Bart - Moe games, JumbleG and Discrepancy

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Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be hypergraphs with a common vertex set V. In a $(p, q, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game, the players take turns selecting previously unclaimed vertices of V. The game ends when every vertex has been claimed by one of the players. The first player, called Bart (to denote his role as Breaker and Avoider together), selects p vertices per move and the second player, called Moe (to denote his role as Maker or Enforcer), selects q vertices per move. Bart wins the game iff he has at least one vertex in every hyperedge $B \in \mathcal{B}$ and no complete hyperedge $A \in \mathcal{A}$. We prove a sufficient condition for Bart to win the $(p, 1)$ game, for every positive integer p. We then apply this criterion to two different games in which the first player's aim is to build a pseudo-random graph of density $\frac{p}{p+1}$, and to a discrepancy game.

1 Introduction

An unbiased positional game is a pair $(X, \mathcal{H})$, where the set X is called the "board", and $\mathcal{H} \subseteq 2^X$ is the family of "winning subsets". During the game two players alternately occupy elements of the board. The first player, called Occupier, wins the game if at the end of the game the subset of the board he occupies is a winning subset, otherwise the second player, called Preventer, wins.

Classical examples of this setting are Maker/Breaker-type games, in which case $\mathcal{H}$ is a monotone increasing family. Maker plays the role of Occupier and Breaker the role of

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Preventer. Once Maker occupies a minimal element of $\mathcal{H}$ with respect to inclusion, the game can be stopped as Maker has already ensured his win. In fact, sometimes we will include an element of 2^X in $\mathcal{H}$ iff it is a minimal winning subset. Not as well studied but equally interesting is the case of a monotone decreasing $\mathcal{H}$ which corresponds to Avoider/Enforcer-type games. In this case Occupier wins if he *avoids* occupying a member of $2^X \setminus \mathcal{H}$, hence plays Avoider in an Avoider/Enforcer-type game $(X, 2^X \setminus \mathcal{H})$.

Frieze et al. [4] studied positional games where the family of winning sets is the intersection of a monotone increasing family and a monotone decreasing family. Here we generalize their results to biased games, that is, when Occupier occupies p elements of the board per move instead of 1. One of the major motivating ideas behind this approach is to try and create pseudo-random graphs of the appropriate edge-density. These graphs can then be used to prove that numerous other natural games of the Maker/Breaker-type can be won by Maker. We will not discuss here the notion of pseudo-random graphs in much detail. the interested reader is referred to a recent survey [6] on the subject. Very generally speaking, a pseudo-random graph is a graph whose edge distribution resembles closely that of a truly random graph of the same density on the same number of vertices.

Our setting is the following. Let $\mathcal{A}$ and $\mathcal{B}$ be hypergraphs with a common vertex set V. In a $(p, q, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game (consult the Simpsons series for the origin of the names; a more mathematical explanation is given later) the players take turns selecting previously unclaimed vertices of V. The first player, called Bart (to denote his role as Breaker and Avoider together), selects p vertices per move and the second player, called Moe (to denote his role as Maker or Enforcer), selects q vertices per move. The game ends when every element of V has been claimed by one of the players. Bart wins the game iff he has at least one vertex in every hyperedge $B \in \mathcal{B}$ and no complete hyperedge $A \in \mathcal{A}$. We prove the following sufficient condition for Bart to win the $(p, 1)$-game.

Theorem 1.1 *For hypergraphs $\mathcal{A}$ and $\mathcal{B}$, if*

$$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} + \sum_{B \in \mathcal{B}} (1 + p)^{-|B|} < \left(1 + \frac{1}{p}\right)^{-p}$$

then Bart has a winning strategy for the $(p, 1, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game.

Remark 1 Theorem 1.1 is a generalization of special cases of several known results. If $\mathcal{A} = \emptyset$ then we get a Maker-Breaker game on $\mathcal{B}$ for which Breaker has a winning strategy if

$$\sum_{B \in \mathcal{B}} (1 + p)^{-|B|} < \left(1 + \frac{1}{p}\right)^{-p}.$$

This is almost as good (and can be made as good by trivial changes to the proof) as a result of Beck for $q = 1$ (c.f. [2]).

If $\mathcal{B} = \emptyset$ then we get an Avoider-Enforcer game on $\mathcal{A}$ for which Avoider has a winning strategy if

$$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} < \left(1 + \frac{1}{p}\right)^{-p}.$$

This is the same as a result we obtain in [5] for $q = 1$.
If $\mathcal{A} = \mathcal{B}$ then we get a sufficient condition for the first player to win the $(p, 1, \mathcal{A})$ 2-coloring game. This generalizes a lemma from [4] which applies only to the case $p = 1$.

One of our main motivations to study Bart-Moe games are Maker/Breaker-type positional games played on the edges of the complete graph K_n. In these games, the goal of Maker is usually to build a graph which satisfies some graph theoretic property. Consider for example, following [7], the Maker-Breaker game where Maker's goal is to occupy $\frac{p}{2(p+q)}(1 + o(1))n$ edge-disjoint Hamiltonian cycles. To handle such tasks, often an indirect approach is more fruitful. In our example, instead of concentrating on building the cycles, Maker creates a pseudo-random graph with the appropriate parameters and then shows (or cites the vast literature on pseudo-random graphs) that any such graph contains the required number of edge-disjoint Hamiltonian cycles.

