2b}{|I_i|}$.

Proof: For part (i), we have that $v_{g-i} \notin I_{i-1}$. Since danger values do not increase during Maker's move we have $\overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1}) \geq \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})$. Before M_{g-i} Maker selected to ease vertex v_{g-i} because its danger was highest among dangerous vertices. Since all vertices of I_{i-1} are dangerous before M_{g-i} we have that $\mathsf{dang}(v_{g-i}) \geq \max(\mathsf{dang}(v_{g-i+1}),\dots,\mathsf{dang}(v_g))$, which implies $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) \geq \overline{\mathsf{dang}}_{M_{g-i}}(I_{i-1})$. Combining the two inequalities establishes part (i).

For part (ii), we have that $v_{g-i} \in I_{i-1}$. In M_{g-i} $\deg_M(v_{g-i})$ increases by 1 and $\deg_M(v)$ does not decrease for any other $v \in I_i$. Besides, the degrees in Breaker's graph do not change during Maker's move. So $\mathsf{dang}(v_{g-i})$ decreases by $2b$, whereas $\mathsf{dang}(v)$ do not increase for any other vertex $v \in I_i$. Hence $\overline{\mathsf{dang}}(I_i)$ decreases by at least $\frac{2b}{|I_i|}$, which implies (ii). □

The next lemma bounds the change of the danger value during Breaker's moves.

Lemma 3.2 *Let i be an integer,* $1 \leq i \leq g-1$.

(i) $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \leq \frac{2b}{|I_i|}$

(ii) $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i}}(I_i) \leq \frac{b+|I_i|-1+a(i-1)-a(i)}{|I_i|}$*, where $a(i)$ denotes the number of edges spanned by I_i which Breaker took in the first $g-i-1$ rounds.*

Proof: Let e_{double} denote the number of those edges with both endpoints in I_i which are occupied by Breaker in B_{g-i}. Then the increase of $\sum_{v\in I_i}\deg_B(v)$ during B_{g-i} is at most $b+e_{\text{double}}$. Since the degrees in Maker's graph do not change during Breaker's move the increase of $\overline{\mathsf{dang}}(I_i)$ (during B_{g-i}) is at most $\frac{b+e_{\text{double}}}{|I_i|}$.

Part (i) is then immediate after noting that $e_{\text{double}} \leq b$.

For (ii), we bound e_{double} more carefully. By definition, Breaker occupied $a(i)$ edges spanned by I_i in his first $g-i-1$ moves. So, all in all, Breaker occupied $a(i)+e_{\text{double}}$ edges spanned by I_i in his first $g-i$ moves. On the other hand, we know that among these edges exactly $a(i-1)$

are spanned by $I_{i-1} = I_i \setminus \{v_{g-i}\}$ and there are at most $|I_i| - 1$ edges in I_i incident to v_{g-i}. Hence $a(i) + e_{\text{double}} \leq a(i-1) + |I_i| - 1$, giving us $e_{\text{double}} \leq |I_i| - 1 + a(i-1) - a(i)$. □

The following estimates for the change of average danger during one full round are immediate corollaries of the previous two lemmas.

Corollary 3.3 *Let i be an integer, $1 \leq i \leq g-1$.*

(i) if $I_i = I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0$.

(ii) if $I_i \neq I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{2b}{|I_i|}$

(iii) if $I_i \neq I_{i-1}$, then $\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{b+|I_i|-1+a(i-1)-a(i)}{|I_i|}$*, where $a(i)$ denotes the number of edges spanned by I_i which Breaker took in the first $g-i-1$ rounds.*

Using Corollary 3.3 we derive that before B_1, $\overline{\mathsf{dang}}(I_{g-1}) > 0$, which contradicts the fact that at the beginning of the game every vertex has danger value 0.

Let $k := \lfloor \frac{n}{\ln n} \rfloor$. For the analysis, we split the game into two parts: The main game, and the end game which starts when $|I_i| \leq k$.

