Construction and Applications of (k, d)-trees

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For $A \in \mathcal{F}$, let $Y_A = 1$ if A is monochromatic, otherwise $Y_A = 0$.

$$\mathbb{E}[\#\text{of m.c. edges of } \mathcal{F}] = \mathbb{E}\left[\sum_{A \in \mathcal{F}} Y_A\right] = \sum_{A \in \mathcal{F}} \mathbb{E} Y_A = \frac{|\mathcal{F}|}{2^{k-1}} < 1$$

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Hence, for sure, **THERE EXISTS** 2-coloring without m.c. edges ("proper 2-coloring") $\square$

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Positional Games

Maker-Breaker Game $(X, \mathcal{F})$:

Board: set X; family of **winning sets**: $\mathcal{F} \subset 2^X$

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Remark: Perfect information game with complementary goals:
1) Exactly one of the players has a winning strategy.
2) Given $\mathcal{F}$, it is clear (at least to an all-powerful computer) which of them has a winning strategy.

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Erdős-Selfridge: $|\mathcal{F}| < 2^{k-1} \Rightarrow$ Breaker has a winning strategy.

Lovász Local Lemma – Neighborhood Conjecture

LLL. $A_1, A_2, \ldots, A_k$ events in some probability space, such that
(1) every A_i is mutually independent from **all but** d other events
(2) $p \geq Pr[A_i]$ for every i
If $ep(d+1) \leq 1$ then $Pr[\wedge_{i=1}^{k} \overline{A_i}] > 0$.

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Application of LLL: $\Delta(L(\mathcal{F})) \leq \frac{2^{k-1}}{e} - 1 \;\Rightarrow\; \mathcal{F}$ is 2-colorable.
$\Delta(\mathcal{F}) \leq \frac{2^{k-1}}{ek} \;\Rightarrow\; \mathcal{F}$ is 2-colorable.

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Neighborhood Conjecture [Beck]

$$\Delta(L(\mathcal{F})) < 2^{k-1} \;\Rightarrow\; \mathcal{F} \text{ is Breaker's win.}$$

Counterexamples to the strongest form of NC

Theorem (Gebauer, '09)

(i) For every large enough k, there is a k-uniform Maker's win hypergraph $\mathcal{H}$ with $\Delta(L(\mathcal{H})) \leq 0.75 \cdot 2^{k-1}$

(ii) For every large enough k there is a k-uniform Maker's win hypergraph $\mathcal{F}$ with $\Delta(\mathcal{F}) < 0.5 \cdot \frac{2^k}{k}$.

$$D(k) := \min\{\Delta(\mathcal{F}) : \ k\text{-uniform, Maker's win } \mathcal{F}\}$$

Best know lower bound $D(k) > \lfloor \frac{k}{2} \rfloor$.
Deciding whether $D(k) = \lfloor \frac{k}{2} \rfloor + 1$ already seems to need new ideas.

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full binary trees $\rightsquigarrow$ k-uniform hypergraphs

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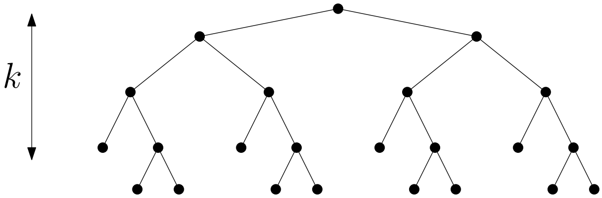

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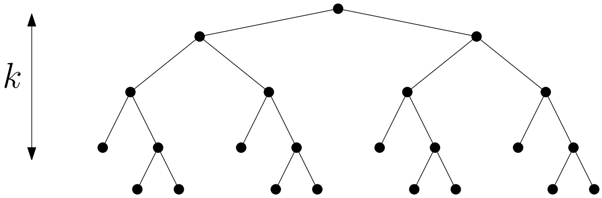

vertices $\leftrightsquigarrow$ elements of the board X

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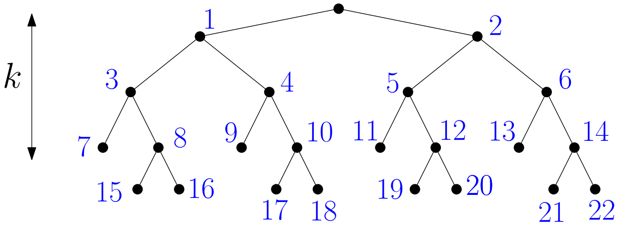