JumbleG We need a few definitions related to pseudo-random graphs. Let $G = (V, E)$, $|V| = n$, be a graph and let $S, T \subseteq V$ be non-empty and disjoint. We say that the pair (S, T) is (α, ε)-*unbiased* if

$$\left| \frac{e_G(S, T)}{|S||T|} - \alpha \right| \leq \varepsilon,$$

where $e_G(S, T)$ is the number of edges with one end in S and the other in T. The graph G is said to be (α, ε)-*regular* if its minimum degree is at least $(\alpha - \varepsilon)n$ and any pair S, T of disjoint subsets of V, such that $|S|, |T| \geq \varepsilon n$, is (α, ε)-unbiased (note that this definition is slightly different than the definition given in [4], but they are essentially the same).
In the (p, q) game of JumbleG (c.f. [4]), two players alternately select unclaimed edges of K_n. The first player, called Jumbler (referring to the pseudo-random "jumbled graphs" of Thomason [8]), wins this game iff he is able to build a graph which is $(\frac{p}{p+q}, \varepsilon)$-regular. A similar game, also presented in [4], is (p, q)-JumbleG2, also played on K_n. The first player, called Jumbler, wins this game iff he is able to build a graph with minimum degree at least $(\frac{p}{p+q} - \varepsilon)n$ and maximum co-degree at most $((\frac{p}{p+q})^2 + \varepsilon)n$. The fact that these properties indeed entail pseudo-randomness is discussed in [4]. Using Theorem 1.1 we prove the following generalizations of Theorems 1 and 2 from [4].

Theorem 1.2 *If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$, $\varepsilon \geq 3\sqrt[3]{\frac{\log n}{np}}$ and n is sufficiently large then Jumbler has a winning strategy for the $(p, 1)$-JumbleG game.*

Theorem 1.3 *If $p < \frac{1}{16}\sqrt[3]{\frac{n}{\log n}}$, $\varepsilon \geq 8\sqrt{\frac{\log n}{np}}$ and n is sufficiently large then Jumbler has a winning strategy for the $(p,1)$-JumbleG2 game.*

The lower bound on ε given in Theorem 1.2 is tight up to a multiplicative constant factor. In fact, for smaller values of ε, the second player wins $(p,1)$-JumbleG no matter how he plays:

Theorem 1.4 *Let n be a sufficiently large positive integer. For every positive integer $p = o\left(\sqrt{\frac{n}{\log n}}\right)$ and for every $\varepsilon \leq c\sqrt[3]{\frac{\log n}{np}}$, where $c < 1/5$, no graph on n vertices is $(\frac{p}{p+1}, \varepsilon)$-regular.*

Discrepancy In a $(p,q,\mathcal{H})$ ε-Discrepancy game the players alternately select previously unclaimed vertices of a hypergraph $\mathcal{H}$ until every vertex has been claimed by some player. The first player, called Balancer, selects p vertices per move and the second player, called Unbalancer, selects q vertices per move. Let B denote the set of vertices selected by Balancer at the end of the game. If $||B \cap A| - \frac{p}{p+q}|A|| < \varepsilon|A|$ for every $A \in \mathcal{H}$ then Balancer wins the game; otherwise Unbalancer wins. The 1:1 version of the Discrepancy game has been recently considered in [1].

We prove a sufficient condition for Balancer to win this game on uniform hypergraphs for $q = 1$:

Theorem 1.5 *Let $\mathcal{H}$ be an n-uniform hypergraph. If $p < \frac{1}{3}\sqrt[3]{\frac{n}{\log(|\mathcal{H}|n)}}$, $\varepsilon \geq 3\sqrt{\frac{\log(|\mathcal{H}|n)}{np}}$ and n is sufficiently large then Balancer has a winning strategy for the $(p,1,\mathcal{H})$ ε-Discrepancy game.*

For the sake of simplicity and clarity of presentation, we make no effort to optimize the constants in Theorems 1.2, 1.3 and 1.5. We also omit floor and ceiling signs whenever these are not crucial. Throughout the paper log stands for the natural logarithm.

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.1. In Section 3 we prove Theorems 1.2,1.3 and 1.4, and discuss their applications to several positional games. In Section 4 we prove Theorem 1.5. Finally, in Section 5 we present some open problems.

2 The criterion

Our proof is based on Beck's proof of a sufficient condition for Breaker to win the $(p,q,\mathcal{H})$ Maker-Breaker game [2], which in turn is based on a method of Erdős and Selfridge [3].

Given a hypergraph $\mathcal{A}$ and disjoint subsets X and Y of the vertex set V let $\varphi_1(X,Y,\mathcal{A}) = \sum'_A (1+\frac{1}{p})^{-|A\setminus X|}$ where the summation $\sum'$ is extended over those $A \in \mathcal{A}$ for which $A \cap Y = \emptyset$. Given $z \in V$, let $\varphi_1(X,Y,\mathcal{A},z) = \sum''_A (1+\frac{1}{p})^{-|A\setminus X|}$ where the summation $\sum''$ is extended over those $A \in \mathcal{A}$ for which $z \in A$ and $A \cap Y = \emptyset$. Similarly, let $\varphi_2(X,Y,\mathcal{B}) = \sum'_B (1+p)^{-|B\setminus Y|}$ where the summation $\sum'$ is extended over those $B \in \mathcal{B}$ for which $B \cap X = \emptyset$. Given $z \in V$ let $\varphi_2(X,Y,\mathcal{B},z) = \sum''_B (1+p)^{-|B\setminus Y|}$ where the summation $\sum''$ is extended over those $B \in \mathcal{B}$ for which $z \in B$ and $B \cap X = \emptyset$.
Now consider a play according to the rules. Let $x_i^{(1)},\ldots,x_i^{(p)}$ and y_i denote the vertices chosen by Bart and Moe on their i^{th} move, respectively.
Let $X_i = \{x_1^{(1)},\ldots,x_1^{(p)},\ldots,x_i^{(1)},\ldots,x_i^{(p)}\}$, $Y_i = \{y_1,\ldots,y_i\}$ where $X_0 = Y_0 = \emptyset$.
Furthermore let $X_{i,j} = X_i \cup \{x_{i+1}^{(1)},\ldots,x_{i+1}^{(j)}\}$ where $X_{i,0} = X_i$.
For every non-negative integer i let $\psi(i) = \psi_1(i) + \psi_2(i)$ where $\psi_1(i) = \varphi_1(X_i,Y_i,\mathcal{A})$ and $\psi_2(i) = \varphi_2(X_i,Y_i,\mathcal{B})$. Bart loses if and only if there exists an integer i such that $A \subseteq X_i$ for some $A \in \mathcal{A}$ or $B \subseteq Y_i$ for some $B \in \mathcal{B}$. In either case $\psi(i) \geq 1$. It follows that if $\psi(i) < 1$ for every $i \geq 0$ then Bart wins the game. Now Bart's strategy is the following: on his i^{th} move, for every $1 \leq k \leq p$, he computes the value of $p\varphi_2(X_{i-1,k-1},Y_{i-1},\mathcal{B},x) - \varphi_1(X_{i-1,k-1},Y_{i-1},\mathcal{A},x)$ for every vertex $x \in V \setminus (Y_{i-1} \cup X_{i-1,k-1})$ and then selects $x_i^{(k)}$ for which the maximum is attained. First, we will prove that $\psi(i+1) \leq \psi(i)$ for every $i \geq 0$. Using the maximum property of $x_{i+1}^{(k)}$ and the simple observations $\varphi_1(X,Y,\mathcal{A},z_2) \leq \varphi_1(X\cup\{z_1\},Y,\mathcal{A},z_2)$ and $\varphi_2(X,Y,\mathcal{B},z_2) \geq \varphi_2(X\cup\{z_1\},Y,\mathcal{B},z_2)$, we get

