Bounded size components - Partitions and transversals

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Abstract

Answering a question of Alon, Ding, Oporowski and Vertigan [4], we show that there exists an absolute constant C such that every graph G with maximum degree 5 has a vertex partition into 2 parts, such that the subgraph induced by each part has no component of size greater than C. We obtain similar results for partitioning graphs of given maximum degree into k parts ($k > 2$) as well.

A related theorem is also proved about transversals inducing only small components in graphs of a given maximum degree.

1 Introduction

In this paper we are concerned with finding (large) induced subgraphs of graphs of given maximum degree, which induce components of size independent of the size of the graph. We will consider two somewhat different but related setups.

First, we aim at *partitioning* the vertex-set into finitely many parts and require *all* parts to induce small components. In the extreme case, when the components are of size one, this formulation corresponds to the usual *proper coloring* of graphs.

In the second approach, we are given a partition of the vertex-set into large enough classes and we would like to select a transversal (i.e. one vertex from each class) which induces small components. This setup is a generalization of a theorem from [10] concerned with *independent transversals*, a topic that has connections to other areas of combinatorics such as graph colouring.

Let us formalize the above. For a graph G and a fixed k, what is the smallest C for which the vertex set of G can be partitioned into k parts, such that the subgraph induced by each part has no components of size larger than C? As mentioned above, this question can be viewed as a

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generalization of the classical problem of coloring a graph, since $C = 1$ would say precisely that G has chromatic number at most k. The general goal is to find conditions on G that guarantee a constant C independent of n, the number of vertices in G.

Earlier work on this subject [1, 2, 6, 11, 3, 8] mainly focused on more specific questions concerning line graphs of 3-regular graphs. These investigations culminated in [13], in which Thomassen proved that the edges of every 3-regular graph can be 2-colored such that each monochromatic component is a path of length at most 5. Alon, Ding, Oporowski and Vertigan [4] proved a number of results showing that C is independent of n under certain conditions involving bounds on the tree-width and maximum degree of G. In particular, they proved that if G has maximum degree 4, and k is taken to be 2, then $C \leq 57$. On the other hand, they give a family of 6-regular graphs for which every 2-partition of the vertices results in arbitrarily large components in one of the induced subgraphs. They therefore asked the following natural question [4, Question 2.4]: is there a constant C such that every graph G with maximum degree 5 has a vertex partition into 2 parts, each part inducing a subgraph with no components of size greater than C? In Section 2.1 we answer this question in the affirmative. In Section 2.2 we discuss the 2-partitioning of graphs of maximum degree 4, and show that here C could be chosen as small as 6. We also note that C must be at least 4; thus in this case it could very well be feasible to determine the constant C exactly.

As in [4], 2-partitioning theorems lead to partitioning results for certain other values of k; these appear in Section 3. In Theorem 3.2, we show that it is possible to partition a graph G of maximum degree at most 8 into 3 parts, such that each part induces components of size at most an absolute constant C. There is a family of 10-regular graphs that do not admit such a 3-partition [4], so only the case of 9-regular graphs remains undecided. In general, we give lower bounds for the largest maximum degree Δ_k which still accommodates a k-partition into parts with bounded components. An asymptotic upper bound of $4k$ for Δ_k was given in [4]. In Theorem 3.5 we improve the asymptotic lower bound to $(3 + \delta)k$, where $\delta > 0$ is a positive constant.

In Section 4 we consider a related problem concerning transversals that induce only components of bounded size. In [10] it was shown that if the vertex set of a graph with maximum degree Δ is partitioned into classes of size at least 2Δ, then it is possible to choose a set of vertices, one from each class, that is an independent set in G. Such a choice of one vertex from each class is called a *transversal* of the partition. In Theorem 4.1 we generalize this result by showing that if each class has size at least $\Delta + \lfloor \Delta/r \rfloor$ then there exists a transversal that induces in G a subgraph with all components bounded in size by r.

Our discussions leave a number of unresolved problems. These loose threads are gathered together in Section 5.

2 Partitioning into two parts

2.1 Graphs of maximum degree 5

Throughout this paper, by *graph* we will mean simple multigraph, i.e., we allow parallel edges but we do not allow loops. For a graph H and a subset V' of its vertex set, $H[V']$ denotes the subgraph of H induced by the vertices of V'.

The main aim of this section is to prove the following theorem.

Theorem 2.1 *There exists an absolute constant C such that the following holds. Let G be a graph with maximum degree at most 5. Then there is a partition $V_1 \cup V_2 = V(G)$ of the vertex set of G, such that for $i = 1, 2$, each component of $G[V_i]$ has at most C vertices.*

Before beginning the proof of Theorem 2.1, we first establish some properties about a special family of vertex partitions that will be important in the proof. Let G be a graph with maximum degree 5, and let (U_1, U_2) be a maximum cut of G (referred to as a *max-cut*), i.e., a partition of the vertex set of G into classes U_1, U_2, such that the number of edges going between the two classes is maximized. In general, for any partition we will refer to these edges as the edges *going across*, or the *crossing edges*. Let $G' = G[U_1] + G[U_2]$, and let $C_1, \ldots, C_s$ be the components of G'. Let $W = \{v \in V(G) : d_{G'}(v) = 2\}$ be the subset of those vertices whose degree in G' (their *G'-degree*) is exactly two. We denote by H the bipartite subgraph of G consisting of the vertices in W and the edges of G going across the partition $(W \cap U_1, W \cap U_2)$. The vertex sets of the components of H will be called *ladders*. The following proposition collects some simple but important facts.

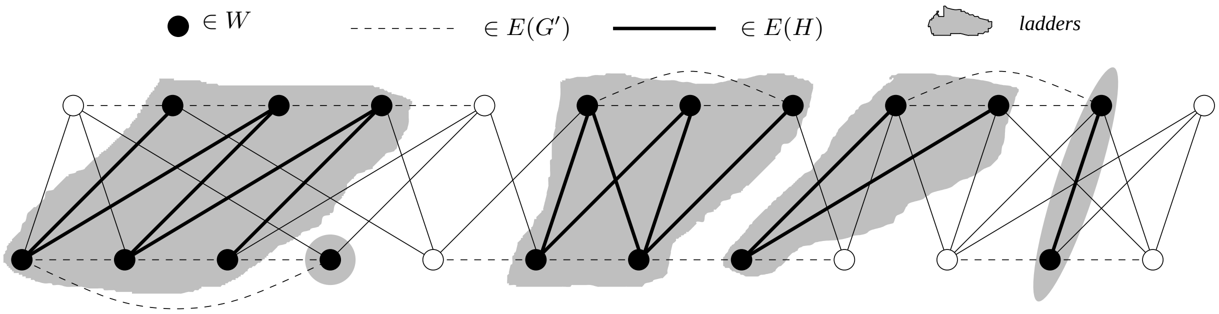

Figure 1: Ladders and such...

