Avoider - Enforcer games

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September 1, 2006

Abstract

Let p and q be positive integers and let $\mathcal{H}$ be any hypergraph. In a $(p, q, \mathcal{H})$ Avoider-Enforcer game two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Avoider selects p vertices per move and Enforcer selects q vertices per move. Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Enforcer loses. We prove a sufficient condition for Avoider to win the $(p, q, \mathcal{H})$ game. We then use this condition to show that Enforcer can win the $(1, q)$ perfect matching game on K_{2n} for every $q = O(n/\log n)$, and the $(1, q)$ Hamilton cycle game on K_n for every $q = O(n \log\log\log\log n/\log n \log\log\log n)$. We also determine exactly those values of q for which Enforcer can win the $(1, q)$ connectivity game on K_n. Our method extends easily to improve a result of Lu [15], regarding forcing an opponent to pack many pairwise edge disjoint spanning trees in his graph. The last two results are somewhat surprising, as they differ strongly from their Maker-Breaker analog.

1 Introduction

Let p and q be positive integers and let $\mathcal{H}$ be any hypergraph. In a $(p, q, \mathcal{H})$ Avoider-Enforcer game two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Avoider selects exactly p vertices per move and Enforcer selects exactly q vertices per move. The only exception to this rule, is the last move, in which a

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player must select all the remaining vertices, which might be less than his share. The game ends when every vertex has been claimed by one of the players. Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Enforcer loses. The integer p is called the bias of Avoider, and q is called the bias of Enforcer. We assume that Avoider starts the game unless stated otherwise, although for the asymptotic nature of our results it is usually irrelevant who starts the game.

The hypergraph $\mathcal{H}$ is sometimes referred to as the *game* (without mentioning the biases). We call the game $(p, q, \mathcal{H})$ an *Avoider's win* (*Enforcer's win*) if Avoider (Enforcer) has a winning strategy in $(p, q, \mathcal{H})$. It is not hard to see that every game $(p, q, \mathcal{H})$ is either an Avoider's win or an Enforcer's win, but not both.

Arguably, the goals of the players in Avoider-Enforcer games are not the most natural ones. The goal of Avoider is defined through a negation, that is, he wins if he does *not* occupy any member of $\mathcal{H}$. The variant of these games with "positive" goals is indeed much more thoroughly studied. In a Maker-Breaker type game, the player called Maker wins if he *does* occupy all the vertices of some member of $\mathcal{H}$; otherwise the other player (Breaker) wins. However, we argue that Avoider-Enforcer games are equally natural. First of all, any game in which the goal of one of the players is to build a graph satisfying some monotone *decreasing* property is an Avoider-Enforcer game (that player being Avoider: his goal is to avoid fully occupying a minimal graph *not* satisfying the property). Furthermore, in discrepancy-type games (see [9], [10]) the goal of one of the players is to claim some fixed percentage (not more and not less) of every winning set, hence he plays as both Breaker and Avoider.

Putting aside a few scattered results, the theory of Maker-Breaker games started with a general criterion of Erdős and Selfridge [8] for Breaker's win in the $(1, 1)$ game. Subsequently, Beck started a systematic study of Maker-Breaker games with a bias. In particular, in [1] he proved the following generalization of the Erdős-Selfridge criterion: If

$$\sum_{D \in \mathcal{H}} (1+q)^{-|D|/p} < \frac{1}{1+q} \tag{1}$$

then Breaker has a winning strategy for the $(p, q, \mathcal{H})$ Maker-Breaker game.

Lu [14] proved that an identical criterion guarantees Avoider's win in the $(1, 1)$ game. One can (somewhat naively) assume that the theory of Avoider-Enforcer games is very similar to that of Maker-Breaker games, and that criterion (1) guarantees a winning strategy for Avoider for every p and q. As it turns out, things are much more complicated, as the case of $(1, 1)$ games is somewhat special and hides the difficulties that arise in biased games.

Further thought reveals that the differences between Maker-Breaker and Avoider-Enforcer games go much deeper. Without giving it much thought, one expects (and rightly so) that Maker's win and Breaker's win will be appropriately monotone in the bias. That is, if for example Maker wins the $(1, 1)$ game on some hypergraph, then he will also win the $(2, 1)$ game on the same hypergraph. The simple-minded reason for this is that "more occupied vertices cannot hurt Maker, and in fact, might even help him". This is indeed true and will be discussed further in Section 6. Now, it is equally plausible to assume that in case Enforcer wins the $(1,1)$ game, he will also win the $(2,1)$ game, since "less occupied vertices cannot hurt Enforcer". It turns out that this intuition fails. For example, it is possible to give examples of $(1,q)$ (resp. $(p,1)$) Avoider-Enforcer games which are won by Enforcer iff q (resp. p) is of a certain parity. We explore these issues of monotonicity in Section 6.

Despite these major differences, one is able to adapt Beck's argument to some extent and to provide an analogous criterion for Avoider's win.

Theorem 1.1 *If Avoider is the last player (i.e. the player to make the last move) and*

$$\sum_{D\in\mathcal{H}}\left(1+\frac{1}{p}\right)^{-|D|} < \left(1+\frac{1}{p}\right)^{-p}$$

then Avoider wins the $(p,q,\mathcal{H})$ game for every $q \geq 1$.
If Enforcer is the last player then the above sufficient condition can be relaxed to

$$\sum_{D\in\mathcal{H}}\left(1+\frac{1}{p}\right)^{-|D|} < 1.$$

Note, that even though we assume for convenience in our paper that Avoider starts the game, the assertion of Theorem 1.1, holds also when Enforcer starts the game.

Our criterion does not take into account the value of q, so it is unlikely to be best possible. For any constant value of q, however, we show that the criterion is "not far" from being best possible. Beck [1] proved that his sufficient condition (1) for Maker-Breaker games is best possible by building explicitly an infinite family of hypergraphs $\mathcal{H}$ such that equality holds in (1) and Maker has a winning strategy for the corresponding game. We think that the problem of finding a useful, and possibly "best possible" criterion for Avoider's win when $q > 1$, is one of the most interesting open problems of the topic.