$$\begin{aligned} p\varphi_2(X_{i,k-1},Y_i,\mathcal{B},x_{i+1}^{(k)}) - \varphi_1(X_{i,k-1},Y_i,\mathcal{A},x_{i+1}^{(k)}) &\geq \\ p\varphi_2(X_{i,k-1},Y_i,\mathcal{B},y_{i+1}) - \varphi_1(X_{i,k-1},Y_i,\mathcal{A},y_{i+1}) &\geq \\ p\varphi_2(X_{i+1},Y_i,\mathcal{B},y_{i+1}) - \varphi_1(X_{i+1},Y_i,\mathcal{A},y_{i+1}) & \end{aligned}$$

for every $1 \leq k \leq p$. So we conclude

$$\begin{aligned} \psi(i+1) &= \psi_1(i) + \frac{1}{p}\sum_{k=1}^{p} \varphi_1(X_{i,k-1},Y_i,\mathcal{A},x_{i+1}^{(k)}) - \varphi_1(X_{i+1},Y_i,\mathcal{A},y_{i+1}) \\ &+ \psi_2(i) - \sum_{k=1}^{p} \varphi_2(X_{i,k-1},Y_i,\mathcal{B},x_{i+1}^{(k)}) + p\varphi_2(X_{i+1},Y_i,\mathcal{B},y_{i+1}) \\ &= \psi(i) + p\varphi_2(X_{i+1},Y_i,\mathcal{B},y_{i+1}) - \varphi_1(X_{i+1},Y_i,\mathcal{A},y_{i+1}) \\ &- \frac{1}{p}\sum_{k=1}^{p} (p\varphi_2(X_{i,k-1},Y_i,\mathcal{B},x_{i+1}^{(k)}) - \varphi_1(X_{i,k-1},Y_i,\mathcal{A},x_{i+1}^{(k)})) \leq \psi(i). \end{aligned}$$

By our assumption $\psi(0) < (1+\frac{1}{p})^{-p}$ and so $\psi(i) < 1$ for every integer i except maybe for $i = r$ which denotes the last round of the game. In this round it is possible that only the first player will participate, but then $\psi(r) \leq (1+\frac{1}{p})^p \psi_1(r-1) + \psi_2(r-1) \leq (1+\frac{1}{p})^p \psi(r-1) \leq (1+\frac{1}{p})^p \psi(0) < 1$ and the theorem follows. $\square$

3 Winning in JumbleG

The following lemma will be useful in the proofs of Theorems 1.2, 1.3 and 1.5:

Lemma 3.1 *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph such that $|V| = N$ and $|E| = M$. Let $0 < l < \frac{k}{3}$ be an integer. Then there exists a collection $\mathcal{X}$ of l-subsets of V, of size at most $s = \left(\frac{N}{k}\right)^l \log M \min\{\exp\{\frac{l^2}{k}\}, \exp\{\frac{l^2}{2k} + \frac{l^3}{k^2} - \frac{l^2}{2N}\}\}$ such that every hyperedge of $\mathcal{H}$ contains an element of $\mathcal{X}$.*

Proof Choose s subsets of V, each of size l, randomly, independently and with replacement, and denote the resulting collection by $\mathcal{X}$. By a simple union bound argument we have $Pr[\exists e \in E \text{ such that } \forall x \in \mathcal{X}, x \nsubseteq e] \leq M \left(1 - \frac{\binom{k}{l}}{\binom{N}{l}}\right)^s$.
We will prove that this probability is strictly less than 1.

$$\begin{aligned} \frac{\binom{k}{l}}{\binom{N}{l}} &= \prod_{i=0}^{l-1} \frac{k-i}{N-i} = \left(\frac{k}{N}\right)^l \prod_{i=0}^{l-1} \left(\frac{1-\frac{i}{k}}{1-\frac{i}{N}}\right) > \left(\frac{k}{N}\right)^l \prod_{i=0}^{l-1} \left(1 - \frac{i}{k}\right) \\ &> \left(\frac{k}{N}\right)^l \exp\{-\sum_{i=0}^{l-1} \frac{2i}{k}\} > \left(\frac{k}{N}\right)^l \exp\{-\frac{l^2}{k}\} \end{aligned}$$

where the second inequality follows since $1 - x > e^{-2x}$ for every $0 < x < \frac{1}{3}$.