Proposition 2.2 *Using the above definitions, the following hold for any max-cut (U_1, U_2).*

(i) $\Delta(G') \leq 2$, *so each component C_i is either a cycle or a path,*

(ii) $\Delta(H) \leq 3$,

(iii) any two H-neighbors of a vertex $w \in W$ are adjacent in G,

(*iv*) *for each ladder L, $L \cap U_j$ consists of consecutive elements of some (path or cycle) component C_k of G', for each $j = 1, 2$. Thus ladders, unless they consists of one vertex, have nontrivial intersection with exactly one component of each side of the partition $U_1 \cup U_2$,*

(*v*) *if $d_H(w) = 3$, and $w \in U_j \cap L$ for some ladder L, then $U_{3-j} \cap L$ consists only of the three H-neighbors of w. Furthermore $|L| \leq 6$.*

Proof.

(*i*) If the degree of a vertex in G' were at least 3, then putting the vertex into the other class would increase the number of edges going across.

(*ii*) This follows immediately from the definition of the vertex set W of H and the fact that $\Delta(G) \leq 5$.

(*iii*) Suppose on the contrary that $w', w'' \in W$ are two H-neighbors of w that are not adjacent in G. Then switching the classes for w, w', w'' would increase the number of edges going across the partition.

(*iv*) Follows directly from (*i*) and (*iii*).

(*v*) by (*iii*), the three H-neighbors of w need to form a triangle in G', which is already a complete component of G'. Thus $U_j \cap L$ can only contain 2 more vertices besides w, since any vertex in $U_j \cap L$ is a neighbor of a neighbor of w, thus (again by (*iii*)) a neighbor of w in G' as well. But w has only two G'-neighbors in U_j. □

The above proposition shows that ladders can consist of just a single vertex, a single edge going across the partition, or, typically, structures like the ones shown in Figure 1.

The next proposition shows that we can find a max-cut that has no long ladders. We remark that the constant 13 can be improved to 10 by a more detailed analysis, but as we do not strive for the optimal constant in Theorem 2.1 this formulation is sufficient.

Proposition 2.3 *Let G be a graph with $\Delta(G) \leq 5$. Then there exists a max-cut $U = (U_1, U_2)$ of the vertex set of G in which each ladder has size at most* 13.

Proof. We say that a max-cut $U = (U_1, U_2)$ has property (M) if $|W|$ is minimized. We fix a partition U having property (M), which minimizes the number

$$l(U) = \sum_L (|L| - 8),$$

where the summation extends over the ladders L of size greater than 8.

We assume U has a ladder of size 14 or more and construct another partition $\bar{U}$ contradicting the choice of U. This contradiction will prove the proposition.

Let L be a ladder of size at least 14. By Proposition 2.2 (*ii*) and (*v*), $\Delta(H[L]) \leq 2$. So we can find vertices $x_1, \ldots, x_{14} \in L$ such that x_i is connected in H to x_{i+1} for $i = 1, \ldots, 13$. We may assume $x_i \in U_1$ for odd i and $x_i \in U_2$ for even i. By Proposition 2.2 (*iii*) we have that x_i and x_{i+2} are adjacent in G (and thus also in G') for $i = 1, \ldots, 12$.

We define the partition $\bar{U} = (\bar{U}_1, \bar{U}_2)$ by switching x_7 and x_8: $\bar{U}_1 = (U_1 \setminus \{x_7\}) \cup \{x_8\}$ and $\bar{U}_2 = (U_2 \setminus \{x_8\}) \cup \{x_7\}$.

First note that the number of crossing edges in $\bar{U}$ is at least the number of crossing edges in U. Hence since U is a max-cut, so is $\bar{U}$, and Proposition 2.2 applies to $\bar{U}$ as well. We denote by $\bar{W}$, $\bar{G}'$ and $\bar{H}$ the analogues (for $\bar{U}$) of W, G' and H, respectively.

Note that the vertices x_5 and x_{10} have degree 1 in $\bar{G}'$, and therefore are not in $\bar{W}$. Since U has property (M), there must be at least two vertices in $\bar{W} \setminus W$. Besides the vertices in W, the only vertices which have a chance to become members of $\bar{W}$ are the neighbors of the displaced vertices x_7 and x_8. Each had four neighbors in W, so both *must* have a fifth one in $\bar{W} \setminus W$. Let $a \in U_1, b \in U_2$ be these neighbors of x_8 and x_7, respectively. Note then that $\bar{U}$ also has property (M). We then have the following (see Figures 2 and 3).

(1) $\bar{W} \cap \bar{U}_1 = (W \cap U_1 \setminus \{x_5, x_7\}) \cup \{x_8, a\}$,

(2) $\bar{W} \cap \bar{U}_2 = (W \cap U_2 \setminus \{x_8, x_{10}\}) \cup \{x_7, b\}$, and

(3) $E(\bar{H}) = (E(H) \setminus \{x_4x_5, x_5x_6, x_6x_7, x_8x_9, x_9x_{10}, x_{10}x_{11}\}) \cup \{x_6x_8, x_7x_9\} \cup E_0$, where E_0 denotes the edges of $\bar{H}$ incident with a or b.

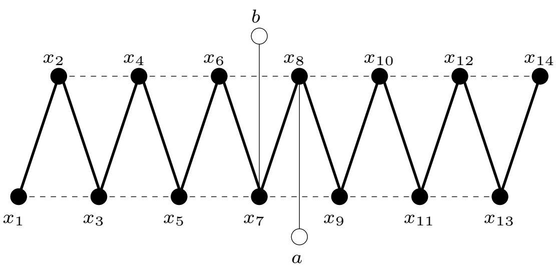

Figure 2: Before...

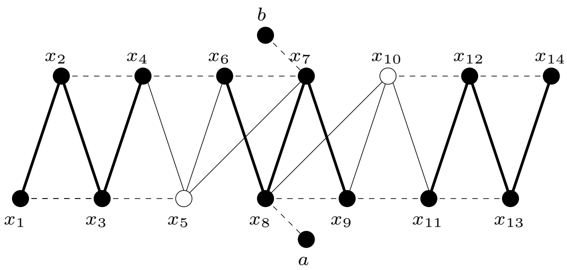

Figure 3: ...and after

We call a ladder of size greater than 8 *long*.