Theorem 1.2 *For every positive integers p and q there are infinitely many hypergraphs $\mathcal{H}$ such that*
$\sum_{D\in\mathcal{H}}(1+\frac{1}{p})^{-|D|} \leq \binom{p+q-1}{q-1}\frac{p+1}{2}$*, and yet Enforcer wins the $(p,q,\mathcal{H})$ game.*

Note that $\binom{p+q-1}{q-1}\frac{p+1}{2}$ is polynomial in p for every fixed q.

In this paper we study more closely three quite natural Avoider-Enforcer games: "connectivity", "perfect matching" and "hamiltonicity". Let $\mathcal{T}_n$ be the set of all spanning trees, $\mathcal{M}_n$ the set of all perfect matchings (here we assume of course that n is even), and $\mathcal{H}_n$ the set of all Hamilton cycles, in the complete graph K_n on n vertices.

It will be convenient to introduce the following notation. For a hypergraph $\mathcal{B}$ we define $b^-_{\mathcal{B}}$ to be the largest integer such that Enforcer can win $(1, b, \mathcal{B})$ for every $b \le b^-_{\mathcal{B}}$, and $b^+_{\mathcal{B}}$ to be the smallest integer such that Avoider can win $(1, b, \mathcal{B})$ for every $b > b^+_{\mathcal{B}}$. Except for certain degenerate cases, $b^-_{\mathcal{B}}$ and $b^+_{\mathcal{B}}$ always exist and satisfy $b^-_{\mathcal{B}} \le b^+_{\mathcal{B}}$. However, as was indicated above, we do not know in general that $b^-_{\mathcal{B}} = b^+_{\mathcal{B}}$, that is, we do not know whether a well-defined threshold exists. In case $b^-_{\mathcal{B}} = b^+_{\mathcal{B}}$, we denote this number by $b_{\mathcal{B}}$ and call it the *threshold bias* of the game $\mathcal{B}$.

For Maker-Breaker games a similar threshold bias, at which a Maker's win turns into a Breaker's win, could be defined and *does exist* for all hypergraphs. It was proved by Chvátal and Erdős [5] and by Beck [2] that the threshold bias for all three Maker-Breaker games $\mathcal{T}_n, \mathcal{M}_n$, and $\mathcal{H}_n$ is of order $n/\log n$.

As a first application of Theorem 1.1 we consider the perfect matching game $(1, q, \mathcal{M}_n)$.

Theorem 1.3 *Enforcer has a winning strategy in $(1, q, \mathcal{M}_{2n})$ if $q < \frac{n}{(2+o(1))\log_2 n}$ and n is sufficiently large. Thus,*

$$b^-_{\mathcal{M}_n} = \Omega\left(\frac{n}{\log n}\right).$$

Although it looks plausible we do not know whether the perfect matching game is monotone. Moreover, even if a threshold does exists, we do not know whether it is of order $n/\log n$.

Next, we will use Theorem 1.1 to prove a sufficient condition for Enforcer to win the Hamilton cycle game $(1, q, \mathcal{H}_n)$. Beck [3] asked whether Enforcer, playing with a bias of $\Theta(n/\log n)$, can force Avoider to build a Hamilton cycle. Although we are not able to solve his question completely, we can get quite close. For every positive integer k, we denote by $\log^{(k)} n$ the k-fold natural logarithm of n, (that is, $\log^{(1)} n = \log n$, $\log^{(2)} n = \log\log n$, etc).

Theorem 1.4 *Enforcer has a winning strategy in $(1, q, \mathcal{H}_n)$ if $q < \frac{n \log 2 \log^{(4)} n}{8500 \log n \log^{(3)} n}$ and n is sufficiently large. Thus,*

$$b^-_{\mathcal{H}_n} = \Omega\left(\frac{n}{\log n} \cdot \frac{\log^{(4)} n}{\log^{(3)} n}\right).$$

Note that, though we are unable to prove it at the present stage, we believe that the answer to Beck's question is positive.

In [5] and [1] it is shown that the threshold bias for the $(1, q)$ Maker-Breaker connectivity game is between $(\log 2 - \varepsilon)\frac{n}{\log n}$ and $(1 + \varepsilon)\frac{n}{\log n}$ for every $\varepsilon > 0$. It is also suggested there that the order of magnitude $n/\log n$ is very reasonable for this problem as, at the end of the game, Maker will have about $\frac{1}{2}n \log n$ edges which is the threshold for the connectivity of a random graph $G(n, m)$ (see e.g. [12] for background on random graphs). Hence, we find it somewhat surprising that this insight fails badly for the Avoider-Enforcer connectivity game. We would also like to stress, that this is the only case where we could establish the monotonicity of a game of interest.

Theorem 1.5 *Avoider wins the* $(1, q)$ *connectivity game* $\mathcal{T}_n$ *iff at the end of the game he has at most* $n - 2$ *edges. In particular the threshold* $b_{\mathcal{T}_n}$ *exists for every* n*. We have* $b_{\mathcal{T}_n} = \left\lfloor \frac{n}{2} \right\rfloor - 1$*, except when* n *is odd and Avoider starts the game, in which case* $b_{\mathcal{T}_n} = \left\lfloor \frac{n}{2} \right\rfloor$*.*

To a certain extent we can generalize the assertion of Theorem 1.5 to the "k-edge connectivity game", in which Avoider loses iff he builds a k-edge connected spanning subgraph of K_n. Unfortunately, if $k > 1$, we do not know the exact bias, nor do we know whether it exists, that is, whether the corresponding game is monotone.

Theorem 1.6 *Playing on* K_n*, if* $q \leq \frac{n}{2k} - 1$ *then Enforcer wins the* k*-edge connectivity game. If* $q \geq \frac{n}{k}$ *then Avoider wins the* k*-edge connectivity game.*

In [15], Lu considered the $(1, 1)$ Avoider-Enforcer game on the edges of K_n, in which Enforcer's goal is to force Avoider to build as many pairwise edge disjoint spanning trees as possible. Clearly $\lfloor n/4 \rfloor$ is an upper bound. Lu proved that for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ then, playing on K_n, Enforcer can force Avoider to build $(1 - \varepsilon)n/4$ pairwise edge disjoint spanning trees. We improve this by showing that the trivial upper bound $\lfloor n/4 \rfloor$ is in fact tight.

Theorem 1.7 *For every positive integer* n*, playing the* $(1, 1)$ *game on* K_n*, Enforcer can force Avoider to build* $\lfloor n/4 \rfloor$ *pairwise edge disjoint spanning trees.*

Theorems 1.5, 1.6 and 1.7 are relatively easy consequences of the following theorem.