Similarly, and since $e^{-x-x^2} < 1 - x < e^{-x}$ for every $0 < x < \frac{1}{3}$, we have

$$\begin{aligned} \frac{\binom{k}{l}}{\binom{N}{l}} &= \left(\frac{k}{N}\right)^l \prod_{i=0}^{l-1} \left(\frac{1-\frac{i}{k}}{1-\frac{i}{N}}\right) > \left(\frac{k}{N}\right)^l \exp\{\sum_{i=0}^{l-1} \frac{i}{N} - \sum_{i=0}^{l-1} (\frac{i}{k} + \frac{i^2}{k^2})\} \\ &> \left(\frac{k}{N}\right)^l \exp\{\frac{l^2}{2N} - \frac{l^2}{2k} - \frac{l^3}{k^2}\}. \end{aligned}$$

Either way, $Pr[\exists e \in E \text{ such that } \forall x \in \mathcal{X}, x \nsubseteq e] < M \exp\{-s\binom{k}{l}/\binom{N}{l}\} < 1$, and so there exists a collection $\mathcal{X}$ with the desired properties. $\square$

The following technical lemma will save us some calculations later on:

Lemma 3.2 *Let m, r and p be positive integer-valued functions of n, such that $mr \to \infty$. Let ε also be a function of n such that $\varepsilon > 3\sqrt{\frac{\log(mr)}{rp}}$ and $p = o(\varepsilon^{-1})$. Furthermore we define $l = \frac{\varepsilon r}{2}$ and $k = (\frac{1}{p+1} + \varepsilon)r$. Then*

$$m \left(\frac{r}{k}\right)^l \log \binom{r}{k} \exp\{\frac{l^2}{k}\} (1+p)^{-l} = o(1).$$

Proof

$$
\begin{aligned}
& m\left(\frac{r}{k}\right)^{l}\log\binom{r}{k}\exp\left\{\frac{l^2}{k}\right\}(1+p)^{-l} \le mr\left(\left(\frac{1}{p+1}+\varepsilon\right)(1+p)\right)^{-l}\exp\left\{\frac{l^2(p+1)}{r}\right\}\\
= \;& mr\left(1+(p+1)\varepsilon\right)^{-l}\exp\left\{\frac{l^2(p+1)}{r}\right\} \le mr\exp\left\{\frac{\varepsilon^2 r^2(p+1)}{4r}-(1-o(1))\frac{\varepsilon^2 r(p+1)}{2}\right\} = o(1)
\end{aligned}
$$

where the last equality follows by our choice of ε and the second inequality follows since $p = o(\varepsilon^{-1})$. $\square$

Proof of Theorem 1.2

We will define an auxiliary Bart-Moe game on the edges of K_n, such that Jumbler, playing in the role of Bart, will win JumbleG once he wins the auxiliary game. We will apply Theorem 1.1 to provide the winning strategy.

Let us set $\varepsilon = 3\sqrt[3]{\frac{\log n}{np}}$. For larger ε the statement then trivially follows. Note that by the bound on p, we have $p\varepsilon = o(1)$; this will be used several times in the proof.

Let $G = (V, E)$ where $V = V(K_n)$ and E is the set of all edges claimed by Jumbler. In order to win, Jumbler would like G to be $(\frac{p}{p+1}, \varepsilon)$-regular. In particular he would like the pair (S, T) to be $(\frac{p}{p+1}, \varepsilon)$-unbiased for every disjoint $S, T \subseteq V$, both of size at least $t = \varepsilon n$. By an averaging argument we can assume that both S and T are of size exactly t. Indeed, let $S', T' \subseteq V$ be disjoint and of size at least t. The expectation of $\frac{e_G(S,T)}{t^2}$, where S and T are random t-subsets of S' and T' respectively is $\frac{e_G(S',T')}{|S'||T'|}$. Clearly if $|\frac{e_G(S,T)}{t^2} - \frac{p}{p+1}| \le \varepsilon$ for every disjoint pair S, T with $|S| = |T| = \varepsilon n$, then so is the expectation.
Let $\mathcal{T}$ consist of all pairs (S, T) of disjoint subsets of V, both of size exactly t. Fix a pair $(S, T) \in \mathcal{T}$. Jumbler would like to have "many" $S-T$ edges (plays as Breaker), but not "too many" (plays as Avoider). Starting with the latter, let $\mathcal{H}^A_{S,T} = (V^A_{S,T}, E^A_{S,T})$ be the hypergraph whose vertices are the edges of K_n with one end in S and the other in T, and whose hyperedges are all the subsets of $V^A_{S,T}$ of size $k_1 = (\frac{p}{p+1} + \varepsilon)t^2$. Jumbler would like to avoid claiming a complete $e \in E^A_{S,T}$.
By Lemma 3.1, there exists an s_1-sized collection $\mathcal{X}^A_{S,T}$ of l_1-subsets of $V^A_{S,T}$, where

$$l_1 = 3n\log n,\; s_1 \le \left(\frac{t^2}{k_1}\right)^{l_1}\log|E^A_{S,T}|\exp\left\{\frac{l_1^2}{2k_1}+\frac{l_1^3}{k_1^2}-\frac{l_1^2}{2|V^A_{S,T}|}\right\}, \tag{1}$$

such that every $e \in E^A_{S,T}$ contains an element of $\mathcal{X}^A_{S,T}$.

Similarly, let $\mathcal{H}^B_{S,T} = (V^B_{S,T}, E^B_{S,T})$ be the hypergraph whose vertices are the edges of K_n with one end in S and the other in T, and whose hyperedges are all the subsets of $V^B_{S,T}$ of size $k_2 = (\frac{1}{p+1} + \varepsilon)t^2$. Jumbler would like to claim an element of every $e \in E^B_{S,T}$.
By Lemma 3.1, there exists an s_2-sized collection $\mathcal{X}^B_{S,T}$ of l_2-subsets of $V^B_{S,T}$, where

$$l_2 = \frac{3n\log n}{p},\; s_2 \le \left(\frac{t^2}{k_2}\right)^{l_2}\log|E^B_{S,T}|\exp\left\{\frac{l_2^2}{k_2}\right\}, \tag{2}$$

such that every $e \in E^B_{S,T}$ contains an element of $\mathcal{X}^B_{S,T}$.

