Finite Volume, Conservative Projection-Type Methods for Low Speed Compressible Flows

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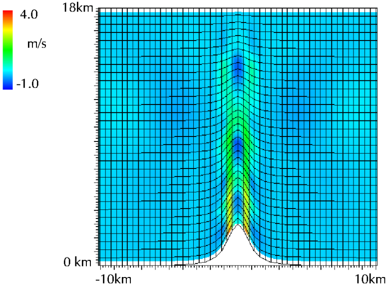
Outline

Accumulation of Unbalanced Truncation Errors

Spurious winds over steep orography:

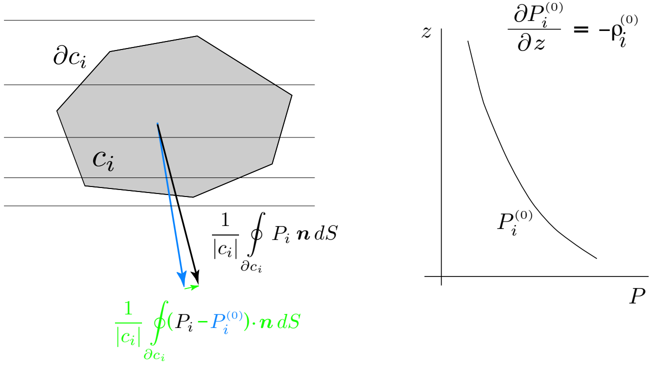
- atmosphere at rest
- 3000 m mountain
- 3D compressible inviscid flow eqns.
- standard finite volume scheme
- 128×32 grid cells
- velocities after 60 min.



Various finite difference / finite volume schemes produce comparable results.

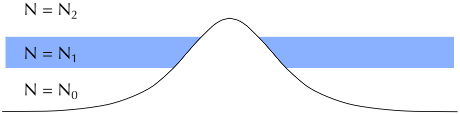
Archimedes' Principle

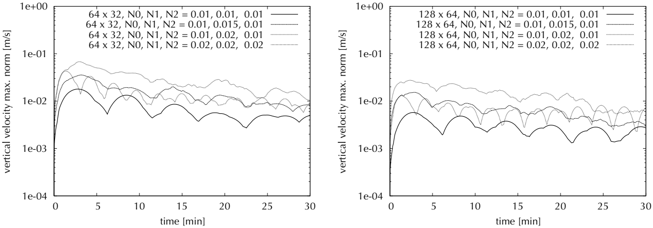
- In Botta et al. [2004] a general applicable solution for this problem has been proposed
- Implementation in the context of FV methods:



Stable Layer Intersecting Steep Orography

Inversion



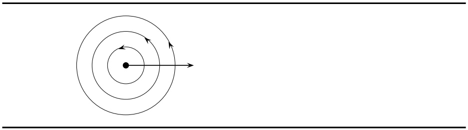


maximum norm of vertical velocity

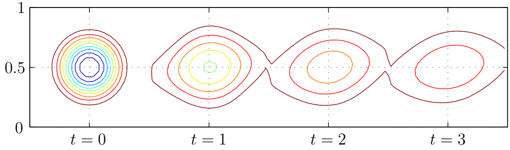
[Botta et al., 2004]

Computation of Nearly Incompressible Flow

Stationary vortex is advected by constant background flow [Gresho and Chan, 1990]:



- rectangular domain with 80×20 grid cells
- periodic BC on left and right side, walls at top / bottom
- explicit Godunov-type method for compressible flows:

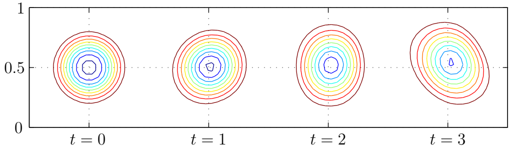


$\mathrm{M} = 0.01$

Construction of Conservative Numerical Methods

The following procedure for the construction of numerical fluxes was proposed [Schneider et al., 1999]:

- compute predictions for convective flux components with a standard method for hyperbolic conservation laws
- correct predictions by two projection steps to guarantee divergence control at new time level



$M = 0$

- problem: second projection step admits a local decoupling of the solution (checkerboading)

Conservative Projection-Type Methods

S. Vater

FV Methods for Geophysical Problems
Well Balanced Finite Volume Methods
Conservative Projection-Type Methods

A New Projection Method
Governing Equations
Formulation of the Scheme
Stability of the Second Projection
Numerical Results

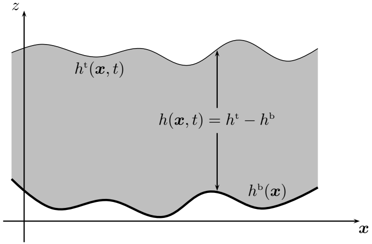
Summary

Outline

The Shallow Water Equations

Non-dimensional form:

$$
\begin{aligned}
h_t \quad + \quad \nabla \cdot (h\boldsymbol{v}) &= 0 \\
(h\boldsymbol{v})_t \quad + \quad \nabla \cdot \left(h\boldsymbol{v} \circ \boldsymbol{v} + \frac{1}{2\,\mathsf{Fr}^2}\, h^2\, \boldsymbol{I} \right) &= \frac{1}{\mathsf{Fr}^2}\, h \nabla h^{\mathrm{b}}
\end{aligned}
$$



- hyperbolic system of conservation laws
- similar to Euler equations, no energy equation

The "Incompressible" Limit
(as Fr $\longrightarrow 0$)

- zero Froude number shallow water equations:

$$\begin{aligned} h_t &+ \nabla \cdot (h\boldsymbol{v}) & & = 0 \\ (h\boldsymbol{v})_t &+ \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v}) &+ h\nabla h^{(2)} & = \mathbf{0} \end{aligned}$$

  $h = h_0(t)$ given through boundary conditions.

- mass conservation becomes a divergence constraint for the velocity field:

$$\int_{\partial V} (h\boldsymbol{v}) \cdot \boldsymbol{n} \, d\sigma = -|V| \frac{dh_0}{dt}$$

Conservation Form

Consider a FV method in conservation form:

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_{\partial V}} |I| \, \mathbf{F}_I$$

$$\mathbf{F}_I(\mathbf{U}_I, \boldsymbol{n}_I) := \begin{pmatrix} h(\boldsymbol{v} \cdot \boldsymbol{n}) \\ h\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n}) & + & h_0 \, h^{(2)} \, \boldsymbol{n} \end{pmatrix}_I$$

Construction of numerical fluxes:

- advective fluxes from standard explicit FV scheme (applied to an auxiliary system)
- (MAC)-projection corrects advection velocity divergence
- second (exact) projection adjusts new time level divergence of cell-centered velocities
- second order accuracy

Auxiliary System

The auxiliary system

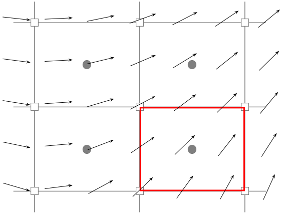
$$\begin{aligned} h_t^* &+ \nabla \cdot (h\boldsymbol{v})^* &= 0 \\ (h\boldsymbol{v})_t^* &+ \nabla \cdot \left((h\boldsymbol{v} \circ \boldsymbol{v})^* + \tfrac{1}{2}(h^*)^2 \boldsymbol{I}\right) &= \boldsymbol{0} \end{aligned}$$

enjoys the following properties:

- It has the same convective fluxes as the zero Froude number shallow water equations.
- The system is hyperbolic.
- Having constant height h^* and a zero velocity divergence at time t_0, solutions satisfy at $t_0 + \delta t$:

$$\nabla \cdot \boldsymbol{v}^* = \mathcal{O}(\delta t) \ , \quad (h^* \nabla h^*) = \mathcal{O}\left(\delta t^2\right)$$