Theorem 1.8 *If* G *contains* $q + 1$ *pairwise edge disjoint spanning trees, then Enforcer, as first or second player, wins the* $(1, q)$ *connectivity game on* G*.*

Observe that the case $q = 1$ of the above theorem can be considered as the Avoider-Enforcer analog of the celebrated Lehman's criterion [13] for Maker's win in connectivity games.

Throughout the paper, for the sake of simplicity and clarity of presentation, we omit floor and ceiling signs whenever these are not crucial. All logarithms are natural unless explicitly stated otherwise. Our graph-theoretic notation is standard and follows that of [6]. In particular, for a graph $G = (V, E)$ and a set $A \subseteq V$, let $N_G(A) = \{u \in V : \exists w \in A, (u, w) \in E\}$ be the neighborhood of A in G. Often, when there is no risk of confusion, we abbreviate $N_G(A)$ with $N(A)$.

The rest of the paper is organized as follows: In Section 2 we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorem 1.3, in Section 4 we prove Theorem 1.4, in Section 5 we prove Theorem 1.8 and then derive Theorems 1.5,1.6 and 1.7. In Section 6 we discuss the non-monotonicity of biased games, and in Section 7 we present several related open problems.

2 A sufficient condition for Avoider's win

Proof of Theorem 1.1: Our argument is based on Beck's proof of a sufficient condition for Breaker to win the $(p, q, \mathcal{H})$ Maker-Breaker game [1], which in turn is based on the potenial function method of Erdős and Selfridge [8].

Given a hypergraph $\mathcal{H}$ and disjoint subsets X and Y, of the vertex set V of $\mathcal{H}$, let $\varphi(X, Y, \mathcal{H}) = \sum'_D (1 + \frac{1}{p})^{-|D \smallsetminus X|}$ where the summation $\sum'$ is extended over those $D \in \mathcal{H}$ for which $D \cap Y = \emptyset$. Given $z \in V$, let $\varphi(X, Y, \mathcal{H}, z) = \sum''_D (1 + \frac{1}{p})^{-|D \smallsetminus X|}$ where the summation $\sum''$ is extended over those $D \in \mathcal{H}$ for which $z \in D$ and $D \cap Y = \emptyset$.

Now, consider a play according to the rules; assume first that Avoider starts the game. Let $x_i^{(1)}, \ldots, x_i^{(p)}$ and $y_i^{(1)}, \ldots, y_i^{(q)}$ denote the vertices chosen by Avoider and Enforcer on their ith move, respectively.

Let $X_i = \{x_1^{(1)}, \ldots, x_1^{(p)}, \ldots, x_i^{(1)}, \ldots, x_i^{(p)}\}$, $Y_i = \{y_1^{(1)}, \ldots, y_1^{(q)}, \ldots, y_i^{(1)}, \ldots, y_i^{(q)}\}$, where $X_0 = \emptyset$ and $Y_0 = \emptyset$. Furthermore let $X_{i,j} = X_i \cup \{x_{i+1}^{(1)}, \ldots, x_{i+1}^{(j)}\}$ and $Y_{i,j} = Y_i \cup \{y_{i+1}^{(1)}, \ldots, y_{i+1}^{(j)}\}$ where $X_{i,0} = X_i$ and $Y_{i,0} = Y_i$. Whenever Avoider claims some vertex x, the "danger" that Avoider will completely occupy a hyperedge that contains x (and therefore lose) increases. On the other hand, if Enforcer claims some vertex y, then Avoider can never completely occupy a hyperedge that contains y, that is, such a hyperedge poses no "danger" at all for Avoider. This leads us to define the following potential function: for every non-negative integer i, let the *potential of a hyperedge* $D \in \mathcal{H}$ after the ith round be $(1 + \frac{1}{p})^{-|D \setminus X_i|}$ if $D \cap Y_i = \emptyset$ and 0 otherwise. Furthermore, we define the function $\psi(i) = \varphi(X_i, Y_i, \mathcal{H})$ which we call the *potential of the game* after the ith round. Observe that the potential of the game is just the sum of the potentials of the hyperedges. Avoider loses if and only if there exists an integer i such that $D \subseteq X_i$ for some $D \in \mathcal{H}$. If this is the case then the potential of D is $(1 + \frac{1}{p})^0 = 1$. It follows that if the potential $\psi(i)$ of the game is less than 1 for every $i \geq 0$ then Avoider wins. Avoider's winning strategy is then the following: on his $(i+1)$st move, for every $1 \leq k \leq p$, he computes the value of $\varphi(X_{i,k-1}, Y_i, \mathcal{H}, x)$ for every vertex $x \in V \smallsetminus (Y_i \cup X_{i,k-1})$ and then selects $x_{i+1}^{(k)}$ for which the minimum is attained. We show that the value of ψ does not increase throughout the game. If Avoider claims a vertex $x_{i+1}^{(k)}$, then the potential of every hyperedge that contains $x_{i+1}^{(k)}$ is multiplied by $1 + \frac{1}{p}$. Hence, every such hyperedge e, which currently has potential $f(e)$, adds an extra $\frac{1}{p} f(e)$ to the potential of the game. On the other hand, if Enforcer claims some vertex y, then the potential of every hyperedge that contains y drops to 0 (equivalently, the potential of such a hyperedge is subtracted from the potential of the game). Thus, we have

$$\psi(i+1) = \psi(i) + \frac{1}{p}\sum_{k=1}^{p} \varphi(X_{i,k-1}, Y_i, \mathcal{H}, x_{i+1}^{(k)}) - \sum_{t=1}^{q} \varphi(X_{i+1}, Y_{i,t-1}, \mathcal{H}, y_{i+1}^{(t)}). \qquad (2)$$