Jumbler would also like to have $deg_G(u) \geq (\frac{p}{p+1} - \varepsilon)n$ for every $u \in V$. For a vertex $u \in V$ let $\mathcal{H}_u = (V_u, E_u)$ be the hypergraph whose vertices are the edges of K_n incident with u, and whose hyperedges are all the subsets of V_u of size $k_3 = (\frac{1}{p+1} + \varepsilon)n$. Jumbler would like to claim an element of every $e \in E_u$.
By Lemma 3.1, there exists an s_3-sized collection $\mathcal{X}_u$ of l_3-subsets of V_u, where

$$l_3 = \frac{\varepsilon n}{2},\ s_3 \leq \left(\frac{n}{k_3}\right)^{l_3} \log|E_u| \exp\left\{\frac{l_3^2}{k_3}\right\}, \tag{3}$$

such that every $e \in E_u$ contains an element of $\mathcal{X}_u$.

Now we are ready to define our auxiliary game. Let $\mathcal{A} = \bigcup_{(S,T)\in\mathcal{T}} \mathcal{X}^A_{S,T}$, $\mathcal{B}_1 = \bigcup_{(S,T)\in\mathcal{T}} \mathcal{X}^B_{S,T}$ and $\mathcal{B}_2 = \bigcup_{u\in V} \mathcal{X}_u$. If Bart can win the $(p, 1, \mathcal{A} \cup (\mathcal{B}_1 \cup \mathcal{B}_2))$ Bart-Moe game, then Jumbler has a winning strategy for the $(p, 1)$ JumbleG game on K_n. By Theorem 1.1 it suffices to prove that

$$\sum_{A\in\mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} + \sum_{B\in\mathcal{B}_1} (1+p)^{-|B|} + \sum_{B\in\mathcal{B}_2} (1+p)^{-|B|} < \frac{1}{e}.$$

By (3) and Lemma 3.2 (with $m = n$, $r = n$, $k = k_3$ and $l = l_3$), $\sum_{B\in\mathcal{B}_2} (1+p)^{-|B|} = o(1)$ and so it suffices to prove that $\sum_{A\in\mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} = o(1)$ and $\sum_{B\in\mathcal{B}_1} (1+p)^{-|B|} = o(1)$. By (1) and since $p = o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
& \sum_{A\in\mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} \leq \binom{n}{t}^2 s_1 \left(1 + \frac{1}{p}\right)^{-l_1} \\
< \ & n^{2t} \left(\frac{t^2}{k_1}\right)^{l_1} \log\binom{t^2}{k_1} \exp\left\{\frac{l_1^2}{2k_1} + \frac{l_1^3}{k_1^2} - \frac{l_1^2}{2t^2}\right\} \left(1 + \frac{1}{p}\right)^{-l_1} \\
< \ & n^{2t} \left(\left(\frac{p}{p+1} + \varepsilon\right)\left(1 + \frac{1}{p}\right)\right)^{-l_1} t^2 \exp\left\{\frac{l_1^3}{k_1^2} + \frac{l_1^2}{2t^2}\left(\frac{1}{\frac{p}{p+1} + \varepsilon} - 1\right)\right\} \\
\leq \ & t^2 n^{2t} (1+\varepsilon)^{-l_1} \exp\left\{2\frac{l_1^2}{2t^2 p}\right\} \leq t^2 n^{2t} \exp\left\{\frac{l_1^2}{t^2 p}\right\} \exp\{-(1 - o(1))\varepsilon l_1\} \\
\leq \ & n^{2\varepsilon n + 2} n^{\frac{9 \log n}{\varepsilon^2 p} - 3(1 - o(1))\varepsilon n} = o(1).
\end{aligned}$$

where the last equality follows from our choice of ε. The fourth inequality follows since $\frac{l_1^3}{k_1^2} < \frac{l_1^2}{2t^2 p}$ which is why we get the upper bound on p.

Similarly, by (2) and since $p = o(\varepsilon^{-1})$ we have:

$$
\begin{aligned}
&\sum_{B\in\mathcal{B}_1} (1+p)^{-|B|} \le \binom{n}{t}^2 s_2 (1+p)^{-l_2} \\
< \; & n^{2t} \left(\frac{t^2}{k_2}\right)^{l_2} \log\binom{t^2}{k_2} \exp\left\{\frac{l_2^2}{k_2}\right\} (1+p)^{-l_2} < n^{2t} \left(\left(\frac{1}{p+1}+\varepsilon\right)(1+p)\right)^{-l_2} t^2 \exp\left\{\frac{l_2^2(p+1)}{t^2}\right\} \\
= \; & t^2 n^{2t} \left(1+(p+1)\varepsilon\right)^{-l_2} \exp\left\{\frac{l_2^2(p+1)}{t^2}\right\} \le t^2 n^{2t} \exp\left\{\frac{l_2^2(p+1)}{t^2}\right\} \exp\left\{-(1-o(1))\varepsilon(p+1)l_2\right\} \\
\le \; & n^{2\varepsilon n+2} n^{\frac{9\log n}{\varepsilon^2 p} - 3(1-o(1))\varepsilon n} = o(1).
\end{aligned}
$$

where the last equality follows from our choice of ε. □

Proof of Theorem 1.3

Again, we will define an auxiliary Bart-Moe game such that Bart's win in this auxiliary game implies Jumbler's win in JumbleG2.

We set $\varepsilon = 8\sqrt{\frac{\log n}{np}}$. Then $p\varepsilon = o(1)$ by the upper bound on p.