Correction of the Fluxes

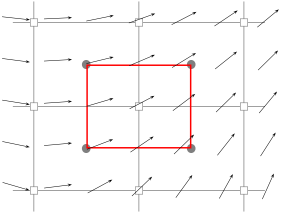


1. Projection:

- divergence constraint imposed on each grid cell
- correct convective fluxes on boundary of volume

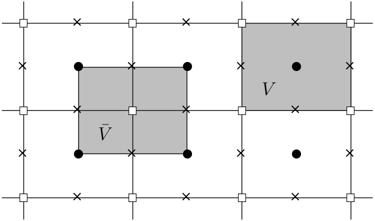
2. Projection:

- divergence constraint imposed on dual discretization
- correct momentum to obtain correct divergence for new velocity field



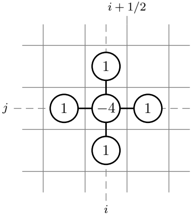
Original Scheme

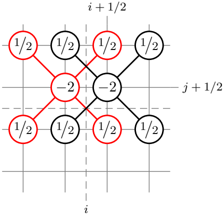
[Schneider et al., 1999]



- both Poisson-type problems are solved for cell averages (i.e. piecewise constant data)

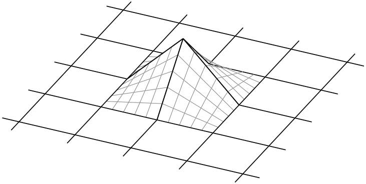
- stencils: standard FD discretizations
- 2nd Poisson-type problem has local decoupling





The New (Second) Projection
Discretization of the Poisson-Type Problem

Let us consider a Petrov-Galerkin finite element discretization [Süli, 1991]:



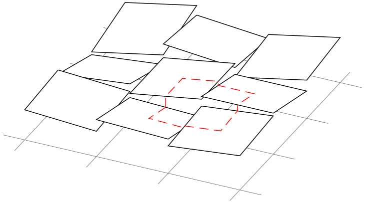
- bilinear trial functions for the unknown $h^{(2)}$
- piecewise constant test functions on the dual discretization

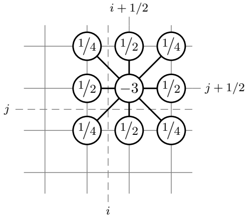
Integration over $\mathbf{\Omega}$ and using the divergence theorem leads to ($h_0 = const.$):

$$\delta t\, h_0 \int\limits_{\partial \bar{V}} \nabla h^{(2)} \cdot \boldsymbol{n}\ d\sigma = \int\limits_{\partial \bar{V}} \left((h\boldsymbol{v})^{**} + (h\boldsymbol{v})^n\right) \cdot \boldsymbol{n}\ d\sigma$$

The New (Second) Projection
Discrete Velocity Space

- velocity components at boundary of the dual cells are piecewise linear!
- discrete divergence can be exactly calculated





- discrete divergence, Laplacian and gradient satisfy $L = D(G)$
- discrete Laplacian has compact stencil

Approximate and Exact Projection

- new discrete divergence also affected by partial derivatives u_y and v_x
- using just the mean values to correct momentum:

$$(h\boldsymbol{v})_V^{n+1} = (h\boldsymbol{v})_V^{**} - \delta t\, h_0\, \overline{\mathsf{G}(h^{(2)})}$$

  we obtain $\mathsf{D}(\boldsymbol{v}^{n+1}) = \mathcal{O}\big(\delta t\, \delta x^2\big)$: approximate projection method
- additional correction of derivatives and their employment in the reconstruction of the predictor step: exact projection method