Using the minimum property of $x_{i+1}^{(k)}$ and the simple observation $\varphi(X, Y, \mathcal{H}, z') \leq \varphi(X \cup \{z''\}, Y, \mathcal{H}, z')$, we get $\varphi(X_{i,k-1}, Y_i, \mathcal{H}, x_{i+1}^{(k)}) \leq \varphi(X_{i,k-1}, Y_i, \mathcal{H}, y_{i+1}^{(1)}) \leq \varphi(X_{i+1}, Y_i, \mathcal{H}, y_{i+1}^{(1)})$ for every $1 \leq k \leq p$. By this, equation (2), and since $\varphi(X_{i+1}, Y_{i,t-1}, \mathcal{H}, y_{i+1}^{(t)}) \geq 0$ for every $2 \leq t \leq q$, we have

$$\begin{aligned}
\psi(i+1) &\leq \psi(i) + \frac{1}{p}\sum_{k=1}^{p} \varphi(X_{i,k-1}, Y_i, \mathcal{H}, x_{i+1}^{(k)}) - \varphi(X_{i+1}, Y_i, \mathcal{H}, y_{i+1}^{(1)}) \\
&\leq \psi(i) + \frac{1}{p} p \varphi(X_{i+1}, Y_i, \mathcal{H}, y_{i+1}^{(1)}) - \varphi(X_{i+1}, Y_i, \mathcal{H}, y_{i+1}^{(1)}) \\
&= \psi(i).
\end{aligned}$$

If Enforcer is the last player to move, then by our assumption $\psi(0) < 1$, we have $\psi(i) < 1$ for every i, implying Avoider's win. Note that on his last move Enforcer might claim strictly less than q vertices (but at least one). This will affect the equality (2), but not the overall inequality $\psi(i+1) \leq \psi(i)$ derived above.

If Avoider is the last player to move, then by our assumption $\psi(0) < \left(1 + \frac{1}{p}\right)^{-p}$, and so $\psi(i) < 1$ for every integer i except maybe for $i = r$ which denotes the last round of the game. In this round only Avoider will participate, but then $\psi(r) \leq (1 + \frac{1}{p})^p \psi(r-1) \leq (1 + \frac{1}{p})^p \psi(0) < 1$.

Finally, assume that Enforcer starts the game, and on his first move he claims, say, vertices $y_1, \ldots, y_q$. Let $\tilde{\mathcal{H}}$ be the hypergraph, obtained from $\mathcal{H}$ by deleting the vertices $y_1, \ldots, y_q$ and deleting every hyperedge $e \in \mathcal{H}$ such that $e \cap \{y_1, \ldots, y_q\} \neq \emptyset$. Clearly, $\sum_{D \in \tilde{\mathcal{H}}} \left(1 + \frac{1}{p}\right)^{-|D|} \leq \sum_{D \in \mathcal{H}} \left(1 + \frac{1}{p}\right)^{-|D|}$. Hence, by the proof above, Avoider wins the game on $\tilde{\mathcal{H}}$ as the first player, entailing his win on $\mathcal{H}$ as the second player. This concludes the proof of the theorem. □

Proof of Theorem 1.2:
For every positive integers p and q, we define an infinite sequence of hypergraphs $\{\mathcal{H}_{p,q}^n\}_{n \geq 1}$. Let $G_{p,q}^n$ be an auxiliary tree consisting of a path of length n on vertices $v_0, \ldots v_n$ with edges

$e_i = (v_{i-1}, v_i)$ for every $1 \leq i \leq n$, and $p+q-1$ new leaves, attached to each vertex of the path except for v_n. The set containing e_i and the $p+q-1$ edges connecting v_{i-1} to leaves, is called the *ith level*.

The vertices of $\mathcal{H}^n_{p,q}$ are the edges of $G^n_{p,q}$. The hyperedges of $\mathcal{H}^n_{p,q}$ are of the form $\{e_1, \ldots, e_{i-1}\} \cup W$, where W is a subset of the ith level, $|W| = p$ and $e_i \notin W$, or of the form $\{e_1, \ldots, e_n\} \cup W$ where W is a subset of the nth level, $|W| = p-1$ and $e_n \notin W$ (see Figure 1).

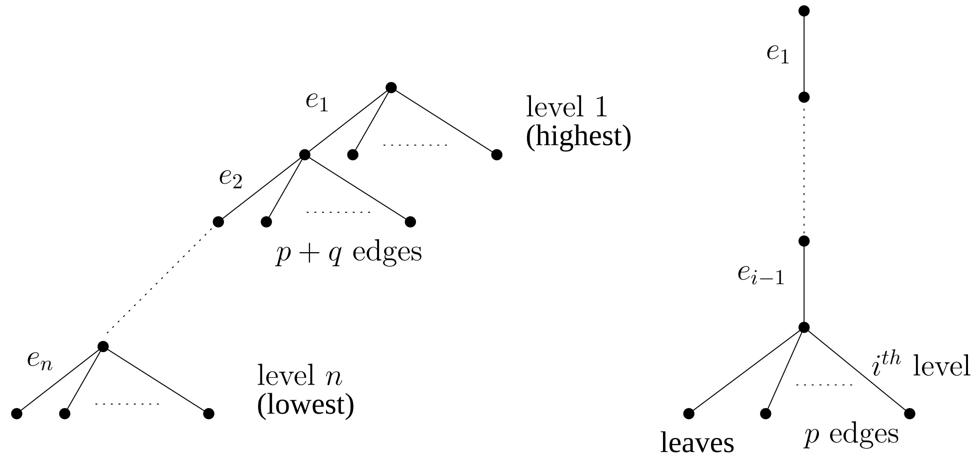

Figure 1: $G^n_{p,q}$ and a typical hyperedge of $\mathcal{H}^n_{p,q}$.

We have:

$$\begin{aligned}
\sum_{D \in \mathcal{H}^n_{p,q}} \left(1+\frac{1}{p}\right)^{-|D|} &= \binom{p+q-1}{p} \sum_{i=0}^{n-1} \left(1+\frac{1}{p}\right)^{-(i+p)} + \binom{p+q-1}{p-1} \left(1+\frac{1}{p}\right)^{-(n-1+p)} \\
&= \binom{p+q-1}{q-1} \left[\left(1+\frac{1}{p}\right)^{-(n-1+p)} \frac{(1+\frac{1}{p})^n - 1}{1+\frac{1}{p}-1} + \frac{p}{q} \left(1+\frac{1}{p}\right)^{-(n-1+p)} \right] \\
&\leq \binom{p+q-1}{q-1} p \left(1+\frac{1}{p}\right) \left(1+\frac{1}{p}\right)^{-p} \\
&\leq \binom{p+q-1}{q-1} \frac{p+1}{2},
\end{aligned}$$

where the first inequality follows since $q \geq 1$.