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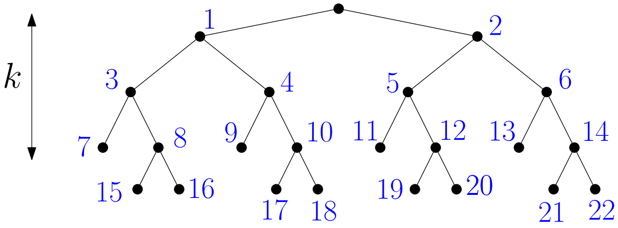

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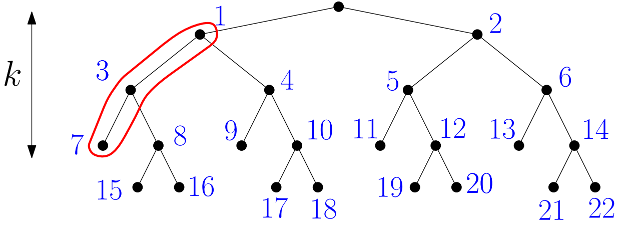

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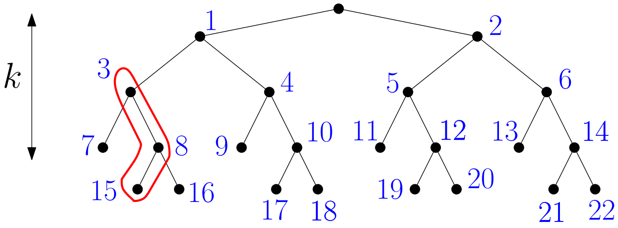

$\mathcal{F} := \{\{1, 3, 7\}, \{3, 8, 15\}\}$

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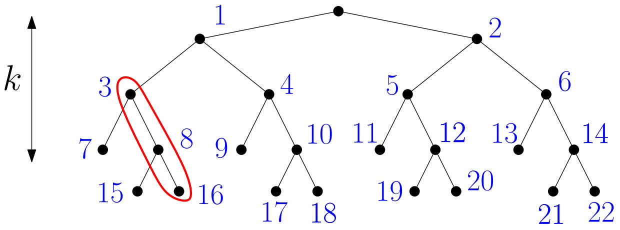

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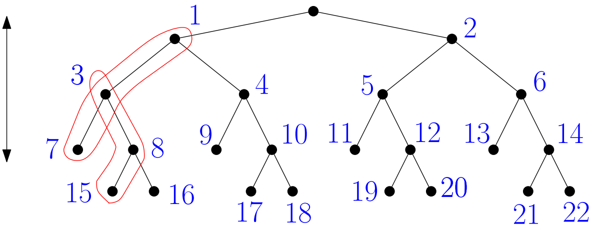

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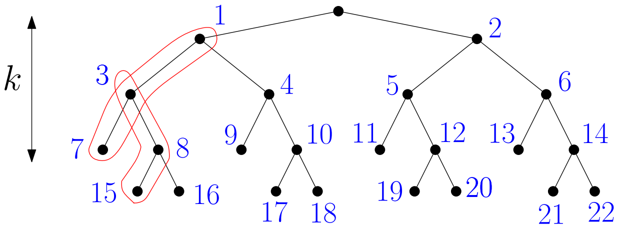

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Proposition

Maker has a winning strategy on the hypergraph $\mathcal{F}$.

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- Every leaf has depth $\geq k$
- Every vertex has $\leq d$ leaf-descendants of distance $\leq k$

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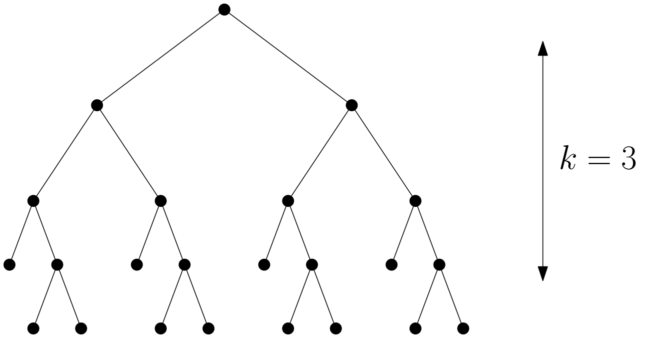
A $(3, 6)$-tree