Generalized Saddle-Point Problems

- Find $(u, p) \in (\mathcal{X}_2 \times \mathcal{M}_1)$, such that

$$\begin{cases} a(u, v) + b_1(v, p) & = & \langle f, v \rangle \quad \forall\, v \in \mathcal{X}_1 \\ b_2(u, q) & = & \langle g, q \rangle \quad \forall\, q \in \mathcal{M}_2 \end{cases} \tag{1}$$

- abstract theory by Nicolaïdes [1982] and Bernardi et al. [1988]: If $b_i(\cdot, \cdot)$ (and similarly $a(\cdot, \cdot)$) satisfies:

$$\inf_{q \in \mathcal{M}_i} \sup_{v \in \mathcal{X}_i} \frac{b_i(v, q)}{\|v\|_{\mathcal{X}_i} \, \|q\|_{\mathcal{M}_i}} \geq \beta_i > 0$$

Then, (1) has a unique solution for all f and g.

Reformulation of the Poisson-Type Problem

- derive a saddle point formulation using momentum update and divergence constraint:

$$
\begin{aligned}
(h\boldsymbol{v})^{n+1} &= (h\boldsymbol{v})^{**} - \delta t\left(h_0 \nabla h^{(2)}\right) \\
\frac{1}{2}\nabla\cdot\left[(h\boldsymbol{v})^{n+1} + (h\boldsymbol{v})^{n}\right] &= -\frac{dh_0}{dt}
\end{aligned}
$$

- variational formulation: multiply with test functions φ and ψ and integrate over Ω
- discrete problem obtained by using piecewise linear vector and piecewise constant scalar test functions; equivalent to Poisson-type equation

Existence & Uniqueness

Continuous Problem

- find solution with $(h\boldsymbol{v})^{n+1} \in H_0(\text{div};\Omega)$ and $h^{(2)} \in H^1(\Omega)/\mathbb{R}$
- test functions in the spaces $(L^2(\Omega))^2$ and $L^2(\Omega)$
- bilinear forms given by:

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}\ ,\quad b_1(\boldsymbol{v},q) := \delta t\, h_0 \int_\Omega \boldsymbol{v}\cdot\nabla q\,d\boldsymbol{x}$$

$$b_2(\boldsymbol{v},q) := \int_\Omega q\,(\nabla\cdot\boldsymbol{v})\,d\boldsymbol{x}$$

Theorem: [Vater, 2005] The continuous generalized saddle point problem has a unique solution $((h\boldsymbol{v})^{n+1}, h^{(2)})$.

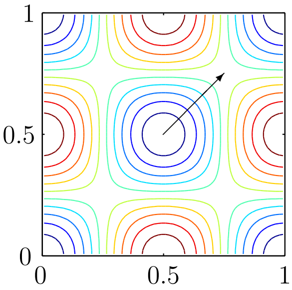
Stability of the Discrete Problem?

- $b_1(\cdot,\cdot)$ satisfies a discrete inf-sup condition
- open question for $a(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$
- problem: piecewise linear vector functions not in $H(\mathrm{div};\Omega)$ in general (nonconforming finite elements)
- common (e.g. Raviart-Thomas) elements do not match with the piecewise linear, discontinuous ansatz functions from the Godunov-Type method
- discretization by Schneider et al. [1999] can also be formulated as saddle point problem; but unstable!

Convergence Studies
Taylor Vortex

Originally proposed by MINION [1996] and ALMGREN ET AL. [1998] for the incompressible flow equations

- smooth velocity field
- nontrivial solution for $h^{(2)}$
- solved on unit square with periodic BC
- 32×32, 64×64 and 128×128 grid cells
- error to exact solution at $t = 3$