Let $\bar{L}$ be the ladder of $\bar{U}$ containing x_7 (and thus x_8). We claim that $\bar{L}$ is not long. Indeed, otherwise $\bar{H}[\bar{L}]$ would be a path or a cycle by Proposition 2.2(*ii*) and (*v*) and it would extend by at least three vertices beyond at least one end of the path $x_6x_8x_7x_9$. By symmetry we may assume it extends by at least three vertices beyond x_9. By (3) the next vertex must be b. By Proposition 2.2(*iii*) the vertex after that must be a $\bar{G}'$ neighbor of x_9, so (as x_8 is already in the path) it must be x_{11}. Again by (3) the next vertex must be x_{12}. Now by Proposition 2.2(*iii*) b and x_{12} must be connected in G, so also in G'. Since the only G'-neighbors of x_{12} (x_{10} and x_{14}) are in W while b is not, b cannot be a neighbor of x_{12}. This contradiction proves the claim.

Now assume that the ladder $\bar{L}_a$ of $\bar{U}$ containing a is long. Then $x_8 \notin \bar{L}_a$ (otherwise we have $\bar{L}_a = \bar{L}$) and thus by Proposition 2.2(*iii*) a must be the last or next-to-last vertex of the path $\bar{H}[\bar{L}_a]$ (as a has at most a single $\bar{G}'$ neighbor in the same ladder). Similarly, if the ladder $\bar{L}_b$ of $\bar{U}$ containing b is long, then b must be the last or next-to-last vertex in the path $\bar{H}[\bar{L}_b]$.

We now try to establish $l(\bar{U}) < l(U)$ for a contradiction. Let us consider all the ladders of the partition $\bar{U}$. By (3), all of these ladders, except $\bar{L}$, $\bar{L}_a$ and $\bar{L}_b$, are either contained in L or are also ladders in the partition U. Ladders which do not change have equal contribution to $l(U)$ and $l(\bar{U})$. The contribution of $\bar{L}$ to $l(\bar{U})$ is zero (as it is not long). The contribution of $\bar{L}_a$ (or $\bar{L}_b$) to $l(\bar{U})$ is at most 2 more than the contribution to $l(U)$ of the U-ladders it contains. Finally, the total contribution to $l(\bar{U})$ of the $\bar{U}$-ladders contained in L is at least 6 less than the contribution of L to $l(U)$, as the six vertices $x_5, x_6, x_7, x_8, x_9, x_{10} \in L$ are not in a long ladder any more, and the contribution of $|L|$ to $l(U)$ is $|L| - 8 \geq 6$. We thus have

$$l(\bar{U}) \leq l(U) + 2 \cdot 2 - 6 < l(U),$$

a contradiction proving the Proposition. □

Besides Proposition 2.3, the other main ingredient in the proof of Theorem 2.1 will be the well-known Lovász Local Lemma from [9] (see also [5]). The version of the Local Lemma we use is as follows. The constant e below is the base of the natural logarithm.

Theorem 2.4 *Let $A_1, \ldots, A_n$ be events (usually called bad events) in an arbitrary probability space. Suppose that for each i, event A_i is independent of a collection of all but at most d of the other events A_j. If $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$, and $\mathrm{e}p(d+1) \leq 1$, then $\Pr(\bigwedge_{i=1}^{n} \bar{A}_i) > 0$.*

We are now ready to prove the main theorem of this section.

Proof of Theorem 2.1. Let a graph G with maximum degree 5 be given. By Proposition 2.3, we may assume that $V(G)$ has a max-cut $U = (U_1, U_2)$ such that each ladder L has size at most 13 and thus by Proposition 2.2(*iv*), $|L \cap U_j| \leq 7$ for $j = 1, 2$. Let W, G', H, and $C_1, \ldots, C_s$ be as defined just before Proposition 2.2.

Our strategy is the following. We randomly select a set of ladders for which we switch the sides of their vertices, in order to break up all the long components in G'. (Note that each component of G' could intersect many ladders, see Figure 1.) Each ladder is selected for a switch with a suitably chosen probability p, the selections being independent of each other. These events are called *elementary events*. The crucial observation is that by performing any number of ladder-switches at once, the vertices of degree 2 in G', that do not switch sides, *do not receive any new neighbor*. This is true simply because, if a vertex $v \in U_i$ has G'-degree 2 and its G-neighbor $w \in U_{3-i}$ is selected for switching over, then v (being in the same ladder as w) is also selected for the switch. Thus, in choosing a switch that breaks up the large components of G', we just need to take care that the vertices of degree at most 1 in G' do not join up a lot of components via the newly switched vertices. This will be done by applying Theorem 2.4, with some suitably chosen "bad" events.

We begin by fixing a positive constant $p < 1$ satisfying $56\mathrm{e}p^2(90\lceil -\log(56p^2)/p\rceil + 1) \leq 1$, and the constant $\ell_0 = \lceil -\log(56p^2)/p\rceil$. Here and later in this paper, log refers to the natural logarithm. The choice $p = 0.000003$ is suitable. Next, we partition each component C_i of G' for which $C_i \cap W$ intersects at least $2\ell_0$ ladders as follows. We partition these C_i into connected *segments* A^i_j, such that no ladder intersects more than one A^i_j on either side U_1 and U_2, and A^i_j intersects a^i_j consecutive ladders, where $\ell_0 \leq a^i_j < 2\ell_0$.

Let us define the set of bad events we would like to avoid.

Bad event type (i). For each segment A^i_j, let E^i_j be the event that no ladder of A^i_j is picked for switching. The probability of E^i_j is $(1-p)^{a^i_j} \leq (1-p)^{\ell_0} < \mathrm{e}^{-p\ell_0}$. Hence by definition of ℓ_0 we see that $\Pr(E^i_j) \leq 56p^2$.

Bad event type (ii). For any path component C_i with endpoints u and v (if C_i has length 0 then $u = v$), let E_{C_i} be the event that at least two ladders, containing a neighbor of u or v on the side of the partition opposite to C_i, are picked for switching. Suppose there are k ladders which contain a neighbor of u or v. As $\Delta(G) \leq 5$, $k \leq 8$. Then

$$Pr(E^i_j) \leq \binom{k}{2} p^2 \leq 28p^2.$$

Bad event type (iii). Finally, fix a numbering of the consecutive ladders of each component C_i, and define the event F^i_j such that the j^{th} and $(j+1)^{st}$ ladder of C_i are *both* picked for switching. The probability of F^i_j is clearly p^2.