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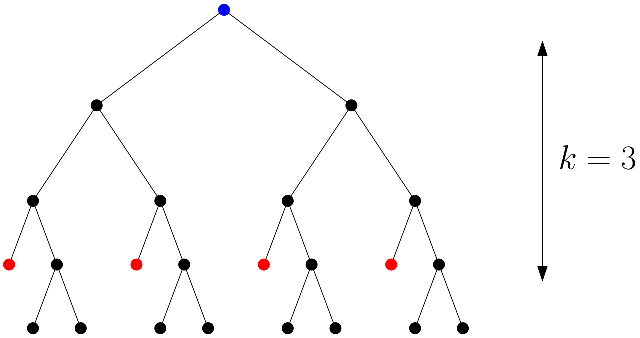

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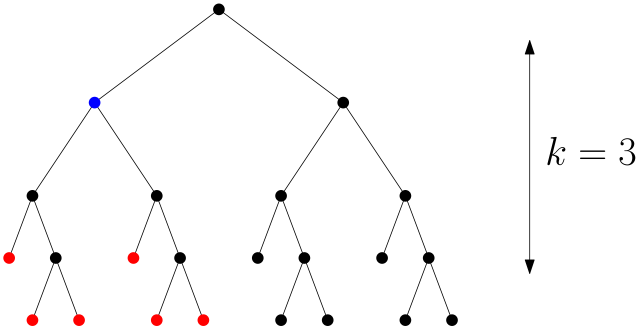

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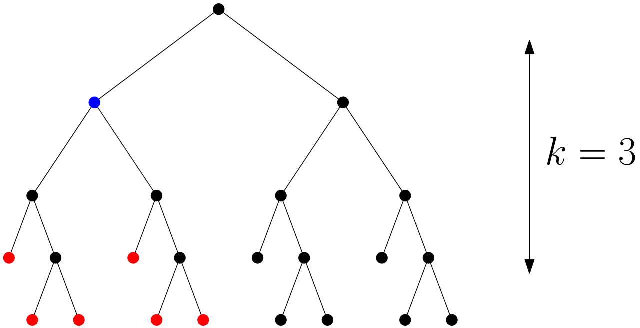

Proposition

There is a $(k-1, d)$-tree $\Rightarrow$ $D(k) \leq d$

Constructing (k, d)-trees

Theorem (Gebauer - Sz. - Tardos, 2011)

There exists a (k, d)-tree with

$$d = \left(\frac{2}{e} + o(1)\right)\frac{2^k}{k}.$$

Corollary

For every positive integer k there exists Maker's win k-uniform hypergraphs $\mathcal{H}$ and $\mathcal{H}'$, such that

(i) $\Delta(L(\mathcal{H})) = \left(1 + O\left(\frac{1}{\sqrt{k}}\right)\right)\frac{2^{k-1}}{e}$,

(ii) $\Delta(\mathcal{H}) = \left(1 + O\left(\frac{1}{\sqrt{k}}\right)\right)\frac{2^k}{ek}$.

Application 2 of LLL: (k,s)-SAT

Application 2 of LLL: Let F be a boolean CNF-formula such that every clause contains exactly k distinct literals. If every variable occurs in less than $\frac{1}{e} \cdot \frac{2^k}{k}$, then F is satisfiable.

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Known values: $f(3) = 3$, $f(4) = 4$, $f(5) = ?$
f is NOT known to be computable

Upper bounds:

	$k \cdot \frac{2^k}{k}$	trivial
	$k^{0.74} \cdot \frac{2^k}{k}$	Savicky-Sgall, '00
	$\log k \cdot \frac{2^k}{k}$	Hoory-Szeider, '06
	$1 \cdot \frac{2^k}{k}$	Gebauer, '09

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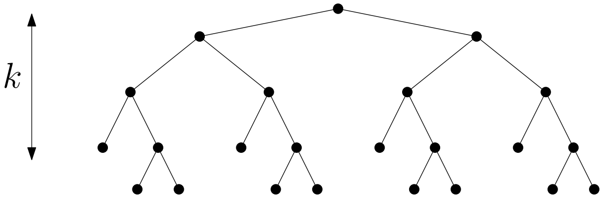

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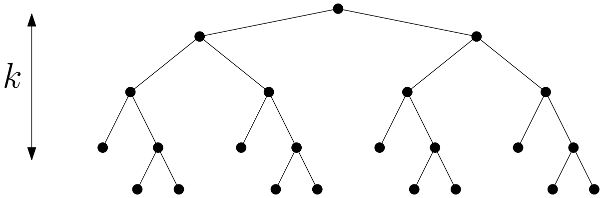