Let $|I_g| = r$. Let $i_1 < \ldots < i_{r-1}$ be those indices for which $I_{i_j} \neq I_{i_j - 1}$. Note that $|I_{i_j}| = j+1$. Observe that by definition $a(i_{j-1}) \geq a(i_j - 1)$.

Recall that the danger value of v_g directly before B_g is at least $n - c - b(2c-1)$.

Assume first that $k > r$.

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1}\Big(\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1})\Big)\\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1}\Big(\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1})\Big) \quad \text{[by Corollary 3.3}(i)]\\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1}\frac{b+j+a(i_j-1)-a(i_j)}{j+1} \quad \text{[by Corollary 3.3}(iii)]\\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - bH_r - r - \frac{a(0)}{2} + \sum_{j=2}^{r-1}\frac{a(i_{j-1})}{(j+1)j} + \frac{a(i_{r-1})}{r} \quad \text{[since } a(i_{j-1}) \geq a(i_j-1)]\\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - bH_k - k \quad \text{[since } a(0)=0 \text{ and } r \leq k]\\
&\geq n - c - b(2c + \ln k) - k\\
&\geq n - \frac{n}{\ln n}(2c + \ln n - \ln\ln n) - \frac{n}{\ln n} - c\\
&\geq \frac{n\ln\ln n}{3\ln n} - \frac{n}{\ln n} - c\\
&> 0. \quad \text{[for large } n] \qquad (2)
\end{aligned}$$

Assume now that $k \leq r$.

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} \Big(\overline{\mathsf{dang}}_{B_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \Big) \\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \Big(\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1}) \Big) \quad \text{[by Corollary 3.3}(i)\text{]} \\
&= \overline{\mathsf{dang}}_{B_g}(I_0) + \sum_{j=1}^{k-1} \Big(\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1}) \Big) + \\
&\quad \sum_{j=k}^{r-1} \Big(\overline{\mathsf{dang}}_{B_{g-i_j}}(I_{i_j}) - \overline{\mathsf{dang}}_{B_{g-i_j+1}}(I_{i_j-1}) \Big) \\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - \sum_{j=1}^{k-1} \frac{b + j + a(i_j - 1) - a(i_j)}{j+1} - \sum_{j=k}^{r-1} \frac{2b}{j+1} \quad \text{[by Corollary 3.3}(iii)\text{ and }(ii)\text{]} \\
&\geq \overline{\mathsf{dang}}_{B_g}(I_0) - b(2H_r - H_k) - k - \frac{a(0)}{2} + \sum_{j=2}^{k-1} \frac{a(i_{j-1})}{(j+1)j} + \frac{a(i_{k-1})}{k} \\
&\geq n - c - b(2c - 1 + 2H_n - H_k) - k \quad \text{[since } n \geq r \text{ and } a(0) = 0\text{]} \\
&\geq n - c - \left(\frac{n}{\ln n} - \frac{n \ln\ln n}{\ln^2 n} - (2c+3)\frac{n}{\ln^2 n} \right) (\ln n + \ln\ln n + 2c + 2) - \frac{n}{\ln n} \\
&\geq \frac{n (\ln\ln n)^2}{\ln^2 n} \quad \text{[for } n \text{ large enough]} \\
&> 0. \qquad (3)
\end{aligned}$$

□

Proof of Theorem 1.3: The previous proof works line by line, we only have to adapt the last few lines of the calculations of (2) and (3). For (2), we have

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &\geq n - c - b(2c + \ln k) - k \\
&\geq n - c - (1-\varepsilon) \cdot \frac{n}{\ln n} \cdot \left(\frac{2\varepsilon}{3(1-\varepsilon)} \cdot \ln n + \ln n - \ln\ln n \right) - \frac{n}{\ln n} \\
&\geq \frac{\varepsilon}{3} n \\
&> 0.
\end{aligned}$$