Convergence Studies

Errors and Convergence Rates

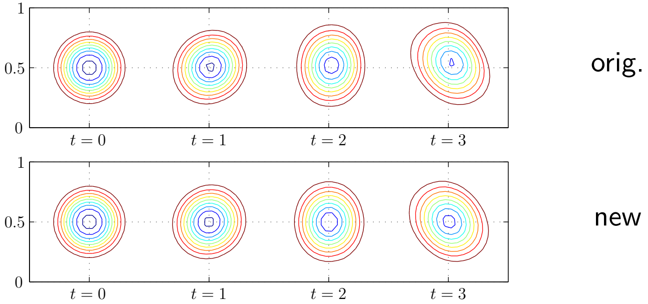
Method	Norm	32x32	Rate	64x64	Rate	128x128
original projection	L^2	0.2929	2.16	0.0656	2.16	0.0146
	L^∞	0.4207	2.15	0.0945	2.18	0.0209
new exact projection	L^2	0.0816	2.64	0.0131	2.17	0.0029
	L^∞	0.1277	2.45	0.0234	2.32	0.0047

- second order accuracy is obtained in the L^2 and the L^∞ norms
- absolute error obtained with the new exact projection about four times smaller on fixed grids

Advection of a Vortex

Results for the New Projection Method

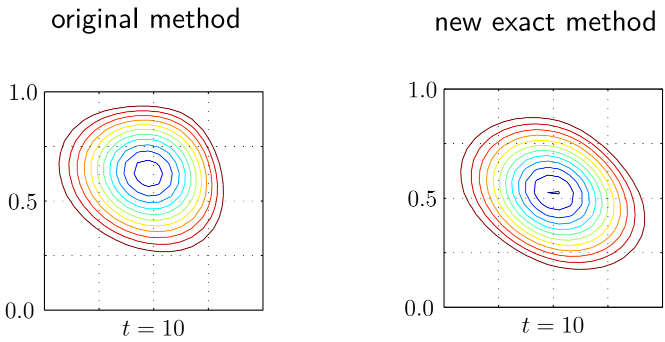
Exact projection, central differences (no limiter):



Less deviation from the center line of the channel, loss in vorticity is slightly reduced.

Advection of a Vortex

Results for the New Projection Method



Summary

A new projection method has been presented. It has the following properties:

- the projection is based on a FE formulation
- numerical results of the new method show considerable accuracy improvements on fixed grids compared to the old formulation
- results supported by theoretical analysis; no local decoupling of the gradient in the 2nd projection

- Outlook
  - stability of the discrete method has to be solved
  - discrete divergence is determined by mean values and u_y and v_x; other partial derivatives give additional degrees of freedom

For Further Reading

Th. Schneider, N. Botta, K.J. Geratz and R. Klein.
Extension of Finite Volume Compressible Flow Solvers to Multi-dimensional, Variable Density Zero Mach Number Flows.
Journal of Computational Physics, 155 : 248–286, 1999.

S. Vater.
A New Projection Method for the Zero Froude Number Shallow Water Equations.
PIK Report No. 97, Potsdam Institute for Climate Impact Research, 2005.

N. Botta, R. Klein, S. Langenberg and S. Lützenkirchen.
Well Balanced Finite Volume Methods for Nearly Hydrostatic Flows.
Journal of Computational Physics, 196 : 539–565, 2004.

Convective Fluxes

Consider the semi-discrete equations:

$$
\begin{aligned}
h^{n+1} &= h^n - \delta t \left[\nabla \cdot (h\boldsymbol{v})^{n+1/2}\right] + \mathcal{O}(\delta t^3) \\
(h\boldsymbol{v})^{n+1} &= (h\boldsymbol{v})^n - \delta t \left[\nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v})^{n+1/2} + \right. \\
&\qquad \left. (h_0 \nabla h^{(2)})^{n+1/2})\right] + \mathcal{O}(\delta t^3)
\end{aligned}
$$

The momentum is given by

$$
(h\boldsymbol{v})^{n+1/2} = (h\boldsymbol{v})^{*,n+1/2} - \frac{\delta t}{2} (h_0 \nabla h^{(2)})^{n+1/4} + \mathcal{O}(\delta t^3)
$$

Impose divergence constraint at $t^{n+1/2}$:

$$
\frac{\delta t}{2} \nabla \cdot (h_0 \nabla h^{(2)})^{n+1/4} = \nabla \cdot (h\boldsymbol{v})^{*,n+1/2} + \frac{dh_0}{dt} + \mathcal{O}(\delta t^3)
$$

Interface Heights

Using $h^{(2)}$ from first Poisson-type problem: instabilities, divergence constraint not satisfied at new time step.