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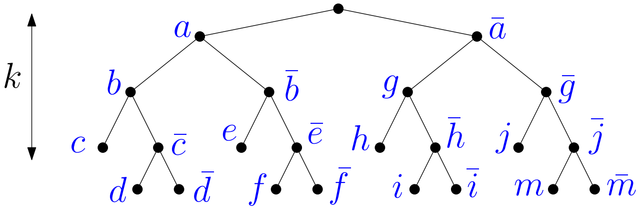

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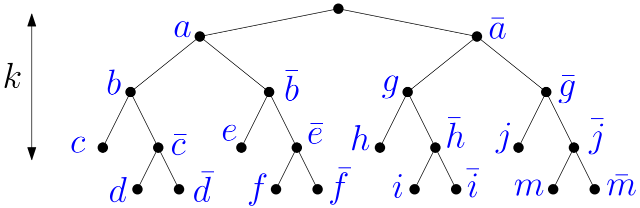

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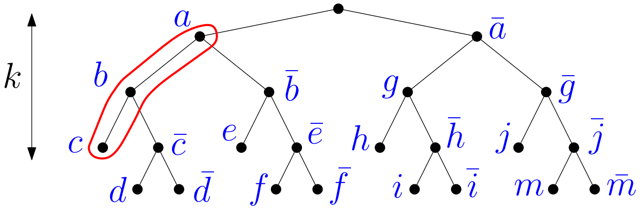

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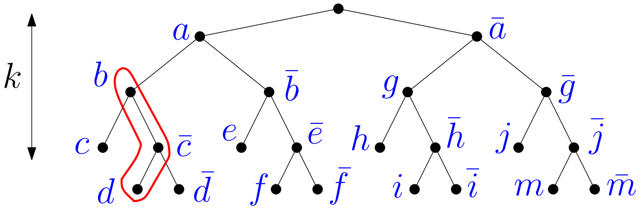

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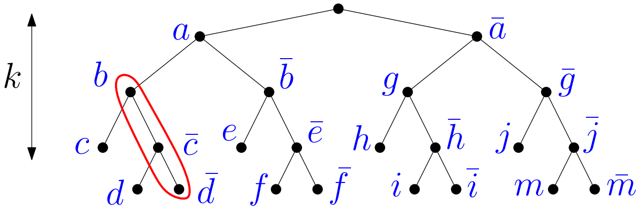

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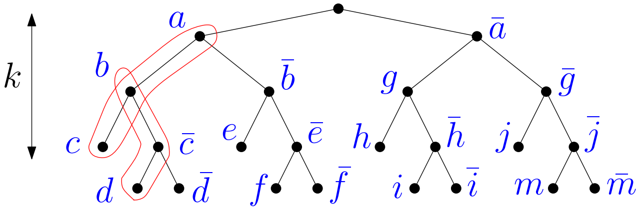

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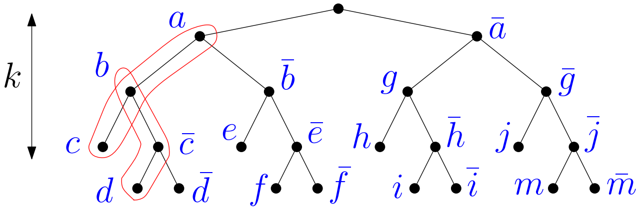

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Proposition

The obtained formula is NOT satisfiable

Asymptotics via (k, d)-trees

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There is a (k, d)*-tree* $\quad \Rightarrow \quad f(k) < d$

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Theorem (Gebauer-Sz.-Tardos, '11)

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Def. The degree of clause C in formula F is the number of those clauses of F that share a variable with C.
$D(F)$ denotes the largest of all clause-degrees in F.

$l(k) := \max\{s : \text{ every } k\text{-CNF formula } F \text{ with } D(F) \leq s \text{ is saitisfiable}\}$

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Def. The degree of clause C in formula F is the number of those clauses of F that share a variable with C.
$D(F)$ denotes the largest of all clause-degrees in F.

$l(k) := \max\{s : \text{ every } k\text{-CNF formula } F \text{ with } D(F) \leq s \text{ is saitisfiable}\}$

Theorem (Gebauer-Sz.-Tardos, '11)

$$l(k) = \left(\frac{1}{e} + o(1)\right)2^k$$

Complexity hardness jump

(k, s)-SAT Problem

- Input: a (k, s)-CNF F
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$\vdots$ $\vdots$

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. . . .

- $(k, f(k) + 1)$-SAT NP-hard

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- (k, ∞)-SAT NP-hard

Improved lower bound with Lopsided LLL

For LLL: set every variable x to $\texttt{true}$ with probability $P_x = \frac{1}{2}$.
Works for every (k, s)-CNF formula F with $s = \left\lfloor \frac{1}{e} \cdot \frac{2^k}{k} \right\rfloor$.