Let $G = (V, E)$ where $V = V(K_n)$ and E is the set of all edges claimed by Jumbler. In order to win, Jumbler would like G to have minimum degree at least $(\frac{p}{p+1} - \varepsilon)n$ (plays as Breaker) and maximum co-degree at most $((\frac{p}{p+1})^2 + \varepsilon)n$ (plays as Avoider). Starting with the latter, for every two vertices $u, w \in V$ and every set $S \subseteq V \setminus \{u, w\}$ of size $k_1 = ((\frac{p}{p+1})^2 + \varepsilon)n$, Jumbler would like to avoid claiming the set of edges $\{(x, y) | x \in S, y \in \{u, w\}\}$.
For every two vertices $u, w \in V$ define a hypergraph $\mathcal{H}_{u,w} = (V_{u,w}, E_{u,w})$ where $V_{u,w} = V \setminus \{u, w\}$ and $E_{u,w}$ is the set of all subsets of $V_{u,w}$ of size k_1. By Lemma 3.1, there exists an s_1-sized collection $\mathcal{X}_{u,w}$ of l_1-subsets of $V_{u,w}$, where

$$l_1 = \frac{\varepsilon np}{2},\ s_1 \le \left(\frac{n}{k_1}\right)^{l_1} \log|E_{u,w}| \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\}, \tag{4}$$

such that every $e \in E_{u,w}$ contains an element of $\mathcal{X}_{u,w}$.

Jumbler would also like to have $deg_G(u) \ge (\frac{p}{p+1} - \varepsilon)n$ for every $u \in V$. Fix $u \in V$ and let $\mathcal{H}_u = (V_u, E_u)$ be the hypergraph whose vertices are the edges of K_n incident with u, and whose hyperedges are all the subsets of V_u of size $k_2 = (\frac{1}{p+1} + \varepsilon)n$. Jumbler would like to claim an element of every $e \in E_u$.
By Lemma 3.1, there exists an s_2-sized collection $\mathcal{X}_u$ of l_2-subsets of V_u, where

$$l_2 = \frac{\varepsilon n}{2},\ s_2 \le \left(\frac{n}{k_2}\right)^{l_2} \log|E_u| \exp\left\{\frac{l_2^2}{k_2}\right\}, \tag{5}$$

such that every $e \in E_u$ contains an element of $\mathcal{X}_u$.

We can now define our auxiliary Bart-Moe game. Let $\mathcal{B} = \bigcup_{u \in V} \mathcal{X}_u$ and

$$\mathcal{A} = \bigcup_{\substack{u,w \in V \\ u \neq w}} \{\{(u,x)|x \in e\} \cup \{(w,x)|x \in e\} : e \in \mathcal{X}_{u,w}\}.$$

If Jumbler can win the $(p, 1, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game as Bart, then he has a winning strategy for the $(p, 1)$ JumbleG2 game on K_n. By Theorem 1.1 it suffices to prove that

$$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} + \sum_{B \in \mathcal{B}} (1+p)^{-|B|} < \frac{1}{e}.$$

By (5) and Lemma 3.2 (with $m = n$, $r = n$,$k = k_2$ and $l = l_2$), $\sum_{B \in \mathcal{B}} (1+p)^{-|B|} = o(1)$ and so it suffices to prove that
$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} = o(1)$. By (4) and since $p = o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
& \sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} \leq \binom{n}{2} s_1 \left(1 + \frac{1}{p}\right)^{-2l_1} \\
\leq \; & \frac{n^2}{2} \left(\frac{n}{k_1}\right)^{l_1} \log \binom{n}{k_1} \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\} \left(1 + \frac{1}{p}\right)^{-2l_1} \\
\leq \; & \frac{n^3}{2} \left(\left(\left(\frac{p}{p+1}\right)^2 + \varepsilon\right) \left(1 + \frac{1}{p}\right)^2\right)^{-l_1} \exp\left\{\frac{l_1^3}{k_1^2} + \frac{l_1^2}{2n} \left(\frac{1}{\left(\frac{p}{p+1}\right)^2 + \varepsilon} - 1\right)\right\} \\
\leq \; & \frac{n^3}{2} (1+\varepsilon)^{-l_1} \exp\left\{\frac{3l_1^2}{2np} + \frac{l_1^2}{4np}\right\} \leq \frac{n^3}{2} \exp\left\{\frac{7\varepsilon^2 n^2 p^2}{16np} - (1 - o(1))\frac{\varepsilon^2 np}{2}\right\} = o(1)
\end{aligned}$$

where the last equality follows from our choice of ε. The fourth inequality follows since $\frac{l_1^3}{k_1^2} < \frac{l_1^2}{4np}$ which is why we get the upper bound on p. $\square$

3.1 The tightness of Theorem 1.2

Proof of Theorem 1.4
Let $G = (V, E)$ be any graph on n vertices. It suffices to prove that there exist disjoint sets $S, T \subseteq V$, both of size $t = \varepsilon n$, such that the pair (S, T) is not $(\frac{1}{p+1}, \varepsilon)$-unbiased (indeed such a pair (S, T) is (α, ε)-unbiased in a graph iff it is $(1 - \alpha, \varepsilon)$-unbiased in the complement of that graph). Assume that $\varepsilon = c\sqrt[3]{\frac{\log n}{np}}$ (this is clearly legitimate as if G

is not $(\frac{p}{p+1}, \varepsilon)$-regular then it is not $(\frac{p}{p+1}, \varepsilon')$-regular for any $\varepsilon' \leq \varepsilon$). Let X be a random t-subset of V chosen uniformly. For every $y \in V$ let $A_{X,y}$ be the event "$y \in V \setminus X$ and $||N(y) \cap X| - \frac{t}{p+1}| > \varepsilon t$", where $N(y) = \{u \in V | (u, y) \in E\}$.