Finally, we need to show that Enforcer wins the (p,q) game on $\mathcal{H}^n_{p,q}$. His strategy is very simple - he always picks edges from the lowest possible levels (level $i+1$ is considered to be lower than level i), breaking ties arbitrarily.

With this strategy, he ensures that the number of edges claimed by Avoider in the first level is at least p. If this number is strictly larger than p, or e_1 was claimed by Enforcer then Avoider lost. Assume then, that Avoider has claimed exactly p edges in the first level and one of them is e_1. Now, Enforcer's strategy ensures that Avoider has claimed at least p edges of the second level. If Avoider did not claim a winning set in the second level then by the same reasoning he has claimed at least p edges of the third level and so on. Since, in the nth level, every p edges form a winning set, Avoider must have claimed one (in some level) and therefore lost. $\square$

3 Enforcing a matching

Proof of Theorem 1.3
Let $0 \leq t \leq q$ be the smallest integer such that $(q+1) \mid n(n-1)+t$. Let $G = (U \cup V, E)$ be a copy of $K_{n,n}$ in K_{2n} and let F be an arbitrary set of t edges from E. Let $E_1 = E \setminus F$ and let E_2 denote the remaining edges of K_{2n}. Whenever Avoider picks an edge of E_2, Enforcer picks q edges of E_2. This is always possible as $|E_2| = n(n-1)+t$ which is divisible by $q+1$. Whenever Avoider picks an edge of E_1, Enforcer, picks q edges of E_1 (this is always possible except for maybe once). It is therefore sufficient to prove that Enforcer can win the $(1, q)$ perfect matching game on E_1.

We will provide Enforcer with a strategy, which guarantees that at the end of the game Avoider's graph will satisfy Hall's condition. To this end we define an auxiliary game which we denote by $HALL$ on E_1 with hypergraph $\mathcal{F}_{2n}$ (which is defined below), where Enforcer takes the role of Avoider (to avoid confusion, Enforcer will be referred to as "HALL-Avoider") and HALL-Avoider's win in $(q, 1, \mathcal{F}_{2n})$ implies Enforcer's win in the $(1, q)$ perfect matching game.

The vertices of $\mathcal{F}_{2n}$ are the elements of E_1 and the hyperedges of $\mathcal{F}_{2n}$ are all the edge-sets $E(X, Y) \subseteq E_1$ between two subsets $X \subseteq U$ and $Y \subseteq V$ for which $|X|+|Y| = n+1$. Clearly, if HALL-Avoider avoids completely occupying any such set $E(X, Y)$, then in his opponent's graph $|N(X)| \geq |X|$ for every $X \subseteq U$, where $N(X) = \{v \in V : \exists u \in X, (u, v) \in E_1\}$.

We apply Theorem 1.1. For $q = cn/\log_2 n$, we have:

$$
\begin{aligned}
\sum_{D\in\mathcal{F}_{2n}}\left(1+\frac{1}{q}\right)^{-|D|} &\le \sum_{D\in\mathcal{F}_{2n}} 2^{-|D|/q} \le \sum_{k=1}^{n}\binom{n}{k}\binom{n}{n-k+1}2^{-\frac{(k(n-k+1)-t)\log_2 n}{cn}} \\
\le\ & 2\sum_{k=1}^{n/2}\binom{n}{k}^2 2^{1-\frac{k(n-k+1)\log_2 n}{cn}} \\
\le\ & 2\sum_{k=1}^{\sqrt{n}}\left[n^2\cdot 2^{1-\frac{(n-k+1)\log_2 n}{cn}}\right]^k + 2\sum_{k=\sqrt{n}}^{n/2}\left[\left(\frac{en}{k}\right)^2 2^{1-\frac{(n-k+1)\log_2 n}{cn}}\right]^k \\
\le\ & 2\sum_{k=1}^{\sqrt{n}}\left[2n^2\cdot n^{-\left(1-\frac{1}{\sqrt{n}}\right)\frac{1}{c}}\right]^k + 2\sum_{k=\sqrt{n}}^{n/2}\left[n\cdot 2e^2\cdot n^{-\left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{c}}\right]^k.
\end{aligned}
$$

Both sums are $o(1)$ provided $c = \frac{1}{2} - o(1)$. Hence Theorem 1.1 applies and the proof of Theorem 1.3 is complete. $\square$

Remark. Theorem 1.3 can be easily adapted to show that playing the $(1, q)$ game on K_{2n+1}, Enforcer can force Avoider's graph to admit a matching which covers all vertices but one for every $q \le \frac{cn}{\log n}$. We omit the straightforward details.

4 Enforcing a Hamilton cycle

Proof of Theorem 1.4
We will use the following special case of a theorem from [11]:

Theorem 4.1 *Let $G = (V, E)$ be a graph on n vertices and let $d = \frac{\log^{(3)} n}{\log^{(4)} n}$. Assume that G satisfies the following two properties:*

P1 *For every $S \subset V$, if $|S| \le \frac{n\log^{(2)} n \log d}{d \log n \log^{(3)} n}$ then $|N(S)| \ge d|S|$.*

P2 *There is an edge in G between any two disjoint subsets $A, B \subseteq V$ such that $|A|, |B| \ge \frac{n\log^{(2)} n\log d}{4130\log n\log^{(3)} n}$.*

Then G is hamiltonian, for sufficiently large n.