In order to estimate the parameter d in Theorem 2.4, we fix for each event E a *determining set* $D(E)$ consisting of elementary events that together determine whether E happens. The independence of the elementary events implies that any event E is mutually independent of the set of all events whose determining sets are disjoint from $D(E)$.

From the definitions we see that $D(E^i_j)$ can consist of the a^i_j elementary events corresponding to the ladders intersecting A^i_j. The determining set $D(E_{C_i})$ can consist of the elementary events corresponding to ladders containing some neighbor of an endpoint of C_i, so $|D(E_{C_i})| \leq 8$. Finally $D(F^i_j)$ can consist of the two elementary events corresponding to the j^{th} and $(j+1)^{st}$ ladders of component C_i. On the other hand, an elementary event E_M, corresponding to a ladder M, is contained in the determining set of at most 2 bad events of type (i), the ones corresponding to segments containing its two sides. Also, E_M is contained in the determining set of at most 4 bad events of type (iii), at most two on each side. Finally, M has at most 13 vertices, each of them is the neighbor of an endpoint of at most 3 different components of G' on the opposite side of the partition, so E_M is contained in the determining set of at most 39 bad events of type (ii).

Thus for any bad event E, there are at most $|D(E)|(2+39+4)$ bad events E' with $D(E) \cap D(E') \neq$

$\emptyset$. This implies that each bad event is independent of the set of all but $45|D(E)| \leq 90\ell_0$ bad events. Set $d = 90\ell_0$.

We may now apply Theorem 2.4 to the set of bad events. Since each bad event occurs with probability at most $56p^2$, and $56ep^2(d+1) \leq 1$ by definition of p, we conclude that there exists a selection of ladders that can be switched without causing any bad event. Let us perform such a switch, and denote the classes of the resulting partition of G by V_1 and V_2.

Claim 2.5 *Each component in $G[V_1]$ or $G[V_2]$ has at most $588\ell_0 + 7$ vertices.*

Proof. Let us stop for a second in the middle of the switch, after the vertices of the chosen ladders were removed from their respective sides, but were not yet placed on the other. Since there are no bad events of type (i), each large component C_i is broken into pieces by the removal of a ladder from each of its segments A^i_j intersecting at most $2\ell_0$ ladders. So at most $28\ell_0$ vertices could stay together from an old component C_i, since each ladder contributes at most 7 vertices.

Now new vertices are coming over from the other side by the switch. Since there are no bad events of type (iii), no two consecutive ladders arrive, thus the vertices coming from the other side arrive in components of size at most 7.

We still have to make sure that not too many "old" and "newly arrived" components stick together. As we noted earlier, if a vertex of G' of degree 2 does not switch sides, then it does not receive any new neighbors. So old and new components can stick together only through a vertex of an old component whose degree in G' was at most 1 (it was the endpoint of a path component of G'). As there are no bad events of type (ii), at most 1 new ladder is connected to any old component. One new ladder brings at most 7 vertices, each of which can be connected to at most 3 old components, thus at most $7 + 21 \cdot 28\ell_0$ vertices stick together to form a component within a class V_i. $\square$

This finishes the proof of Theorem 2.1. $\square$

Remark In the proof of Theorem 2.1 we do not attempt to obtain the smallest possible value of C. By making more careful estimates, and using Theorem 4.1 with $r = 1$ instead of Theorem 2.4, one can show that $C \leq 17617$. However, as this value is almost certainly very far from being optimal, we do not include the details here.

2.2 Graphs of maximum degree 4

In this subsection we improve on a result of [4]. We reduce, from 57 to 6, the maximum size of the components one can guarantee when 2-partitioning graphs of maximum degree 4. Our argument depends on the following useful lemma about partitioning a pair of graphs on the same set of vertices. This same lemma will be applied also in Section 4 to obtain a result on transversals that induce only small components.

Lemma 2.6 *Let G_1 and G_2 be multigraphs with maximum degree at most 2 on the same vertex set X. Then there exists a partition of X into two parts, X_1 and X_2, such that for each $i \in \{1,2\}$ we have $\Delta(G_i[X_i]) \leq 1$.*

Proof. First we assign an arbitrary orientation to each path or cycle in G_1 and G_2, so from now on we consider them as directed graphs. We denote by v_i^+ and v_i^- the out-neighbor and in-neighbor of v in G_i respectively, if they exist. We construct the partition one vertex at a time, beginning by placing an arbitrary vertex in X_1. We never remove a vertex from its part of the partition once it has been placed. In general, after having placed a vertex v in X_i we do the following.

(a) If v_i^+ exists and is not already placed, we place it in X_{3-i}.

(b) Otherwise if v_i^- exists and is not placed yet, we place v_i^- in X_{3-i}.

(c) If neither (a) nor (b) applies, we select an arbitrary unplaced vertex and place it in X_1.

We claim that this procedure produces a partition $X_1 \cup X_2$ with the desired property. To see this, first suppose on the contrary that three distinct consecutive vertices x, y, and z in G_i are all placed in part X_i of the partition, where $y = x_i^+$ and $z = y_i^+$. Then by the construction, the first vertex of $\{x, y, z\}$ to be placed in X_i must have been z, otherwise by (a) the very next step would have been to place y or z in X_{3-i}. For the same reason, the next to be placed in X_i was y. But then at this point $z = y_i^+$ is already placed, so the next step is to place x in X_{3-i}, contradicting our assumption.

Now suppose that x and y are the two vertices of a two-vertex cycle in G_i. Then without loss of generality, x is placed in X_i first. But then by (a), the next step is to place $y = x_i^+$ in X_{3-i}. This completes the proof of the lemma. □

We are now ready to turn to the main result of this subsection.

Theorem 2.7 *Let G be a graph with maximum degree 4. Then the vertex set of G can be partitioned into two parts $V_1 \cup V_2 = V(G)$ such that each part induces components of size at most 6.*

Proof. Let us start with a max-cut $U_1 \cup W_2 = V(G)$, with the additional property that it has the minimum number of vertices in U_1.

Let $G_1 = G[U_1]$ and $G_2' = G[W_2]$. Since the number of edges going across is maximum, every vertex has degree at most 2 in each of G_1 and G_2'. The minimality of $|U_1|$ implies that G_1 has maximum degree at most 1 as switching a degree 2 vertex of G_1 over to W_2 does not decrease the number of edges going across.