Claim 3.3 *$Pr[A_{X,y}] \geq \frac{2t}{n}$ for every $y \in V$.*

Proof of Claim 3.3 Let $d = d(y)$ denote the degree of y in G. Assume that $d \leq \frac{n-1}{p+1}$. We wish to find a lower bound on

$$Pr[y \in V \setminus X, |N(y) \cap X| \leq \frac{t}{p+1} - \varepsilon t] = \sum_{i=0}^{\frac{t}{p+1}-\varepsilon t} \binom{d}{i}\binom{n-1-d}{t-i}\binom{n}{t}^{-1}. \tag{6}$$

A lower bound on $Pr[y \in V \setminus X, |N(y) \cap X| \geq \frac{t}{p+1} + \varepsilon t]$ for $d \geq \frac{n-1}{p+1}$ will follow by an analogous argument. Note that by our choice of p, the sum on the right hand side of (6) is not empty. The probability (6) is decreasing as a function of d (as for larger values of d it is more likely that y will have many neighbours in X) and so it suffices to bound it for $d = \frac{n-1}{p+1}$. For every $\varepsilon t \leq k \leq \frac{t}{p+1}$ let s_k be the summand corresponding to $i = \frac{t}{p+1} - k$ in (6). First, we will estimate

$$s'_k = \binom{\frac{n-1}{p+1}}{\frac{t}{p+1} - k}\binom{n-1-\frac{n-1}{p+1}}{t - \frac{t}{p+1} + k}\binom{n-1}{t}^{-1}.$$

Let $R \sim H\left(t; \frac{n-1}{p+1}, n-1\right)$ be a random variable with a hypergeometric distribution, that is, $R = |A \cap B|$, where A is a fixed $\frac{n-1}{p+1}$-subset of a given set C of size $n-1$, and B is formed by drawing t elements of C at random without replacement. Then $\mu = \mathbb{E}[R] = \frac{t}{p+1}$ and $\sigma^2 = Var(R) \leq \frac{t}{p+1}$. By Chebychev's inequality we have

$$Pr[|R - \mu| \leq 2\sigma] = 1 - Pr[|R - \mu| > 2\sigma] \geq 3/4. \tag{7}$$

The function $p(x) = Pr[R = x]$ attains its maximum value at $x = \mu$ (or more accurately at the upper or lower integer part of μ) and so by (7) we have $s'_0 = Pr[R = \frac{t}{p+1}] \geq \frac{3}{4} \cdot \frac{1}{4\sigma} \geq$

$\frac{1}{6\sigma} \geq \sqrt{\frac{p+1}{36t}}$. For every $0 \leq k \leq 2\varepsilon t$ we have

$$
\begin{aligned}
\frac{s'_{k+1}}{s'_k} &= \frac{\binom{\frac{n-1}{p+1}}{\frac{t}{p+1}-k-1}\binom{n-1-\frac{n-1}{p+1}}{t-\frac{t}{p+1}+k+1}}{\binom{\frac{n-1}{p+1}}{\frac{t}{p+1}-k}\binom{n-1-\frac{n-1}{p+1}}{t-\frac{t}{p+1}+k}} \\
&= \frac{(t-k(p+1))\left(n-1-\frac{n-1}{p+1}-t+\frac{t}{p+1}-k\right)}{(n-1-t+(p+1)(k+1))\left(t-\frac{t}{p+1}+k+1\right)} \\
&\geq \frac{t-(p+1)k}{\frac{pt}{p+1}+k+1}\cdot\frac{p}{p+1}\left(1-\frac{3(p+1)(k+1)}{n}\right) \\
&= \frac{t-(p+1)k}{t+\frac{(p+1)(k+1)}{p}}\left(1-\frac{3(p+1)(k+1)}{n}\right) \\
&\geq \left(1-\frac{2(p+1)(k+1)}{t}\right)\left(1-\frac{3(p+1)(k+1)}{n}\right) \\
&\geq 1-\frac{3(p+1)(k+1)}{t},
\end{aligned}
$$

where the second equality follows by a straightforward calculation and the first and last inequalities follow since $t = o(n)$.

Now, for every $0 \leq k \leq 2\varepsilon t$ we have

$$
s'_k = s'_0 \prod_{j=0}^{k-1} \frac{s'_{j+1}}{s'_j} \geq s'_0 \prod_{j=0}^{2\varepsilon t-1} \frac{s'_{j+1}}{s'_j} \geq s'_0 \left(1-\frac{6(p+1)\varepsilon t}{t}\right)^{2\varepsilon t} \geq s'_0 \exp\left\{-15(p+1)\varepsilon^2 t\right\}.
$$

Moreover $\frac{s_k}{s'_k} = \frac{n-t}{n} = 1-\varepsilon$ and so

$$
\begin{aligned}
\sum_{k=\varepsilon t}^{\frac{t}{p+1}} s_k &\geq \sum_{k=\varepsilon t}^{2\varepsilon t} s'_k(1-\varepsilon) \geq (1-\varepsilon)\varepsilon t s'_0 \exp\left\{-15(p+1)\varepsilon^2 t\right\} \geq (1-\varepsilon)\varepsilon\frac{1}{3}\sqrt{t}\exp\left\{-15(p+1)\varepsilon^3 n\right\} \\
&\geq \frac{2t}{n},
\end{aligned}
$$

where the last inequality follows by our choice of c. This concludes the proof of the claim. □

Let Y_X consist of the vertices $y \in V$ for which $A_{X,y}$ holds. By Claim 3.3 we have $E(|Y_X|) = \sum_{y \in V} E(A_{X,y}) \geq 2t$ and so there exists a t-subset S of V such that $|Y_S| \geq 2t$. Assume without loss of generality that $|N(y) \cap S| < \frac{t}{p+1} - \varepsilon t$ for at least half the vertices of Y_S. Let $T \subseteq Y_S$ consist of any t of these vertices, then the pair (S,T) is not $(\frac{1}{p+1}, \varepsilon)$-unbiased. □

3.2 Applications

From Theorems 1.2 and 1.3 we immediately get generalizations of all the corollaries obtained in [4] (the bounds on p result from the use of Theorems 1.2 and 1.3):

- If $p < \frac{1}{16}\sqrt[3]{\frac{n}{\log n}}$ then Maker can build a graph with minimum degree at least $\frac{pn}{p+1} - 8\sqrt{\frac{n\log n}{p}}$.
- If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$ then Maker can build a $\left(\frac{pn}{p+1} - 8\sqrt{\frac{n\log n}{p}}\right)$ vertex connected graph.
- If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$ then Maker can build a graph that contains at least $(\frac{p}{2(p+1)} - 3\varepsilon)n$ edge disjoint hamiltonian cycles for every $\varepsilon > 10(\frac{\log n}{n})^{\frac{1}{6}}$.
- If $p < \frac{1}{2}\sqrt[5]{\frac{n}{\log n}}$ then Maker can build an r-universal graph, in the sense that it contains an induced copy of every graph on r vertices, for $r = (1+o(1))\log_{p+1} n$. Note that r is in inverse ratio to p as when Maker's graph gets more dense it's harder to find sparse induced subgraphs in it.