Let $\mathcal{H}_n^1$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are all the copies of $K_{r,r}$ in K_n where $r = \frac{n\log^{(2)} n\log d}{4130\log n\log^{(3)} n}$. Let $\mathcal{H}_n^2$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are all the copies of $K_{s,t}$ in K_n for every $1 \leq s \leq \frac{n \log^{(2)} n \log d}{d \log n \log^{(3)} n}$ and $t = n - d \cdot s$. In order to win, Enforcer will make Avoider build a graph that satisfies the properties of Theorem 4.1, that is, Enforcer would like to avoid selecting all the edges connecting any two disjoint subsets of V of size at least $\frac{n \log^{(2)} n \log d}{4130 \log n \log^{(3)} n}$ each, and to avoid selecting all the edges connecting any two disjoint subsets of V, one of size $1 \leq s \leq \frac{n \log^{(2)} n \log d}{d \log n \log^{(3)} n}$ and the other of size $n - d \cdot s$. Thus, by Theorem 1.1 it suffices to prove that $\sum_{D \in \mathcal{H}_n^1 \cup \mathcal{H}_n^2} (1 + \frac{1}{q})^{-|D|} < (1 + \frac{1}{q})^{-q}$. We have

$$
\begin{aligned}
&\sum_{D \in \mathcal{H}_n^1} \left(1 + \frac{1}{q}\right)^{-|D|} \leq \sum_{D \in \mathcal{H}_n^1} 2^{-|D|/q} \\
\leq\ & \binom{n}{\frac{n \log^{(2)} n \log d}{4130 \log n \log^{(3)} n}}^2 \exp\left\{ -\log 2 \left(\frac{n \log^{(2)} n \log d}{4130 \log n \log^{(3)} n} \right)^2 \frac{8500 \log n \log^{(3)} n}{n \log 2 \log^{(4)} n} \right\} \\
\leq\ & \left(\frac{4130 e \log n \log^{(3)} n}{\log^{(2)} n \log d} \right)^{2 \frac{n \log^{(2)} n \log d}{4130 \log n \log^{(3)} n}} \exp\left\{ - \frac{(1 - o(1)) 8500 n (\log^{(2)} n)^2 \log d}{(4130)^2 \log n \log^{(3)} n} \right\} \\
\leq\ & \exp\left\{ (2 + o(1)) \frac{n (\log^{(2)} n)^2 \log d}{4130 \log n \log^{(3)} n} - \frac{8500 n (\log^{(2)} n)^2 \log d}{(4130)^2 \log n \log^{(3)} n} \right\} \\
=\ & o(1).
\end{aligned}
$$

Similarly, for every $1 \leq s \leq \frac{n \log^{(2)} n \log d}{d \log n \log^{(3)} n}$ and $D \in \mathcal{H}_n^2$ of size s we have

$$
\begin{aligned}
\binom{n}{s} \binom{n}{n - d \cdot s} \left(1 + \frac{1}{q}\right)^{-|D|} &\leq n^s n^{d \cdot s} 2^{-|D|/q} \\
&\leq \exp\left\{ s(1 + d) \log n - \log 2 (n - d \cdot s) s \frac{8500 \log n \log^{(3)} n}{n \log 2 \log^{(4)} n} \right\} \\
&= o\left(\frac{1}{n}\right).
\end{aligned}
$$

Thus $\sum_{D \in \mathcal{H}_n^2} (1 + \frac{1}{q})^{-|D|} = o(1)$. It follows that $\sum_{D \in \mathcal{H}_n^1 \cup \mathcal{H}_n^2} (1 + \frac{1}{q})^{-|D|} \leq \sum_{D \in \mathcal{H}_n^1} (1 + \frac{1}{q})^{-|D|} + \sum_{D \in \mathcal{H}_n^2} (1 + \frac{1}{q})^{-|D|} < (1 + \frac{1}{q})^{-q}$. □

5 Connectivity-related games

Proof of Theorem 1.8

Let $T_1, T_2, \ldots, T_{q+1}$ be pairwise edge disjoint spanning trees of $G = (V, E)$. Let $I = \bigcup_{i=1}^{q+1} T_i$

and let $L = E \smallsetminus I$. Enforcer's strategy is the following: he maintains acyclic graphs $G_1, G_2, \ldots, G_{q+1}$. In the beginning $G_i = T_i$ for every $1 \leq i \leq q+1$. Whenever Avoider picks some edge $e \in G_j$, Enforcer picks one edge $f_i \in G_i$ for every $1 \leq i \neq j \leq q+1$ (hence a total of q edges). If $G_i \cup \{e\}$ is acyclic then f_i is chosen arbitrarily. Otherwise $G_i \cup \{e\}$ contains a unique cycle C_i and then Enforcer picks some unclaimed $f_i \in C_i$. In both cases Enforcer replaces G_i with $G_i \cup \{e\} \smallsetminus \{f_i\}$. If Avoider picks an edge of L then Enforcer picks any q previously unclaimed edges of L. If there are only $r < q$ edges left in L then Enforcer picks these r edges and another single arbitrary edge $f_i \in G_i$ for every $1 \leq i \leq q-r$. Finally, if Enforcer starts the game then on his first move he picks any q edges of L if $|L| \geq q$ and otherwise all the edges of L and one arbitrary edge $f_i \in G_i$ for every $1 \leq i \leq q - |L|$. In any case Enforcer removes f_i from G_i.
We will prove that Enforcer's strategy is a winning strategy. First, note that every unclaimed edge of I is in exactly one G_i, every edge of I claimed by Avoider is in every G_i and every edge claimed by Enforcer is in no G_i. This is clearly true in the beginning, and then an edge is removed from G_i iff it is chosen by Enforcer and added to every G_i iff it is chosen by Avoider. Furthermore, after every round (a move by Avoider and a counter move by Enforcer) G_i is either a spanning tree or a spanning tree minus one edge for every $1 \leq i \leq q+1$. This is clearly true in the beginning. Assume it is still true after the kth round. If on his $(k+1)$st move Avoider picks $e \in G_j$, then Enforcer picks $f_i \in G_i$ according to his strategy. If $G_i \cup \{e\}$ is acyclic then it must be a spanning tree and so $G_i \cup \{e\} \smallsetminus \{f_i\}$ is a spanning tree minus one edge. Otherwise $G_i \cup \{e\}$ contains a cycle C_i and since $f_i \in C_i$ (such an f_i must exist as all the G_i's were acyclic on the kth round) $G_i \cup \{e\} \smallsetminus \{f_i\}$ is the same as G_i was (both are spanning trees or both are spanning trees minus an edge). If both players play in L then there is nothing to prove. It is possible (as was mentioned above) that there will be one (and only one) round in which Avoider does not pick any edges of I and Enforcer does. Clearly (by the above argument), before that round every G_i was a spanning tree. Now several G_i's will still be spanning trees and the rest will be spanning trees minus one edge. Thus, in the end $G_i = G_A \cap I$ for every $1 \leq i \leq q+1$, where G_A is the graph built by Avoider. It follows that $G_A \cap I$ is either a spanning tree or a spanning tree minus an edge, and since $|G_A \cap I| = |V| - 1$, the former must hold. □