Let S be a maximum size independent set of degree 2 vertices of G_2', and let us define $U_2 = W_2 \setminus S$, $W_1 = U_1 \cup S$, $G_2 = G[U_2]$ and $G_1' = G[W_1]$. Clearly, every element of S has degree 2 in G_1' and the partition (W_1, U_2) is also a max-cut. So G_1' has maximum degree at most two. The set S is a maximum size independent set of the degree 2 vertices of G_1' because if S' is another independent set, then $(W_1 \setminus S', U_2 \cup S')$ is another max-cut of $V(G)$, so $|W_1 \setminus S'| \geq |U_1|$. By the choice of S, G_2 has maximum degree at most one.

Thus G_1 and G_2 consist of disjoint edges and vertices, while G_1' and G_2' are the disjoint union of cycles and paths (possibly of length 0).

Our strategy is to split S between the two sides using Lemma 2.6.

We define the auxiliary graphs H_i for $i = 1, 2$ on the vertex set S by letting two vertices of S be adjacent in H_i if they are at distance 2 or 3 in G_i'. We have $\Delta(H_i) \leq 2$ as S is an independent set of the graph G_i' and $\Delta(G_i') \leq 2$.

We now apply Lemma 2.6 to H_1 and H_2 to obtain a partition $X_1 \cup X_2$ of S for which $\Delta(H_i[X_i]) \leq 1$ for $i = 1, 2$. We let the classes of the final partition be $V_i = U_i \cup X_i$ for $i = 1, 2$. Notice that $V_i \subseteq W_i$, and since W_i spans the graph G_i' of maximum degree at most 2, each component of the graph $G[V_i]$ is a path or a cycle. Suppose such a component D is of size 7 or more. As S is a maximum size independent subset of the degree two vertices of G_i', it must contain at least three vertices of D. To be in $D \subseteq V_i$ all of these vertices must be in X_i and they are in a component of $H_i[X_i]$ contradicting the choice of X_i. The contradiction proves that all components of $G[V_i]$ are of size 6 or less, as claimed. $\square$

Remark Even the improved bound on the component size in Theorem 2.7 is not known to be optimal. The complement of the seven-cycle shows that the same statement with component size less than four is false. It is a 4-regular graph, and one can easily verify that any subset of the vertices of size four or more span a connected subgraph.

3 Partitioning into several parts

In this section we discuss a problem analogous to the one considered in Theorem 2.1, but now we partition the vertices into more than two parts. In several cases we utilize ideas from [4]. We also need the following partitioning result of Lovász [12].

Theorem 3.1 *Let G be a graph and let $k_1, \ldots, k_m$ be non-negative integers such that $k_1 + \ldots + k_m \geq \Delta(G) - m + 1$. Then $V(G)$ has a partition $V_1 \cup \ldots \cup V_m$ such that $\Delta(G[V_i]) \leq k_i$ for each i.*

Our first result is the analogue of Theorem 2.1 for partitioning into 3 parts.

Theorem 3.2 *There exists a constant C' such that the vertex set of any graph of maximum degree at most 8 can be 3-partitioned such that each part spans subgraphs with components of size at most C'.*

Proof. Let G be a graph of maximum degree at most 8. Using Theorem 3.1, we partition the vertex set of G into two parts $U_1 \cup U_2 = V(G)$ such that $\Delta(G[U_1]) \leq 5$ and $\Delta(G[U_2]) \leq 2$. By Theorem 2.1, U_1 can be partitioned into two parts $U_1 = V_1 \cup W$, each spanning components of size bounded by the constant C guaranteed by the theorem. As in [4], we apply the theorem on independent transversals from [10] (which is the special case of our Theorem 4.1 with $r = 1$) to get rid of the long paths and cycles in U_2. We define an auxiliary graph H on the vertex set U_2, by connecting two vertices with an edge if they are adjacent in G or connected to the same component of $G[W]$. Clearly, $\Delta(H) \leq 64C$. We split each component of $G[U_2]$ into pairwise disjoint path-segments of length $128C$, such that in any component at most $128C$ vertices are not covered by these segments. By Theorem 4.1, there exists a transversal T of the segments which is an independent set of H. We define the classes of the 3-partition of $V(G)$ to be V_1, $V_2 = W \cup T$ and $V_3 = U_2 \setminus T$. Observe that all components of $G[V_2]$ are of size at most $8C + 1$, since T is independent in H. The components of $G[V_3]$ are of size less than $384C$, since T is a transversal of the above defined segment-partition. As all components of $G[V_1]$ are of size at most C, the theorem is proved with constant $C' = 384C$. $\square$

In [4] it is proved that every graph with maximum degree Δ can be partitioned into $\lceil(\Delta + 2)/3\rceil$ classes, such that each class has components of size at most $f(\Delta)$. In the following we slightly improve this result in two ways. On one hand we show that $\lceil(\Delta + 1)/3\rceil$ classes suffice for any Δ, and that only $(1/3 - \epsilon)\Delta$ classes are needed for suitably large values of Δ, where ϵ is a small positive constant. On the other hand for these results we obtain partitions where the size of the components is independent of Δ.

In general, our next theorem contains a slightly weaker statement than its follow-up, but besides being instructional, it provides a stronger result for small values of Δ.

Theorem 3.3 *There exists a constant C' such that the following holds. Let G be a graph of maximum degree Δ. Then it is possible to $\lceil(\Delta + 1)/3\rceil$-partition the vertex set such that each part spans components of size at most C'.*

Proof. First suppose $\lceil(\Delta + 1)/3\rceil = 2k$ is even.

Let us partition the vertex set into k classes $V_1 \cup \ldots \cup V_k = V(G)$, such that the number of edges going *within* the classes is minimized. As $\Delta \leq 6k - 1$ the maximum degree of the graph $G[V_i]$ is at most 5 for every $i = 1, \ldots, k$. By Theorem 2.1 each V_i can be separated into two parts $V_i' \cup V_i'' = V_i$, such that both parts induce graphs with largest component size bounded by C, where C is as in Theorem 2.1.

Thus $\cup_{i=1}^{k}(V_i' \cup V_i'')$ is an appropriate partition into $2k$ classes.