We omit the straightforward proofs.

4 Proof of Theorem 1.5

Let us fix $\varepsilon = 3\sqrt{\frac{\log(|\mathcal{H}|n)}{np}}$. In order to win the game, Balancer would like to have "many" vertices in every hyperedge of $\mathcal{H}$ (plays as Breaker), but not "too many" (plays as Avoider). Starting with the latter, for every $e \in \mathcal{H}$ define a hypergraph $\mathcal{H}_e^A = (V_e^A, E_e^A)$ where V_e^A is the set of vertices of e and E_e^A is the set of all subsets of V_e^A of size $k_1 = (\frac{p}{p+1} + \varepsilon)n$. By Lemma 3.1, there exists an s_1-sized collection $\mathcal{X}_e^A$ of l_1-subsets of V_e^A, where

$$l_1 = \frac{\varepsilon np}{2},\ s_1 \leq \left(\frac{n}{k_1}\right)^{l_1} \log|E_e^A| \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\}, \tag{8}$$

such that every hyperedge of $\mathcal{H}_e^A$ contains an element of $\mathcal{X}_e^A$.

Similarly, for every $e \in \mathcal{H}$ define a hypergraph $\mathcal{H}_e^B = (V_e^B, E_e^B)$ where V_e^B is the set of vertices of e and E_e^B is the set of all subsets of V_e^B of size $k_2 = (\frac{1}{p+1} + \varepsilon)n$. By Lemma 3.1, there exists an s_2-sized collection $\mathcal{X}_e^B$ of l_2-subsets of V_e^B, where

$$l_2 = \frac{\varepsilon n}{2},\ s_2 \leq \left(\frac{n}{k_2}\right)^{l_2} \log|E_e^B| \exp\left\{\frac{l_2^2}{k_2}\right\}, \tag{9}$$

such that every hyperedge of $\mathcal{H}_e^B$ contains an element of $\mathcal{X}_e^B$.

Let $\mathcal{A} = \bigcup_{e \in \mathcal{H}} \mathcal{X}_e^A$ and $\mathcal{B} = \bigcup_{e \in \mathcal{H}} \mathcal{X}_e^B$. If Balancer, playing as Bart, can win the $(p, 1, \mathcal{A} \cup \mathcal{B})$ Bart-Moe game, then he has a winning strategy for the $(p, 1, \mathcal{H})$ ε-Discrepancy game. By Theorem 1.1 it suffices to prove that

$$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} + \sum_{B \in \mathcal{B}} (1+p)^{-|B|} < \frac{1}{e}.$$

By (9) and Lemma 3.2 (with $m = |\mathcal{H}|$, $r = n$, $k = k_2$ and $l = l_2$), $\sum_{B \in \mathcal{B}} (1+p)^{-|B|} = o(1)$ and so it suffices to prove that
$\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} = o(1)$. By (8) and since $p = o(\varepsilon^{-1})$ we have:

$$\begin{aligned}
\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|} &\leq |\mathcal{H}| \left(\frac{n}{k_1}\right)^{l_1} \log \binom{n}{k_1} \exp\left\{\frac{l_1^2}{2k_1} - \frac{l_1^2}{2n} + \frac{l_1^3}{k_1^2}\right\} \left(1 + \frac{1}{p}\right)^{-l_1} \\
\leq\ & |\mathcal{H}| n \left(\left(\frac{p}{p+1} + \varepsilon\right)\left(1 + \frac{1}{p}\right)\right)^{-l_1} \exp\left\{\frac{l_1^3}{k_1^2} + \frac{l_1^2}{2n}\left(\frac{1}{\frac{p}{p+1} + \varepsilon} - 1\right)\right\} \\
\leq\ & |\mathcal{H}| n (1+\varepsilon)^{-l_1} \exp\{2\frac{l_1^2}{2np}\} \leq |\mathcal{H}| n \exp\left\{\frac{\varepsilon^2 n^2 p^2}{4np} - (1 - o(1))\frac{\varepsilon^2 np}{2}\right\} = o(1)
\end{aligned}$$

where the last equality follows from our choice of ε. The third inequality follows since $\frac{l_1^3}{k_1^2} < \frac{l_1^2}{2np}$ by the upper bound on p. $\square$

5 Concluding remarks and open problems

- It would be interesting to find a sufficient condition for Bart to win the (p, q) Bart-Moe game for $q > 1$, and apply it to several specific combinatorial games.
- It would be interesting to analyze $(p, 1)$-JumbleG, JumbleG2 and Discrepancy for every value of p. Note that we can consider greater values of p at the cost of enlarging ε; that is, if $m \leq p = o(m^2)$, where m denotes the upper bound on p given in Theorem 1.2, then by a similar argument we can prove that the assertion of Theorem 1.2 holds for $\varepsilon > const \sqrt[3]{\frac{\log n}{n\sqrt{p}}}$. The same can be done with Theorems 1.3 and 1.5. Note that having a certain upper bound on p is reasonable, as $\sum_{A \in \mathcal{A}} \left(1 + \frac{1}{p}\right)^{-|A|}$ grows with p.

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