Remark. The opposite implication of Theorem 1.8 is "almost" true, in the sense that it is true if we add restrictions on the number of edges and the identity of the first player. Indeed if G does not contain $q+1$ pairwise edge disjoint spanning trees then by the famous theorem of Nash-Williams [16] and independently Tutte [17], for some $r \geq 2$ there exists a partition of the vertices of G into r parts with at most $(q+1)(r-1)-1$ crossing edges. So Avoider (depending of course on who starts the game and how many edges are there in G) may claim less than $r-1$ of them and thus win.
Note that something quite different happens in the $(1, q)$ Maker-Breaker connectivity game. Here if $q \geq 2$ then the existence of $q+1$ pairwise edge disjoint spanning trees does not guarantee Maker's win. In fact there are graphs with an arbitrarily large number s of such trees which are a win for Breaker in the $(1, 2)$ game (as was already mentioned in [5]). Such

a graph is for example m copies of K_{2s} such that copy i is connected by s edges to copy $i+1$ for every $1 \leq i \leq m-1$ and m is sufficiently large.
If $q = 1$ then two edge disjoint spanning trees are enough to ensure Maker's win (c.f. [13]). If a graph G does not contains $q+1$ pairwise edge disjoint spanning trees then the outcome depends on the identity of the first player (but not necessarily on the number of edges, as Breaker wins if he starts the game). Again this follows from the theorem of Nash-Williams and Tutte.

Remark. There is a polynomial time algorithm for finding $q+1$ pairwise edge disjoint spanning trees in a graph G, in case they exist, (c.f. [7]). Thus, combined with our proof it yields an efficient explicit winning strategy for Enforcer.

Remark. The proof of Theorem 1.8 can be generalized to a game on any matroid (the matroid contains $q+1$ pairwise disjoint bases and Avoider loses iff he selects all the elements of some basis). We omit the straightforward details.

Proof of Theorem 1.5
If $q \leq \lfloor \frac{n}{2} \rfloor - 1$ then Enforcer wins the game by Theorem 1.8 as K_n contains $\lfloor \frac{n}{2} \rfloor$ pairwise edge disjoint spanning trees. If Avoider is the first player in the $(1, \lfloor \frac{n}{2} \rfloor)$ game on K_n, where n is odd, then at the end of the game he will have exactly $n-1$ edges. So he will win iff he will claim all the edges of some cycle in K_n. This Maker-Breaker game where Maker's goal is to build a cycle, was studied by Bednarska and Pikhurko [4] in a more general context. In the particular case that we are interested in, their result shows that Breaker can break all the cycles and so Enforcer can force Avoider to build a spanning tree. Finally, in any other case, Avoider will not have enough edges to build a spanning tree so he will win no matter how he plays. $\square$

Proof of Theorem 1.6
If $q \geq \frac{n}{k}$ then at the end of the game Avoider will have at most $\lceil \frac{n(n-1)}{2(\frac{n}{k}+1)} \rceil \leq \frac{n(n-1)}{2(\frac{n}{k}+1)} + 1$ edges. Thus the minimum degree in Avoider's graph, regardless of his strategy, will be at most $\frac{n-1}{\frac{n}{k}+1} + \frac{2}{n} < k$ where the last inequality holds for every $k \geq 2$. It follows that Avoider's graph will not be k-edge connected.

Let $q \leq \frac{n}{2k} - 1$ and let $T_1, \ldots, T_{\lfloor n/2 \rfloor}$ be pairwise edge disjoint spanning trees of K_n. For every $1 \leq i \leq k$, let $G_i = \bigcup_{j=(i-1)\lfloor \frac{n}{2k} \rfloor + 1}^{i \lfloor \frac{n}{2k} \rfloor} T_j$. Enforcer plays k separate games in parallel, that is, whenever Avoider claims an edge of G_i, for some $1 \leq i \leq k$, Enforcer plays all his q edges in G_i (of course Avoider might sometimes play in $E(K_n) \setminus \bigcup_{i=1}^{k} G_i$, but, as in the proof of Theorem 1.8, it does not affect the outcome of the game). By Theorem 1.8, Enforcer can make Avoider build a spanning tree in G_i for every $1 \leq i \leq k$ (as in every G_i there are $\lfloor \frac{n}{2k} \rfloor$ pairwise edge disjoint spanning trees and $q \leq \lfloor \frac{n}{2k} \rfloor - 1$). The G_i's are pairwise edge disjoint and so Avoider's graph will be k-edge connected. $\square$

Proof of Theorem 1.7
Similarly to the second part of the proof of Theorem 1.6, Enforcer plays $\lfloor n/4 \rfloor$ separate games in parallel. The board of each of these games consists of two edge disjoint spanning trees, and so, by Theorem 1.8, Enforcer can make Avoider build one spanning tree on every board and hence a total of $\lfloor n/4 \rfloor$ trees. □

6 Non-monotonicity of biased games

In this section we give two examples which show that Avoider-Enforcer games are not monotone in general. It would be extremely interesting to give at least a sufficient condition for an Avoider-Enforcer game to be monotone.

Though quite straightforward, for the sake of completeness, we prove that Maker-Breaker games are indeed monotone.

Proposition 6.1 *If Maker wins the $(p, q, \mathcal{H})$ game, where $\mathcal{H}$ is any hypergraph, then he also wins the $(p+1, q, \mathcal{H})$ game and the $(p, q-1, \mathcal{H})$ game. Similarly, If Breaker wins the $(p, q, \mathcal{H})$ game, then he also wins the $(p-1, q, \mathcal{H})$ game and the $(p, q+1, \mathcal{H})$ game.*

Proof Assume first that Maker has a winning strategy $\mathcal{S}_m$ for the $(p, q, \mathcal{H})$ game. When playing the $(p, q-1, \mathcal{H})$ game, Maker plays according to $\mathcal{S}_m$. Whenever Breaker picks his $q-1$ vertices, Maker (in his mind) chooses an arbitrary unclaimed vertex and 'gives' it to Breaker. If Breaker picks an unclaimed vertex that already 'belongs' to him in Maker's mind, then Maker 'gives' him another arbitrary unclaimed vertex. By the end (in Maker's mind) of the game he has already won (as he played according to $\mathcal{S}_m$). Clearly, no matter how they proceed, Maker will win the game.
When playing the $(p+1, q, \mathcal{H})$ game, Maker plays according to $\mathcal{S}_m$, where in every turn he picks one additional arbitrary unclaimed vertex. At a certain point during the game it might happen that a vertex that Maker should pick according to $\mathcal{S}_m$ already belongs to him; he will then pick another arbitrary unclaimed vertex. Since Maker played according to $\mathcal{S}_m$, at the end of the game, if we remove all the 'additional' vertices picked by Maker, we get a position from which Maker can win. Clearly, picking every remaining vertex is a winning strategy.