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How to improve? Take the particular formula F into account
[idea of Berman-Karpinski-Scott, '04]
LLLL: Cares only about conflicting occurrences of variables

Lemma

(Lopsided Local Lemma) Let $\{A_C\}_{C\in I}$ *be a finite set of events in some probability space. Let* $\Gamma(C)$ *be a subset of* I *for each* $C \in I$ *such that for every subset* $J \subseteq I \setminus (\Gamma(C) \cup \{C\})$ *we have*

$$Pr(A_C | \wedge_{D\in J} \bar{A}_D) \leq Pr(A_C).$$

Suppose there are real numbers $0 < x_C < 1$ *for* $C \in I$ *such that for every* $C \in I$ *we have*

$$Pr(A_C) \leq x_C \prod_{D\in\Gamma(C)} (1 - x_D).$$

Then

$$Pr(\wedge_{C\in I} \bar{A}_C) > 0.$$

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Set variable x to $\mathtt{true}$ with probability $P_x = \frac{1}{2} + \frac{2d_{\bar{x}} - s}{2sk}$,
where for literal v let $d_v :=$ # of occurrences of v in F.

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Liar Game Player A thinks of an integer $x \in [N]$ and Player B tries to figure it out by asking Yes/No questions of the sort "Is $x \in S$?", where S is a subset of $[N]$ picked by B.

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Ulam's problem for binary search with k lies: A is allowed to lie a total of k times What is the smallest number $q(N, k)$ of questions that allows B to figure out the answer.

Instead of limiting the total number of lies, now the number of consecutive lies is limited: A is not allowed to lie k consecutive times

This restriction on the lies is not enough for B to find the value x with certainty, but he will be able to narrow the set of possibilities.

The IMO problem asked for estimates on how small B can guarantee this set of possibilities will eventually be.

Problem 3, 2012 International Mathematics Olympiad

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Theorem

(Gebauer-Sz.-Tardos) Let $N > d$ and k be positive integers. Assume A and B play the game in which A thinks of an element $x \in [N]$ and then answers an arbitrary number of B's questions of the form "Is $x \in S$?". Assume further that A is allowed to lie, but never to k consecutive questions. Then B can guarantee to narrow the number of possibilities for x with his questions to at most d distinct values if and only if a $(k, d+1)$-tree exists, that is, if $d \gtrapprox \frac{2^{k+1}}{ek}(1 + o(1))$.

Tenure Game (J. Spencer)

Two players: the (good) chairman of the department, and the (vicious) dean of the school
The pieces: d non-tenured faculty of the department each at one of k pre-tenured rungs
Winner: The chairman if a faculty is promoted to tenure, otherwise the dean. (A non-tenured faculty becomes tenured if she has rung k and is promoted.)
Procedure: Once each year, the chairman proposes to the dean a subset S of the non-tenured faculty to be promoted by one rung. The dean has two choices: either he accepts the suggestion of the chairman, promotes everybody in S by one rung and fires everybody else, or he does the complete opposite: fires everybody in S and promotes everybody else by one rung.

If all d faculties are at rung 1, then chairman wins iff $k \leq \lfloor \log d \rfloor$.

European Tenure Game (B. Doerr)

Modified Rules: the non-promoted part of the non-tenured faculty is not fired, rather demoted back to rung 1. Assume that all non-tenured faculty are at the lowest rung in the beginning For fixed d let v_d stand for the largest number k of rungs such that the chairman wins.

Doerr (2004) showed

$$\lfloor \log d + \log\log d + o(1) \rfloor \leq v_d \leq \lfloor \log d + \log\log d + 1.73 + o(1) \rfloor.$$

Theorem

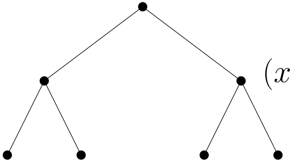
(Gebauer-Sz.-Tardos) The chairman wins the European Tenure Game with d faculty and k rungs if and only if there exists a (k, d)-tree. In particular,

$$v_d = \lfloor \log d + \log\log d + \log e - 1 + o(1) \rfloor.$$

The construction

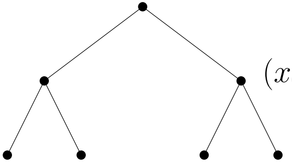
How to construct (k,d)-trees?