Intermediate momentum update:

$$(h\boldsymbol{v})^{**} := (h\boldsymbol{v})^{n} - \delta t \; \nabla \cdot (h\boldsymbol{v} \circ \boldsymbol{v})^{n+1/2}$$

Momentum at time t^{n+1} can be expressed as:

$$(h\boldsymbol{v})^{n+1} = (h\boldsymbol{v})^{**} - \delta t \, (h_0 \nabla h^{(2)})^{n+1/2} + \mathcal{O}\left(\delta t^3\right)$$

A second application of the divergence constraint yields:

$$\delta t \, \nabla \cdot (h_0 \nabla h^{(2)})^{n+1/2} = \nabla \cdot (h\boldsymbol{v})^{**} + \nabla \cdot (h\boldsymbol{v})^{n} + 2 \, \frac{dh_0}{dt} + \mathcal{O}\left(\delta t^2\right)$$

Find $((h\boldsymbol{v})^{n+1}, h^{(2)}) \in (H_0(\mathrm{div};\Omega) \times H^1(\Omega)/\mathbb{R})$, s.th.

$$\begin{cases} a\big((h\boldsymbol{v})^{n+1}, \boldsymbol{\varphi}\big) + b_1\big(\boldsymbol{\varphi}, h^{(2)}\big) & = & \langle (h\boldsymbol{v})^{**}, \boldsymbol{\varphi}\rangle \\ b_2\big((h\boldsymbol{v})^{n+1}, \psi\big) & = & \langle -\nabla\cdot(h\boldsymbol{v})^{n}, \psi\rangle \end{cases}$$

$\forall\, \boldsymbol{\varphi} \in (L^2(\Omega))^2$ and $\forall\, \psi \in L^2(\Omega)$

Bilinear forms given by:

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\; d\boldsymbol{x}\;,\;\; b_1(\boldsymbol{v}, q) := \delta t\, h_0 \int_\Omega \boldsymbol{v}\cdot\nabla q\; d\boldsymbol{x}\;,$$

$$b_2(\boldsymbol{v}, q) := \int_\Omega q\,(\nabla\cdot\boldsymbol{v})\; d\boldsymbol{x}$$

Theorem: The generalized saddle point problem has a unique solution $((h\boldsymbol{v})^{n+1}, h^{(2)})$.

Convergence Studies

Taylor Vortex

- Originally proposed by MINION [1996] and ALMGREN ET AL. [1998] for the incompressible flow equations.
- Initial conditions: Constant height h_0 and

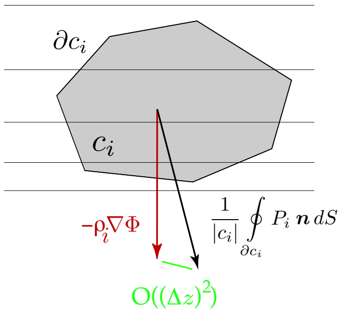
$$\begin{aligned} u_0(x,y) &= 1 - 2\cos(2\pi x)\sin(2\pi y) \\ v_0(x,y) &= 1 + 2\sin(2\pi x)\cos(2\pi y) \end{aligned}$$

  for $(x,y) \in [0,1]^2$, periodic boundary conditions.
- Exact solution of the zero Froude number SWE:

$$\begin{aligned} u(x,y,t) &= 1 - 2\cos(2\pi(x-t))\sin(2\pi(y-t)) \\ v(x,y,t) &= 1 + 2\sin(2\pi(x-t))\cos(2\pi(y-t)) \\ h^{(2)}(x,y,t) &= -\cos(4\pi(x-t)) - \cos(4\pi(y-t)) \end{aligned}$$