Next we consider the case when $\lceil(\Delta + 1)/3\rceil = 2k + 1$ is odd. Let us partition the vertex set into k classes $V_1 \cup \ldots \cup V_k = V(G)$, such that $\Delta(G[V_i]) \leq 5$ for $i = 1, \ldots, k - 1$ and $\Delta(G[V_k]) \leq 8$. Such a partition exists by Theorem 3.1. Now we use Theorem 3.2 to 3-partition V_k and Theorem 2.1 to 2-partition each of the other classes. Then all components spanned by any of the resulting $2k + 1$ parts are bounded in size by the constant from Theorems 2.1 or 3.2. $\square$

In order to improve on the constant multiplier 1/3 of Δ, first we show that in fact, for large k, any $6k$-regular graph can be partitioned into $2k$ parts with bounded size components.

Theorem 3.4 *There exist constants K and C'' such that the following holds. Let G be a graph with maximum degree at most $6k$, $k \geq K$. Then it is possible to $2k$-partition the vertex set such that each part contains components of size at most C''.*

Proof. We choose C as in Theorem 2.1 and set $K = 450C^3$ and $C'' = 6C + 1$. Let us start again with a partition into k classes $V_1 \cup \ldots \cup V_k = V(G)$, such that the number of edges going *within* the classes is minimized. Now we cannot say that the maximum degree of each graph $G[V_i]$ is at most 5 for every $i = 1, \ldots, k$; there *could be* some vertices whose degree within their class is six. Let M be the set of these vertices. By choosing our partition such that $|V_k|$ is maximal, we can assume that all of M is contained in V_k. (A vertex $v \in M$ has exactly six neighbors in each class, so it could be moved to V_k without increasing the number of edges within the classes.) Therefore $\Delta(G[V_i]) \leq 5$ for $i = 1, \ldots, k-1$.

Let $W \subseteq M$ be a maximum independent set in $G[M]$. Clearly, $G[V_k \setminus W]$ has maximum degree at most 5. By Theorem 2.1 each V_i, $i = 1, \ldots, k-1$, and $V_k \setminus W$ can be partitioned into two parts V_i' and V_i'', and respectively V_k' and V_k'' such that all $G[V_i']$ and $G[V_i'']$ have components bounded by C.

Our goal is to distribute the vertices of W among these $2k$ classes, such that they don't glue too many existing components together. We put each vertex into a certain class with probability $p = 1/(2k)$, the choices for distinct vertices being mutually independent.

One vertex $v \in W$ has at most 6 neighbors in a class, so it can glue together at most 6 components in that class. Thus if we can make sure that no component receives more than 1 neighbor, after W is distributed the largest component in each class will have size at most $6C + 1$. It is important to note here that the vertices arriving from W are independent, so arrive in components of size 1.

We plan to use the Lovász Local Lemma, Theorem 2.4. For each component F of $G[V_i']$ or $G[V_i'']$ we define a bad event E_F: that at least two neighbors of vertices of F from W are put in the class of F. Suppose there are f neighbors of the vertices of F in W. Then $Pr(E_F) \leq \binom{f}{2}p^2$.

An event E_F is independent of the set of all events $E_{F'}$ where F and F' have no common neighbor in W. A vertex $u \in F$ with degree d_u within its class V_i, has at most $6 - d_u$ neighbors in W. Otherwise moving these neighbors into V_i would increase d_u above 6, implying that the number of edges within the classes is not minimal (the moving of a subset of W does not change that; again independence of W is critical). Thus F has at most $6C$ neighbors in W, each of those possibly having $6k - 1$ other adjacent components F'. Therefore the parameter d in Theorem 2.4 can be taken to be $d = 36kC - 6$.

By Theorem 2.4, if $\mathrm{e}(36kC - 5)\binom{f}{2}p^2 < 1$, then with positive probability *none* of the bad events happen. In particular there is an assigment of the vertices of W to the classes, such that no component larger than $6C + 1$ is created. Since $f \leq 6C$ the above condition is satisfied. This completes the proof. $\square$

Theorem 3.5 *There exist constants $\epsilon > 0$, C'' and Δ_0 such that the following holds. Every graph*

G of maximum degree $\Delta \geq \Delta_0$ can be partitioned into $\Delta(1/3-\epsilon)$ classes, such that each class spans a graph with components bounded by C''.

Proof. Let K and C'' be the constants claimed by the preceding theorem. Let $t = \lceil(\Delta+1)/(6K+1)\rceil$. Partition $V(G)$ into t parts $U_1, \ldots, U_t$ such that the number of edges going within the classes is minimal. Then each graph $G[U_i]$ is of maximum degree at most $6K$. By the previous theorem we can partition it into $2K$ parts such that each part spans a graph with maximum component size at most C''. This therefore gives an appropriate partitioning into

$$2Kt < \frac{2K}{6K+1}\Delta + 2K$$

parts. Thus $\epsilon = 1/(36K+6)$ and $\Delta_0 = 200K^2$ are appropriate choices. $\Box$

Remark Let us say that degree Δ allows d-partitioning if the following is true: There is a constant C such that the vertices of any graph of maximum degree at most Δ can be d-partitioned with each part spanning components of size at most C. The idea of the previous proofs easily generalizes to the following. If degree Δ allows d-partitioning, then degree $k(\Delta+1)-1$ allows kd-partitioning for any $k \geq 1$ and degree $k(\Delta+1)$ allows kd-partitioning for large enough k.

4 Transversals inducing bounded size components

For a vertex v, we denote by $C(v, H)$ the component of v in the graph H. Whenever there is no ambiguity about the base graph, we write $C(v, V')$ instead of $C(v, H[V' \cup \{v\}])$.

Let G be a graph and let $\mathcal{P}$ be a partition of $V(G)$ into sets $V_1, \ldots, V_m$. A *transversal* of $\mathcal{P}$ is a subset $\{v_1, \ldots, v_m\}$ of $V(G)$ for which $v_i \in V_i$ for each i. In this section we are concerned with the problem of finding transversals T with the property that $G[T]$ has only small components. The following theorem was proved for component size $r = 1$ in [10]. Here we prove a generalization for arbitrary r.

Theorem 4.1 *Let r, d be arbitrary positive integers. Let G be a graph of maximum degree d, and let $\mathcal{P}$ be a partition $V_1 \cup \ldots \cup V_m = V(G)$ of $V(G)$ such that $|V_i| \geq d + \lfloor d/r \rfloor$ for $i = 1, \ldots, m$. Then there exists a transversal T of $\mathcal{P}$ such that the induced subgraph $G[T]$ has components of size at most r.*

Proof. Let T_0 be a maximal size partial transversal of $\mathcal{P}$ such that all components of $G[T_0]$ have size at most r. We assume for contradiction that T_0 is not a complete transversal. Let $\mathcal{T}$ be the set of partial transversals T of $\mathcal{P}$ which span only components of size at most r and satisfy $|T \cap V_i| = |T_0 \cap V_i|$ for $i = 1, \ldots, m$.