For (3), we obtain

$$\begin{aligned}
\overline{\mathsf{dang}}_{B_1}(I_{g-1}) &\geq n - c - b(2c + 2\ln n - \ln k + 1) - k \\
&\geq n - c - (1-\varepsilon) \cdot \frac{n}{\ln n} \cdot \left(\frac{2\varepsilon}{3(1-\varepsilon)} \cdot \ln n + \ln n + \ln\ln n + 2 \right) - \frac{n}{\ln n} \\
&\geq \frac{\varepsilon}{3} n \quad \text{[for large } n\text{]} \\
&> 0.
\end{aligned}$$

□

4 Building a Connected Graph with high Minimium Degree

Proof of Theorem 1.4: To establish a suitable strategy for Maker we basically merge his strategies for occupying a spanning tree and achieving a graph of min-degree c. We will adopt most of the terminology used in the two corresponding proofs, but sometimes with a slightly modified content. We assume that Breaker starts the game and denote by $\deg_M(v)$ and $\deg_B(v)$ the degree of a vertex v in Maker's graph and in Breaker's graph, respectively. We adopt the concept of active vertices as well, at the beginning each vertex is *active*. After each of his moves Maker deletes one or two vertices from the set of active vertices. The corresponding vertices are called *deactivated*.

By a *component*, we always refer to a connected component of Maker's graph. A component is called *dangerous* if it contains at most $2bc$ vertices. In contrast to Section 2 we assign active vertices to dangerous components only. We call a vertex v *dangerous* if it is either active or has degree at most $c-1$ in Maker's graph.

We define a *danger function* on the vertex set. Let

$$\mathsf{dang}(v) = \begin{cases} \deg_B(v) & \text{if } v \text{ is active} \\ \deg_B(v) - 2b \cdot \deg_M(v) & \text{otherwise} \end{cases}$$

Maker's Strategy S_M. If there are no dangerous vertices left then Maker occupies an arbitrary free edge connecting two components. Otherwise, for his ith move Maker identifies a dangerous vertex v_i with the largest danger value (ties are broken arbitrarily) and *eases* v_i by doing the following. If v_i is active then Maker claims an arbitrary edge connecting $C(v_i)$ to another component C' and deactivates v_i. In case C' also had an active vertex and $|C(v_i)| + |C'| > 2bc$, then Maker deactivates the active vertex of C' as well. If v_i is not active then Maker claims an arbitrary edge e incident to v_i. In case a new component C emerges upon the selection of e, Maker deactivates some of the (at most two) active vertices of C arbitrarily such that C has one or zero active vertex depending on whether C is dangerous or not, respectively.

Note that Maker can always make a move according to his strategy unless his graph is connected and has minimum degree at least c (thus he won) or Breaker occupied either a cut or an $(n-c)$-star (and Breaker won).

The following is immediate consequence of the strategy of S_M.

Observation 4.1 *Every dangerous component contains exactly one active vertex whereas other components do not have active vertices.*

Since v_i is only defined for moves when there are still dangerous vertices, we have the following.

Observation 4.2 *Vertex v_i is dangerous any time before Maker's ith move.*

Proof of Maker's win. Suppose, for a contradiction, that Breaker, playing with a bias b, has a strategy S_B to win the game in question against Maker who plays with bias 1. Let B_i and M_i denote the ith move of Breaker and Maker, respectively, in the game where they play against each other using their respective strategies S_B and S_M. Let g be the length of this game, i.e., g is the smallest integer that in move B_g Breaker finished occupying either all edges in a cut $(K, V\backslash K)$ or all edges of a $(n-c)$-star. We call this the end of the game.

Proposition 4.3 $g < (c+1) \cdot n$

Proof: Each move of Maker is used either to decrease the number of components or to ease a vertex (occasionally both). Since the number of components can be decreased at most $n-1$ times and each vertex can be eased at most c times (thereafter it stops being dangerous), Maker can make at most $n-1+cn$ moves. □

Analogously to Section 3, for a set $I \subseteq V$ of vertices we let $\overline{\mathsf{dang}}(I)$ denote the average danger value $\frac{\sum_{v \in I} \mathsf{dang}(v)}{|I|}$ of the vertices of I. When there is risk of confusion we again add an index and write $\mathsf{dang}_{B_i}(v)$ or $\mathsf{dang}_{M_i}(v)$ to emphasize that we mean the danger-value of v *directly before* B_i or M_i, respectively.