Leaf-vectors



$(x_0, x_1, \ldots, x_i, \ldots, x_k)$

Leaf-vectors



$$(x_0, x_1, \ldots, x_i, \ldots, x_k)$$

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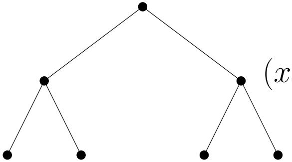

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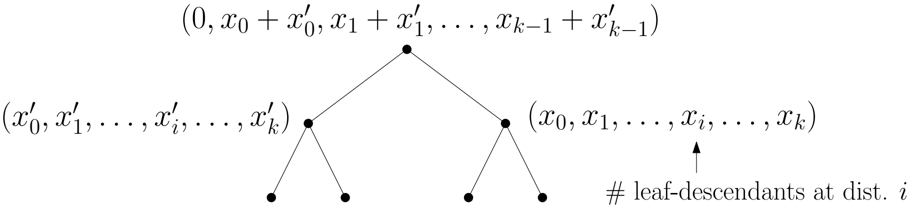
$\sum x_i \leq d$

Leaf-vectors

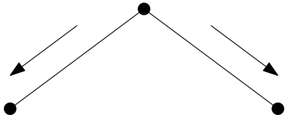


Parents and children

Building the tree from top to bottom by "distributing the debt"

The Fair — SPLIT

$$(x_0, x_1, \ldots, x_k)$$



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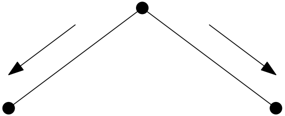

$$\left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_k}{2}, \frac{d}{2}\right) \qquad \left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_k}{2}, \frac{d}{2}\right)$$

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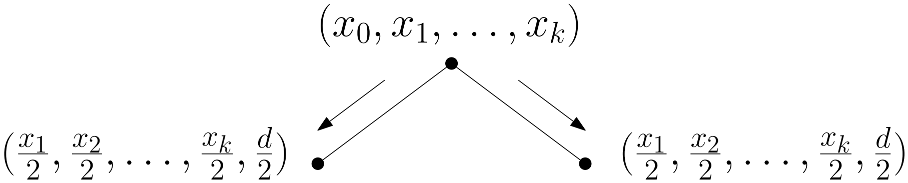
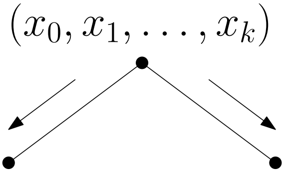


The Unfair — CUT



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The Unfair — CUT

$$(x_0, x_1, \dots, x_k)$$

$$(x_1, \dots, x_r, 0, \dots, 0) \qquad (0, \dots, 0, x_{r+1}, \dots, x_k, 0)$$

(k,d)-constructible leaf-vectors

For the leaf-vector $\vec{\ell_w} = (x_0, x_1, \ldots, x_k)$ of any vertex w of a (k,d)-tree we have $|\vec{\ell_w}| := \sum x_i \leq d$

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Def. For a vector $\vec{x}$ with $|\vec{x}| \leq d$ we say that a tree T with root r is a $(k,d,\vec{x})$-tree if

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Observation

There is a (k, d)-tree $\Leftrightarrow$ $(0, 0, \ldots, 0)$ *is (k, d)-constructible*

Payoff Lemma

Some vectors that are $(k, 2^i)$-constructible:

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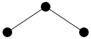

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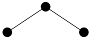
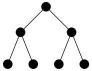


$(0, 0, 4, 0, \ldots, 0)$



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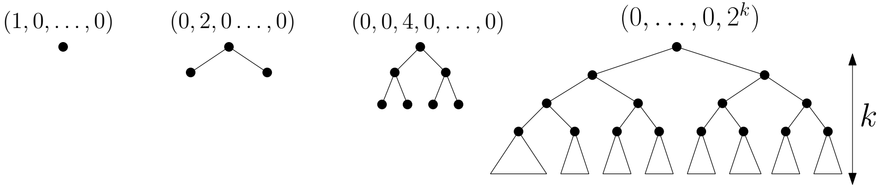

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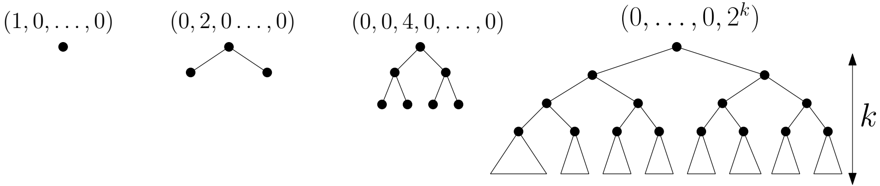