We call a pair (W, T) with $T \in \mathcal{T}$ and $W \subseteq V(G) \setminus T$ *feasible* if

(a) the sets $C(v, T)$ are pairwise disjoint for $v \in W$ and each of them is of size at least $r+1$, and

(b) there is no $v_0 \in W$ and $T' \in \mathcal{T}$ with $T' \cap W = \emptyset$ such that $|C(v_0, T')| < |C(v_0, T)|$ and $C(v, T') = C(v, T)$ for every $v \in W \setminus \{v_0\}$.

Clearly, $(\emptyset, T_0)$ is feasible. We choose a feasible pair (W, T) with $|W|$ being maximal. Our goal is to construct another feasible pair contradicting the maximality of $|W|$ and by this contradiction proving the theorem.

We let $H = \cup_{v \in W} C(v, T)$ and $S = \{j \in [m] : V_j \cap T \subseteq H\}$. By (a) we have $|H| \geq (r+1)|W|$. Each vertex in $H \setminus W$ is in T, thus we have $|S| > |H| - |W|$. (The strict inequality follows from our assumption that T is not a complete transversal of $\mathcal{P}$, since S contains each index $i \in [m]$ with $V_i \cap T = \emptyset$.)

We claim that there exists a vertex $v' \in \cup_{i \in S} V_i \setminus H$ that is not connected to any vertex in H. We prove this by simple counting of the number of possible choices for v' and the number of vertices excluded by being neighbors of some vertices in H. The number of choices is $|\cup_{i \in S} V_i \setminus H| \geq |S|(d + \lfloor d/r \rfloor) - |H| > (|H| - |W|)(d + \lfloor d/r \rfloor) - |H|$. Each vertex in H has at most d neighbors to exclude. But $G[H]$ consists of at most $|W|$ components, so there are at least $|H| - |W|$ edges between vertices of H. These edges contribute to the degree of vertices in H, but they do not exclude any vertices to be considered as v'. The number of excluded vertices is thus at most $d|H| - 2(|H| - |W|)$. To conclude the proof of this claim we need $(|H| - |W|)(d + \lfloor d/r \rfloor) - |H| \geq d|H| - 2(|H| - |W|)$, which follows from simple rearrangement of the inequality $|H| \geq (r+1)|W|$. Note that $v' \notin T$ by definition of S.

We now choose the partial transversal T' that minimizes $|C(v', T')|$, among all partial transversals $T' \in \mathcal{T}$ satisfying $T' \cap (W \cup \{v'\}) = \emptyset$ and $C(v, T') = C(v, T)$ for all $v \in W$. (Notice that we are choosing from a nonempty set, as T is a partial transversal satisfying these properties.) We claim that $(W \cup \{v'\}, T')$ is a feasible pair, contradicting the choice of (W, T).

For condition (a), consider the sets $C(v, T') = C(v, T)$ for $v \in W$; these are pairwise disjoint and of size at least $r + 1$. The last set $C(v', T')$ is disjoint from any set $C(v, T')$ $(v \in W)$, as otherwise a neighbor of v' would be in $C(v, T') \subseteq H$. Now assume for contradiction that $|C(v', T')| \leq r$. Let V_i be the class in partition $\mathcal{P}$ that contains v'. If $V_i \cap T' = \emptyset$ then $T' \cup \{v'\}$ is a partial transversal of $\mathcal{P}$ spanning components of size at most r, contradicting the maximality of T_0. Otherwise $V_i \cap T' \neq \emptyset$, and hence $V_i \cap T \neq \emptyset$. Since $i \in S$, we must have $V_i \cap T = \{u\}$ with some vertex $u \in C(w, T)$ for some $w \in W$ (see Figure 4). Since $C(w, T') = C(w, T)$, and $w \notin T'$, we see that $T' \cap C(w, T) = T \cap C(w, T)$ so we also have $u \in T'$. Therefore $T'' = (T' \setminus \{u\}) \cup \{v'\}$ is a partial transversal in $\mathcal{T}$. Since v' does not have neighbors in H we get that $C(v, T'') = C(v, T)$ for all $v \in W \setminus \{w\}$ and $C(w, T'') \subseteq C(w, T) \setminus \{u\}$. This contradicts property (b) of the feasibility of (W, T) and thus proves property (a) of the feasibility of $(W \cup \{v'\}, T')$.

Finally, for condition (b) in the definition of feasibility of $(W \cup \{v'\}, T')$ notice that for $v_0 \in W$ this condition simply follows from the corresponding condition of the feasibility of (W, T). For $v_0 = v'$ the condition follows from the choice of T'.

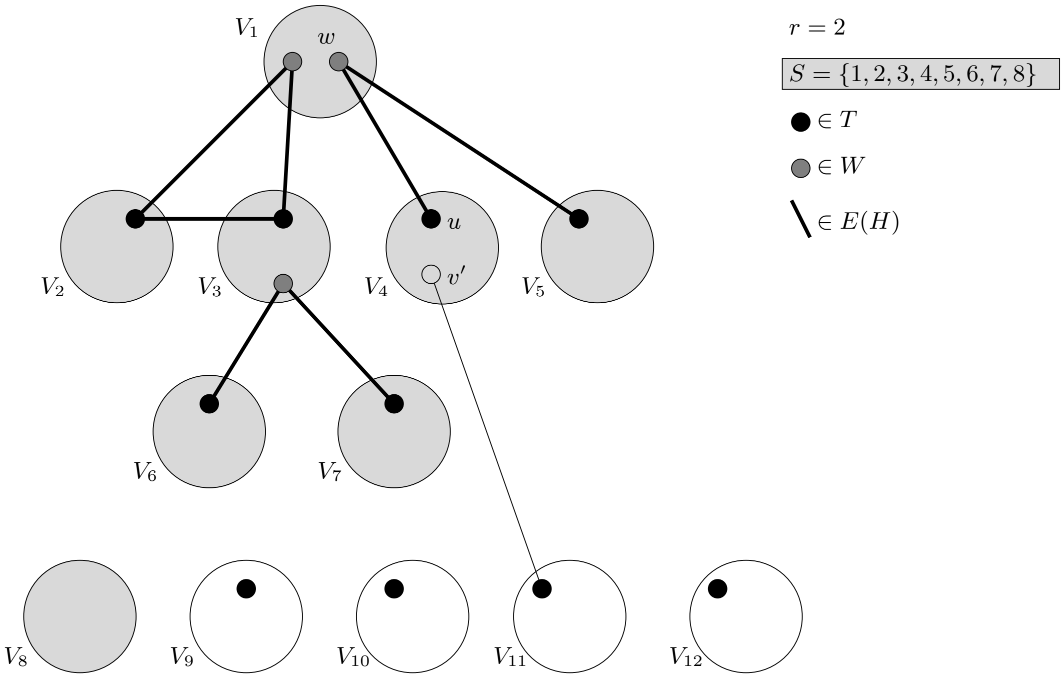

Figure 4: A feasible pair extremum?