Observation 4.4 *Before Breaker's last move* B_g *there is a dangerous vertex* v_g *with* $\mathsf{dang}(v_g) \geq n - b \cdot (2c+1)$

This can be seen by distinguishing two cases

Case 1. After B_g Breaker has completely occupied an $(n-c)$-star
In order to be able to do that, directly before B_g there must be a vertex v_g with $\deg_M(v_g) < c$ and $\deg_B(v_g) \geq n-c-b$. Thus $\mathsf{dang}_{B_g}(v_g) \geq n-c-b-2b(c-1) > n-b \cdot (2c+1)$.

Case 2. After B_g Breaker has completely occupied a cut $(K, V\backslash K)$ with $|K| \leq |K\backslash V|$.
In order to be able to do that, directly before B_g all vertices of K must have degree at least $n-|K|-b$ in Maker's graph. Observe that $|K| \leq 2bc$ since otherwise Breaker would had to occupy at least $2bc(n-2bc) > gb$ (by Proposition 4.3) edges in g rounds, a contradiction for large n. Hence by Observation 4.1 K contains an active vertex v_g, whose danger value is at least $\deg_B(v_g) \geq n-|K|-b \geq n-2bc-b = n-b(2c+1)$.

□

Since v_g is dangerous before Breaker's last move it is dangerous throughout the whole game. So before each move of Maker there is at least one dangerous vertex, implying that in each move Maker eases a vertex and $v_1, \ldots, v_{g-1}$ are all defined during the game. For $0 \leq i \leq g-1$, let $I_i = \{v_{g-i}, \ldots, v_g\}$.

For an estimate of the change in the average danger during Maker's move the statement of Lemma 3.1 is valid; we still copy it here since its proof has to be slightly adapted.

Lemma 4.5 *Let i, $1 \le i \le g-1$,*

(*i*) *if $I_i \neq I_{i-1}$, then* $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \ge 0$.

(*ii*) *if $I_i = I_{i-1}$, then* $\overline{\mathsf{dang}}_{M_{g-i}}(I_i) - \overline{\mathsf{dang}}_{B_{g-i+1}}(I_{i-1}) \ge \frac{2b}{|I_i|}$.

Proof: Part (i) can be shown by following the proof of part (i) of Lemma 3.1 word by word. For part (ii) we have that $v_{g-i} \in I_{i-1} = I_i$. We distinguish two cases

Case 1. v_{g-i} was active before M_{g-i}.

Due to Maker's strategy v_{g-i} is deactivated after M_{g-i}, implying that during M_{g-i} $\mathsf{dang}(v_{g-i})$ decreases by $2b \cdot \deg_M(v_{g-i})$ (where $\deg_M(v_{g-i})$ refers to the moment directly after M_{g-i}). Since after M_{g-i} v_{g-i} is not isolated in Maker's graph (otherwise it would still be active) it has positive degree in Maker's graph, implying that $\mathsf{dang}(v_{g-i})$ decreased by at least $2b$. No vertices increased their danger value during Maker's move, hence $\overline{\mathsf{dang}}(I_i)$ decreases by at least $\frac{2b}{|I_i|}$, which implies (*ii*).

Case 2. v_{g-i} was already deactivated before M_{g-i}.

In this case we can proceed along similar lines as in the proof of part (ii) of Lemma 3.1. □

The rest of the proof agrees with the one of Theorem 1.2 mutatis mutandis, the only difference being in the calculation that $\mathsf{dang}(v_g)$ is lower bounded by $n - b(2c+1)$ instead of $n - c - b(2c-1)$. □

The proof of Theorem 1.5 follows similarly to Theorem 1.3.

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