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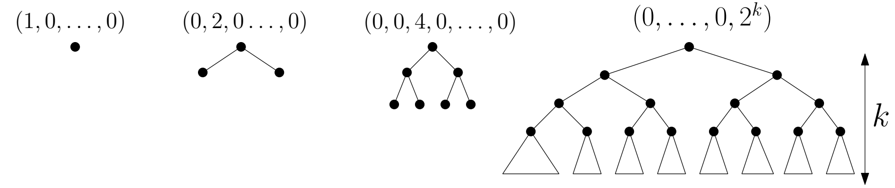

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Lemma (Payoff Lemma)

Let $|\vec{x}| \leq d$. If $w(\vec{x}) \geq 1$, then $\vec{x}$ is (k, d)-constructible.

Inverse of Kraft's Inequality

The Gebauer-trees

For simplicity assume that $d = 2^{s+1}$ is a power of 2.
Then $\left(0, \ldots, 0, 1, 2, 4, \ldots, \frac{d}{4}, \frac{d}{2}\right)$ is (k, d)-constructible if

$$\sum_{i=1}^{s+1} \frac{d/2^i}{2^{k+1-i}} = (s+1)\frac{d}{2^{k+1}} \geq 1$$

That is, when $d \log_2 d \geq 2^{k+1}$. Holds for $d \approx \frac{2^{k+1}}{k}$.

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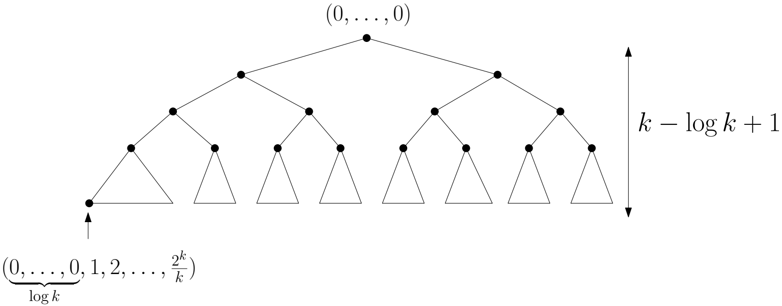

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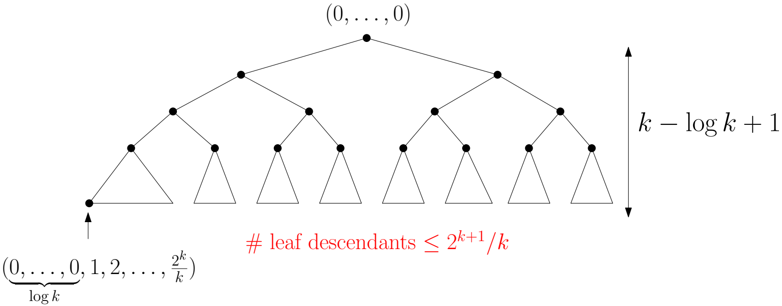

First idea: Cut-and-Split, Left Child pays off

How else can we prove constructibility of vector $(x_0, x_1, \ldots, x_r, x_{r+1}, \ldots, x_k)$?

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Repeat this Operation "Cut-and-Split then Left Child pays off"

Question Will the sequence of Right Child vectors ever converge to one that can pay off?
How to analyse?

Normalized analytic setting

Set $d = \frac{2}{T} \cdot \frac{2^k}{k}$. Eventually we want to get to $T = e - \epsilon$.

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leaf-vector

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Payoff: $w(\vec{x}) \geq 1$

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<u>leaf-vector</u> <u>normalized leaf-vector</u>

$$\vec{x} = \left(0, \ldots, 0, 1, \ldots, \tfrac{d}{4}, \tfrac{d}{2}\right) \rightsquigarrow \vec{y} = (0, \ldots, 0, 1, \ldots, 1, 1)$$

$$\rightsquigarrow y_i = x_i \frac{2^{k+1-i}}{d}$$

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<u>leaf-vector</u>		<u>normalized leaf-vector</u>		<u>leaf-function</u>
$\vec{x} = (0, \ldots, 0, 1, \ldots, \frac{d}{4}, \frac{d}{2})$	$\rightsquigarrow$	$\vec{y} = (0, \ldots, 0, 1, \ldots, 1, 1)$	$\rightsquigarrow$	$f \equiv 1$
	$\rightsquigarrow$	$y_i = x_i \frac{2^{k+1-i}}{d}$		$f : [0,1] \to \mathbb{R}$
Payoff: $w(\vec{x}) \geq 1$				$\int_0^1 f(x)dx \geq T$

For the leaf-function ignore $o(k)$ long segments of the normalized leaf-vector.
(Like the $\Theta(\log k)$ long segment of 0 at the beginning.)