The contradiction of the feasibility of $(W \cup \{v'\}, T')$ with the maximality of (W, T) implies the theorem. $\square$

We state the $r = d + 1$ special case of the above result separately:

Corollary 4.2 *Let G be a graph of maximum degree d. Let $V_1 \cup \ldots \cup V_k = V(G)$ be a partition of the vertex set into subsets with $|V_i| \geq d$ for each i. Then it is possible to choose a transversal T such that $G[T]$ has components of size at most $d + 1$.*

The above corollary is optimal in terms of the class size. No upper bound can be given on the component size of a transversal if the classes are size $d - 1$. This can be seen by considering the complete $(d-1)$-ary tree H with root w. Partition the vertex set of $H \setminus \{w\}$ by letting the sets of $d - 1$ sibling vertices be the classes. This way the largest component in *any* transversal will be the depth of the tree, which can be arbitrarily large.

We remark however that the above corollary is probably *not* optimal in the component size. Indeed, the following corollary tells us that in the special case $d = 2$ one can have components of size at most 2. We do not know if a similar statement limiting the component size by 2 instead of $d + 1$ holds for larger d.

Corollary 4.3 *Let G be graph with maximum degree at most 2 with its vertex set partitioned into 2-element subsets. Then it is possible to select a transversal T of this partition such that $\Delta(G[T]) \leq 1$.*

Proof. Apply Lemma 2.6 to the graphs $G_1 = G$ and G_2 constructed on the same vertex set $V(G)$ by placing two parallel edges between each pair of vertices belonging to the same set V_i. The resulting set X_1 satisfies $\Delta(G[X_1]) \leq 1$ and X_1 contains at least one vertex from each set V_i. Taking one vertex from each $V_i \cap X_1$ gives a transversal of the required type. □

5 Remarks and Open Problems

1. Theorem 4.1 leads us to the following question. For a fixed degree d and component size r, let us define $p(d, r)$ to be the smallest integer such that any d-regular graph partitioned into classes of size at least $p(d, r)$ has a transversal that spans only components of size at most r. In Section 4 we showed $d \leq p(d, r) \leq d + \lfloor d/r \rfloor$. Tight results are known for $r = 1$ [7, 14], when $p(d, 1) = 2d$. We also have $p(2, 2) = 2$ by Corollary 4.3. Right now it is even possible that $p(d, 2) = d$ for all d. Any asymptotic tightening of the gap between the upper and lower bounds would be very interesting. The smallest unknown case is $p(3, 2)$; that is, how big must the partition classes of a 3-regular graph be, to guarantee the existence of a transversal that spans at most a matching? The answer is either 3 or 4.

2. With a more detailed analysis we can prove a maximum component size $C = 17617$ in Theorem 2.1, but it is definitely far from the truth. The determination of the smallest possible such C would be of interest but might be out of reach. Not so for Theorem 2.7; there the required maximum component size is between 4 and 6.

3. There are lots of questions concerning the partitioning of graphs into more than two parts. The most general one is to determine for every fixed k the largest maximum degree Δ_k, such that every graph with maximum degree Δ_k can be partitioned into k parts, where each part spans components of size bounded by a constant. In Section 2 we proved $\Delta_2 = 5$. As shown in [4], $\Delta_k < 4k - 2$ for any k, while for large enough k Theorem 3.5 implies $(3 + \delta)k < \Delta_k$ with a positive constant $\delta > 0$. It would be of great interest to determine Δ_k asymptotically.

The smallest unknown case is interesting in its own right: we don't know whether Δ_3 is 8 or 9. In other words, is it possible to color the vertex set of a graph with maximum degree 9 by three colors such that every monochromatic component is bounded by a constant?

4. In the following we define a density version of the results of Section 2. We intend to weaken the maximum degree condition by bounding the density of the graph, which allows a few very large degree vertices. We find this question interesting but can only show modest results.

Let $\mu(G) = \max\{|E(G[W])|/|W| : W \subseteq V(G)\}$. We raise the problem of determining the supremum value α, such that every graph G with $\mu(G) < \alpha$ has a partition into two parts spanning components of bounded size. Here we can only show that $1 \leq \alpha \leq 2$.

To see the upper bound, consider the following construction. Let $n \geq 1$ and let A_n be the graph with $2n + 1$ vertices and $4n - 1$ edges, such that two vertices of A_n have degree $2n$. Notice that whenever we 2-partition the vertex set of A_n such that each part spans components of size at most

n, the two full-degree vertices must be placed in the same part. Now consider the graph B_n that is the union of n isomorphic copies of A_n sharing a single common vertex x that has full degree in each of the graphs. Notice that $\mu(B_n) < 2$ for all n. If the vertex set of B_n is 2-partitioned then either one part spans a component of size more than n of a copy of A_n, or the part containing x contains the other n high-degree vertices and they form a component of size greater than n. Therefore $\alpha \leq 2$.

For the lower bound, we claim that if $\mu(G) \leq 1$, then G can be 2-partitioned, $V_1 \cup V_2 = V(G)$, such that for $i = 1, 2$, each component of $G[V_i]$ has at most 2 vertices. Indeed, by the density condition each component of G is a tree or has unique cycle. Therefore it is possible to remove a matching M from G, such that $G - M$ is bipartite. Then $G - M$ could be two-partitioned into two independent sets. Adding back the edges of the matching will create components of size at most two.

Similar problems could be raised for partitioning into k parts, $k > 2$, as well.

Note added in proof. Recently A. Kostochka improved the lower bound on α from Remark 4. He showed that for every $\epsilon > 0$ there exists a constant $C = C(\epsilon)$, such that any graph with $\mu(G) \leq 3/2 - \epsilon$ can be partitioned into two parts spanning components of at most C vertices.

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