Matching transversals and m-connectedness

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(joint work with Gábor Tardos)

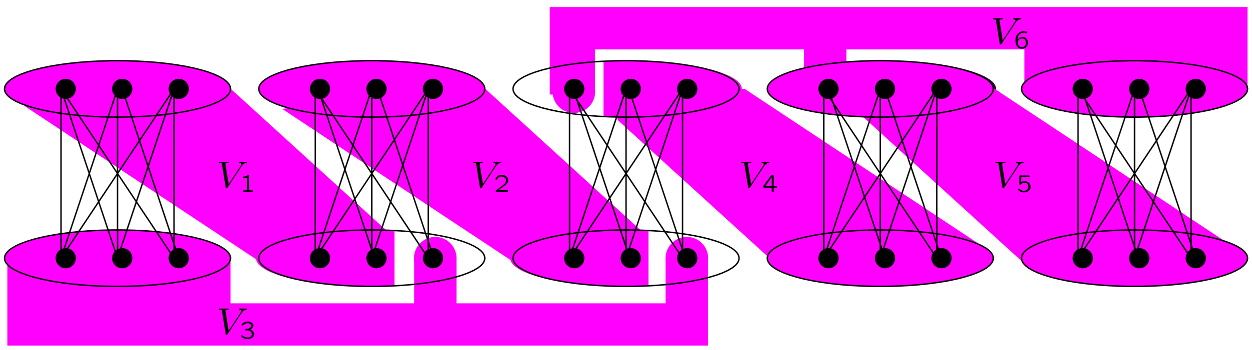
Independent transversals

$T \subseteq V(G)$ is called an independent transversal of the partitioned graph G, $V(G) = V_1 \cup \cdots \cup V_n$, if

- T spans no edge
- $|T \cap V_i| = 1$ for every $i = 1, \ldots n$.

Theorem (Haxell, 1995) Let G be a graph such that $\Delta(G) \leq d$.
Then for every partition $V(G) = V_1 \cup \cdots \cup V_n$ of G with $|V_i| \geq 2d$ there is an independent transversal.

Best possible for every d.
Example: $d = 3$, $|V_i| = 2 \cdot 3 - 1$



Appearences of independent transversals

- The linear arboricity conjecture (Akiyama, Exoo, Harary) is asymptotically correct (Alon)

- Strong chromatic number (Alon)

- List coloring (Reed's Conjecture, Haxell)

- Vertex partitions with bounded components (Alon, Ding, Oporowski and Vertigan)

Independent set complex

Aharoni and Haxell
Aharoni, Chudnovsky and Kotlov
Meshulam

$\mathcal{K}_1(G)$ is the simplicial complex on $V(G)$ which consists of simplices corresponding to independent sets of G.

Corollary If $|V(G)| > 2m\Delta(G)$, then $\mathcal{K}_1(G)$ is $(m-1)$-connected.

Best possible: $G = mK_{d,d}$

$\Delta(G) = d$

$|V(G)| = 2md$

$\mathcal{K}_1(G) \simeq (\mathcal{K}_1(K_{d,d}))^{*m} \simeq (S^0)^{*m} \simeq S^{m-1}$

Matching transversals

$T \subseteq V(G)$ is called an matching transversal of the partitioned graph G, $V(G) = V_1 \cup \cdots \cup V_n$, if

- T induces a matching of G
- $|T \cap V_i| = 1$ for every $i = 1, \ldots n$.

Theorem (Haxell, Sz., Tardos, 2003) Let G be a graph such that $\Delta(G) \leq d$. For every partition $V(G) = V_1 \cup \cdots \cup V_n$ of G with $|V_i| \geq \left\lfloor \frac{3}{2} d \right\rfloor$ there is a matching transversal.

Theorem is not best possible.

For $d = 2$, parts of size 2 are enough.

For $d \geq 3$, the optimal part size is not known. It is known to be between d and $\left\lfloor \frac{3}{2} d \right\rfloor$.

Appearences of matching transversals

- vertex partitions with bounded components
  (Haxell-Sz.-Tardos)

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Induced matching complex

$\mathcal{K}_2(G)$ is the simplicial complex on $V(G)$ which consists of simplices corresponding to vertex sets of G which induce a matching of G.

Theorem (Sz.- Tardos) If $|V(G)| > \frac{3}{2}m\Delta(G) - \frac{1}{2}m$, then $\mathcal{K}_2(G)$ is $(m-1)$-connected.

The constant multiplier $\frac{3}{2}$ is not known to be optimal.

Example:

$G = kH_{d/2}$, where $H_{d/2}$ is the blown up C_5.

$\Delta(G) = d$, $|V(G)| = \frac{5}{4}md$, where $m = 2k$

$$\begin{aligned} \mathcal{K}_2(G) &\simeq (\mathcal{K}_2(H_{d/2}))^{*k} \simeq (\mathcal{K}_1(H_{d/2}))^{*k} \\ &\simeq (S^1)^{*k} \simeq S^{2k-1} \end{aligned}$$

Notation

Given a graph G, a simplicial complex $\mathcal{K}$, a mapping $l : V(\mathcal{K}) \to V(G)$,
the pair $(\mathcal{K}, l)$ is called a *G-labeled simplicial complex*.

