On nearly regular co-critical graphs

Tibor Szabó

The Ohio State University, Columbus, Ohio,
231 W 18th Ave, Columbus, Ohio, 43210, USA and
Eötvös Loránd University, Budapest, Hungary

Abstract

A graph G is called (K_3, K_3)-co-critical if the edges of G can be coloured with two colours without getting a monochromatic triangle, but adding any new edge to the graph, this kind of 'good' colouring is impossible. In this short note we construct (K_3, K_3)-co-critical graphs of maximal degree $O(n^{3/4})$.

1. INTRODUCTION

In [2] Galluccio, Simonovits and Simonyi dealt with the concept of (K_3, K_3)-co-critical graphs. They called a graph G (K_3, K_3)-co-critical (or just simply co-critical) if the edges of G can be coloured with two colours (say RED and BLUE) without getting a monochromatic triangle, but adding any arbitrary new edge to the graph, this kind of 'good' colouring is impossible.

Among several other results they looked for co-critical graphs with low edge-density. It is easy to construct (K_3, K_3)-co-critical graphs with a linear number of edges (see [2]). But those examples all have vertices of degree $\geq cn$. The natural question arises: what about the 'nearly regular' co-critical graphs with low edge-density or what is the smallest possible maximal degree a co-critical graph can have. In [2], using a random graph construction, the authors proved the existence of (K_3, K_3)-co-critical graphs of maximal degree $O(n^{3/4}\text{logn})$.

In this note we give a simpler and constructive example of co-critical graphs of maximal degree $O(n^{3/4})$. Unfortunately we still have a big gap between the trivial lower bound $c\sqrt{n}$ and this new upper bound. The lower bound follows from the fact that a co-critical graph must contain a K_3-saturated graph and that in a K_3-saturated graph the distance of any two points is ≤ 2 (see [1]). (By a K_3-saturated graph we mean a graph G which does not contain a triangle, but adding any new edge to G results in a triangle.) A drawback of our construction is that locally it has a lot of edges, so it can not answer any other of the open problems discussed at the end of [2].

2. THE CONSTRUCTION

Following the notation of [2] if G and H are two graphs, we denote by $G \otimes H$ the graph what we obtain by joining a copy of G to a copy of H completely (each vertex of G to each vertex of H). To keep this note self-contained we prove a variant of a lemma used in [2].

Lemma: Let C be a non-bipartite graph. If we colour the edges of $K_3 \otimes C$ with two colours, then we get a monochromatic K_3.

Proof: Let $V(K_3) = \{p, q, r\}$.

Suppose to the contrary, that there is a 2-colouring of $K_3 \otimes C$ without monochromatic triangles.

Thus we can assume that two of the edges of the K_3 have the same colour (say RED) and the third one is different (BLUE). Let, say, r be the common endpoint of the two RED edges.

Suppose that one of the edges between r and C, say $\{r, c_1\}$, is RED. Then either one of the edges $\{c_1, p\}$, $\{c_1, q\}$, say $\{c_1, p\}$, is RED and r, c_1 and p form a RED triangle or both $\{c_1, p\}$ and $\{c_1, q\}$ are BLUE making c_1, p and q a BLUE triangle.

But this is impossible, thus we can assume that all the edges between C and r are BLUE. Therefore all the edges of C must be RED, otherwise we would have a BLUE triangle $\{r, c, d\}$, for some $c, d \in C$.

The neighbours of p in C, which are connected by a RED edge to p must form an independent set. The BLUE neighbours are independent also, since they must be the subset of the RED neighbours of q. (If not, there would be a vertex in $V(C)$ which together with p and q would form a BLUE triangle.)

This is a contradiction, since we partitioned $V(G)$ into two independent subsets. $\square$

Definition: Let G and H be two graphs. We define their *or*-product $G \vee H$ by the following:

$V(G \vee H) = V(G) \times V(H)$ and

$\{(g, h), (g', h')\} \in E(G \vee H)$ if either $\{g, g'\} \in E(G)$ or $\{h, h'\} \in E(H)$.

Theorem: If G and H are non-bipartite K_3-saturated graphs then $G \vee H$ is (K_3, K_3)-co-critical.

Proof: We can give a trivial good colouring of the edges of $G \vee H$ by colouring an edge RED if the first coordinates of the two vertices formed an edge in G and colouring BLUE all the remaining edges.

Let's add a new edge to $G \vee H$: $\{(g,h),(g',h')\}$. Originally it is not an edge of $G \vee H$ which means $\{g,g'\} \notin E(G)$ and $\{h,h'\} \notin E(H)$. Since G and H are *maximal* K_3-free graphs there exist $g_1 \in V(G)$ and $h_1 \in V(H)$ such that $\{g,g_1\},\{g_1,g'\} \in E(G)$ and $\{h,h_1\},\{h_1,h'\} \in E(H)$.

So $(g,h),(g',h')$ and (g_1,h) form a triangle and also each of these points are connected to every vertex of the form (x,h_1), $x \in V(G)$. Thus, by our Lemma, this subgraph can not be coloured with two colours without getting a monochromatic triangle. □

Theorem: There exists an infinite sequence of (K_3, K_3)-co-critical graphs with

$$d_{max}(G_n) = O(n^{3/4})$$

Proof: Füredi and Seress [1] constructed K_3-saturated graphs with maximal degree $\frac{2}{\sqrt{3}}\sqrt{n} + O(n^{7/24})$. This quantity is sufficiently small for our purposes. Taking the or-product of two such graphs we get a (K_3, K_3)-co-critical graph by the previous theorem. The maximal degree of the product graph is $O(n^{3/4})$. □

Acknowledgment: I would like to thank Ákos Seress for help and encouragement, Miklós Simonovits for improving the presentation of this paper.

[1] Z. Füredi, Á. Seress: Maximal triangle-free graphs with restrictions on the degrees. *Journal of Graph Theory* Vol. 18, No. 1, 11-24 (1994).

[2] A. Galluccio, M. Simonovits, G. Simonyi: On the structure of co-critical graphs. Proceedings of the Seventh International Conference on Graph Theory, Combinatorics, Algorithms and Applications, Kalamazoo, 1992, to appear.