Analytic Cut-and-Split

Let $v \in (0, 1)$.
Operation `Cut-at-`v`-and-Split`
Input function $f : [0, 1] \to \mathbb{R}$

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Left Child

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Right Child

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$$2\int_0^v f \geq T$$

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should be "better" than parent

does **not** mean "greater integral"

Where to cut?

We perform a series of Cut-and-Splits, cutting at
$1-\delta, 1-2\delta, \dots, 1-N\delta$
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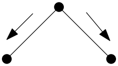
$T = 2-\epsilon$ is the limit of the simple Cut-and-Split.

How to get to $T = e - \epsilon$? — r-deep cuts

$$v_1 := (x_0, x_1, \ldots, x_k)$$



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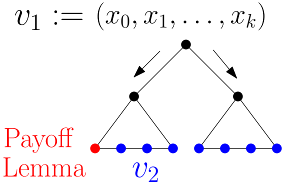

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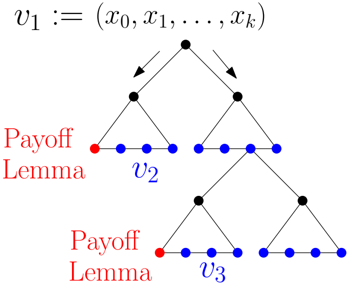

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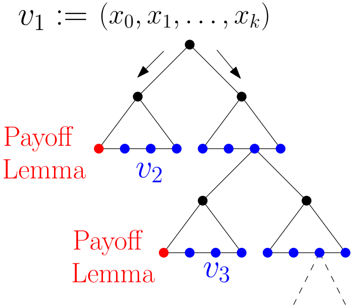

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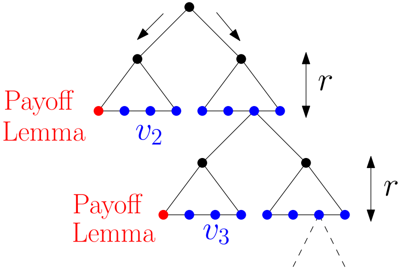

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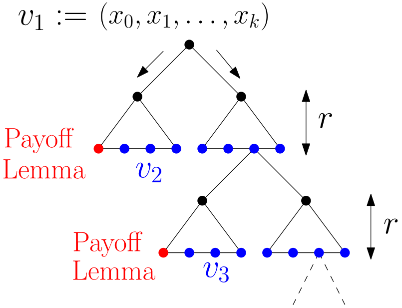

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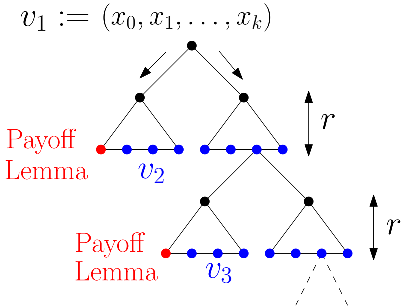

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Look at Right Child $f_{right}(x)$ after "time" $t \quad \rightsquigarrow \quad F(t, x)$
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THEN:

$$F_{right}(t,x) \approx F(t,x) \cdot \frac{2^r}{2^r - 1} \approx F(t,x)\left(1 + \frac{\delta F(t,0)}{T}\right)$$

That is $F(t+\delta, x-\delta) \approx F(t,x)\left(1 + \frac{\delta F(t,0)}{T}\right)$

A differential equation

From the previous page: $F(t+\delta, x-\delta) \approx F(t,x)\left(1+\frac{\delta F(t,0)}{T}\right)$
For some time s, introduce $F_s(t) := F(t, s-t)$.
Rewritten, for any $s-1 \leq t \leq s$:

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\ln F(s,0) &\gtrapprox \frac{F(s-1,0)}{T}
\end{aligned}$$

If $F(s,0)$ converged to a finite limit a, we would have $T \geq \frac{a}{\ln a} \geq e$.

So $\int_0^1 F(s,x)dx \approx F(s,0) \to \infty$ and the right child pays off. $\square$

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Open problems — The Neighborhood Conjecture

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Neighborhood Conjecture There is an $\epsilon > 0$ such that $D(k) > (1+\epsilon)^k$.
More modest goal: $D(k) > 0.51k$
It is still possible that some $\epsilon = \epsilon(k) \to 1$ could be chosen.