Archimedes' Principle

- In Botta et al. [2004] a general applicable solution for this problem has been proposed.
- Implementation in the context of FV methods:



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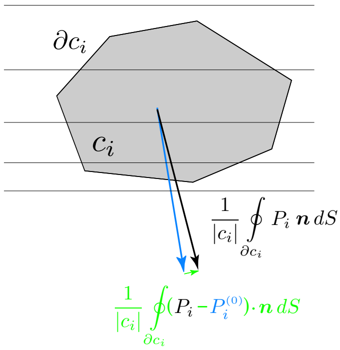
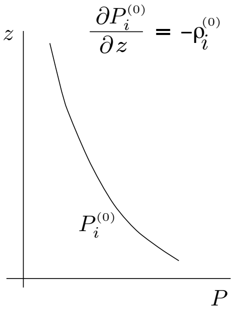
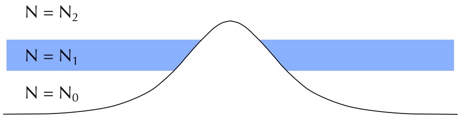
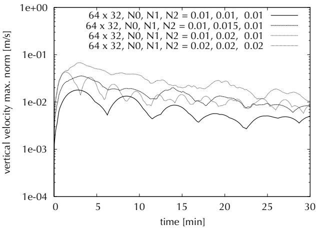


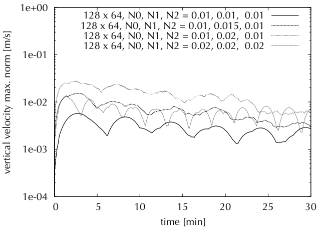


Well Balanced Finite Volume Methods

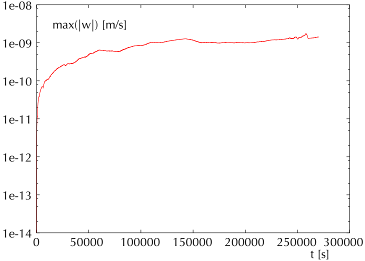
Inversion

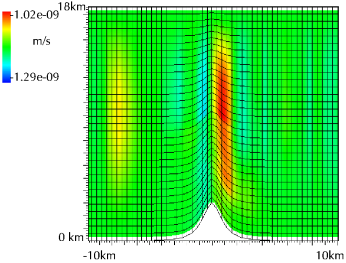






Piecewise Linear Potential Temperature





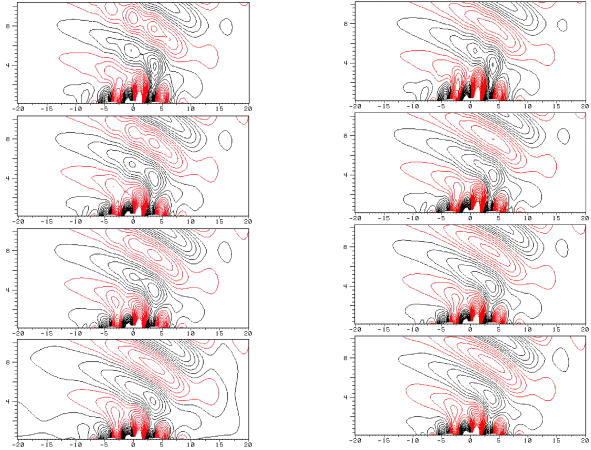
No accumulation of unbalanced truncation errors!

[Botta et al., 2004]

Well Balanced Finite Volume Methods

Schaer's Test

piecewise constant entropy | piecewise linear entropy



16

20

24

32

resolution
(pts / orogr.-wavelen.)