A 1-dimensional simplex $\{x, y\}$ of $\mathcal{K}$ is called ruined if $l(x)$ and $l(y)$ are adjacent in G.

An r-dimensional simplex S of $\mathcal{K}$ is called ruined if the graph of ruined 1-dimensional faces of S is connected and spans all $r + 1$ vertices of S.

The link of a simplex σ in a simplicial complex is
$lk_{\mathcal{K}}(\sigma) := \{\tau \in \mathcal{K} : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \mathcal{K}\}$.

The join of two simplicial complexes with disjoint vertex sets is
$\mathcal{K} * \mathcal{K}' := \{\tau \cup \tau' : \tau \in \mathcal{K}, \tau' \in \mathcal{K}'\}$.

The filling theorem

Theorem (Sz.-Tardos) Let $m \geq 0$ and $r \geq 1$ be arbitrary integers. Let G be a graph of maximum degree $d \geq r-1$ and designated subset $W \subseteq V(G)$ of size $|W| > m(d-1+(d+1)/r)$. Then a G-labeled triangulation $(\mathcal{T}, l)$ of S^{m-1} admits a G-labeled filling $(\mathcal{T}', l')$ satisfying the following properties:

(a) $l'(v) \in W$ for every $v \in V(\mathcal{T}') \setminus V(\mathcal{T})$.

(b) There is no ruined edge between $V(\mathcal{T})$ and $V(\mathcal{T}') \setminus V(\mathcal{T})$.

(c) The ruined r-simplices of $(\mathcal{T}', l')$ are contained in $\mathcal{T}$.

How to fill a sphere?

Induction on m.

Three phases:

1. Creating an inner crust where no ruined edges go to the boundary using induction hypothesis for $(m-1)$.

2. Filling up the inside more or less arbirarily.

3. One by one, delete the ruined 2-simplices and fill up the remaining hole using induction for $(m-2)$.

Sperner's Lemma

Sperner's Lemma. Let $\mathcal{T}$ be a triangulation of the n-dimensional simplex σ. Suppose that the vertices of $\mathcal{T}$ are colored with $n + 1$ colors, such that

- The vertices of σ receive distinct colors
- The vertices of $\mathcal{T}$ on any face τ of σ are colored with the colors of the vertices of τ.

Then there exists a multicolored n-dimensional simplex in $\mathcal{T}$.

Finding a matching transversal

Proof of HST Thm. Consider the $(n-1)$-dimensional simplex σ with vertex set $\{v_1, \ldots, v_n\}$. We construct a labeled triangulation $(\mathcal{T}, l)$ of σ, such that

- for every vertex x of $\mathcal{T}$ and face τ of σ, $l(x) \in \cup_{i:v_i \in \tau} V_i$
- there are no ruined 2-simplices.

Such a labeled triangulation is possible to construct by the previous theorem if $|V_i| \geq \left\lfloor \frac{3}{2} \Delta(G) \right\rfloor$.

Apply Sperner's Lemma to this triangulation, where the color of a vertex is defined to be the index of the part where its label lies. The labels of a multicolored simplex of this triangulation provide a matching transversal. $\square$

Bonus from topology

Theorem (Sz.-Tardos) Let r be an arbitrary positive integer. Let G be a graph of maximum degree d, together with a partition $V_1 \cup \ldots \cup V_m = V(G)$ such that $|V_i| \geq d + \lfloor d/r \rfloor$ for $i = 1, \ldots, m$. Then there exists a transversal T such that the connected components of the induced subgraph $G[T]$ are trees on at most r vertices.

Open problems about transversals

Let $p(d,r)$ be the smallest integer such that any graph with maximum degree at most d, partitioned into classes of size at least $p(d,r)$, has a transversal that induces components of order at most r.

$$
\begin{aligned}
p(d,1) &= 2d \\
p(2,2) &= 2 \\
d \leq p(d,r) &\leq d + \left\lfloor \frac{d}{r} \right\rfloor
\end{aligned}
$$

Questions.

- What is $p(3,2)$? Is it 3 or 4???
  I.e. can a matching transversal of a cubic graph be selected whenever an adversary choses a partition into parts of size three?

- Give any decent lower bound for $p(d,2)$.
  Or could it be that $p(d,2) = d$ for every d???

Open problems about connectedness

$\mathcal{K}_r(G)$ is the simplicial complex on $V(G)$ which consists of simplices T, such that $G[T]$ has components of order at most r.

Theorem (Sz.-Tardos) Let $m \geq 0$, $r \geq 1$, and $\Delta(G) = d \geq r-1$.
If $|V(G)| > \left(1 + \frac{1}{r} - \frac{1}{d} + \frac{1}{rd}\right) md$, then $\mathcal{K}_r(G)$ is $(m-1)$-connected.

Problem. Let r be fixed. What is the smallest constant c_r, such that more than $c_r m \Delta(G)$ vertices guarantee the $(m-1)$-connectedness of $\mathcal{K}_r(G)$, provided $\Delta(G) \geq r-1$.

$$\begin{aligned} c_1 &= 2 \\ \frac{5}{4} \leq c_2 &\leq \frac{3}{2} \\ c_r &\leq 1 + \frac{1}{r} \end{aligned}$$