The proof of monotonicity for Breaker's win is analogous. □

Next, we give examples showing that Avoider-Enforcer games need not be monotone.

Example 1: Consider the $(1, q, \mathcal{H}_t)$ game, where the vertices of $\mathcal{H}_t$ are the vertices of $t \cdot K_2$ (i.e. t vertex disjoint edges), and the hyperedges of $\mathcal{H}_t$ are the edges of $t \cdot K_2$. We claim that, for sufficiently large t, Enforcer (as first or second player) wins this game iff q

is even. Indeed, if q is even then in every turn (for as long as possible) Enforcer picks $\frac{q}{2}$ edges (unclaimed pairs of vertices). Clearly, if $t > q(\frac{q}{2}+1)$ then Avoider will lose. If q is odd, then assuming that Enforcer is the first player, Avoider can always pick an unclaimed vertex whose single neighbor was picked by Enforcer, and therefore win. Finally, if Avoider is the first player, then in every turn, either Enforcer picks all unclaimed vertices which are in the neighborhood of Avoider's vertices, or he picks a vertex whose single neighbor w is unclaimed. In the former case, Avoider claims an arbitrary vertex, whereas in the latter, he claims w. Either way, after every move of Avoider, there is at most one unclaimed vertex in the neighborhood of Avoider's vertices. Since the number of vertices is even and q is odd, Avoider is not the last player and so he wins.
It follows that this game in not monotone in q.

Example 2: Consider the $(p, 1, \mathcal{H}'_t)$ game, where the vertices of $\mathcal{H}'_t$ are the vertices of $t \cdot K_2$ and the hyperedges of $\mathcal{H}'_t$ are the minimal vertex covers of $t \cdot K_2$. We claim that, for sufficiently large t, Avoider (as first or second player) wins this game iff p is even. This follows immediately from the analysis of **Example 1** if Avoider (Enforcer) adopts Enforcer's (Avoider's) strategy from that example.

It follows that this game in not monotone in p.

Note that though in general Avoider-Enforcer games are not monotone, the sufficient condition given in Theorem 1.1 guarantees monotonicity in both p and q. Indeed, $\left(1+\frac{1}{p}\right)^{-1}$ is monotone increasing in p and so if $\sum_{D\in\mathcal{H}}\left(1+\frac{1}{p}\right)^{-|D|} < e^{-1}$ then $\sum_{D\in\mathcal{H}}\left(1+\frac{1}{r}\right)^{-|D|} < e^{-1}$ for every $r \leq p$. Hence, it follows from Theorem 1.1 that if $\sum_{D\in\mathcal{H}}\left(1+\frac{1}{p}\right)^{-|D|} < e^{-1}$, then Avoider wins the (r, q) game for every $r \leq p$ and $q \geq 1$.

7 Concluding remarks and open problems

1. **General criterion for Avoider's win.** It was already indicated in the introduction that our criterion for Avoider's win in the $(p, q, \mathcal{H})$ game is not effective when q is large. Such a criterion might help us improve our bound on $b^{-}_{\mathcal{H}_n}$. It would also have a potentially significant impact on traditional Maker-Breaker type games. Often Maker can achieve his goal in some game by creating a pseudo-random graph of a certain edge-density (see e.g. [9], [10]). Such a graph might need to have a property of "at most" type. Maker could try to achieve such conditions by playing as Avoider and trying to "avoid" occupying too many elements of the winning sets.

2. **General criterion for monotonicity.** We say that an Avoider-Enforcer game $\mathcal{B}$ is *monotone*, if Enforcer's winning strategy for $(p, q, \mathcal{B})$ implies his win in $(p+1, q, \mathcal{B})$ and $(p, q-1, \mathcal{B})$, while Avoider's winning strategy for $(p, q, \mathcal{B})$ implies his win in $(p-1, q, \mathcal{B})$ and $(p, q+1, \mathcal{B})$.

Problem 7.1 *Find a sufficient (and possibly also necessary) condition for an Avoider-Enforcer game to be monotone.*

3. (Asymptotic) monotonicity of $\mathcal{M}_n$ and $\mathcal{H}_n$. For both the Hamilton cycle game and the perfect matching game there is a significant gap between the corresponding thresholds b^-, b^+ shown in this paper (the only bounds on $b^+_{\mathcal{M}_n}$ and $b^+_{\mathcal{H}_n}$ that we know are the ones derived from the trivial lower bound on the number of edges in a perfect matching and a Hamilton cycle respectively). It would be interesting to close, or at least to reduce, these gaps.
We believe that even the following holds.

Conjecture 7.2 *Both the perfect matching game and the Hamilton cycle game are monotone. In particular $b^-_{\mathcal{M}_n} = b^+_{\mathcal{M}_n}$ and $b^-_{\mathcal{H}_n} = b^+_{\mathcal{H}_n}$.*

The function $f(n)$ is called an *asymptotic threshold bias* of the game $\mathcal{B}_n$ if both $b^-_{\mathcal{B}_n} = \Theta(f(n))$ and $b^+_{\mathcal{B}_n} = \Theta(f(n))$. If an asymptotic threshold bias exists, that is, if $b^-_{\mathcal{B}_n} = \Theta\left(b^+_{\mathcal{B}_n}\right)$, then the game $\mathcal{B}_n$ is called *asymptotically monotone*.

It would be a significant step towards proving Conjecture 7.2 if one could establish that the perfect matching and Hamilton cycle games are asymptotically monotone and determine the order of magnitude of the asymptotic threshold bias. Recall that currently we do not even know whether Avoider can win $(1, o(n), \mathcal{M}_n)$ or $(1, o(n), \mathcal{H}_n)$.

Acknowledgment. We are grateful to József Beck; his question about Enforcer's strategy in the biased Hamilton cycle game, greatly motivated this research.

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