A Semi-Implicit Projection Method for the Zero Froude Number Shallow Water Equations

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Outline

1. Governing Equations
   - Shallow Water Equations
   - The “Incompressible” Limit

2. Formulation of the Scheme
   - Conservation Form
   - Discretization of Projection Step
   - Exact Projection Method

3. Stability of the Projection Step
   - Generalized Saddle-Point Problems
   - Discrete Inf-Sup Conditions

4. Numerical Results
The Shallow Water Equations

Non-dimensional form:

\[
\begin{align*}
    h_t + \nabla \cdot (h \mathbf{v}) &= 0 \\
    (h \mathbf{v})_t + \nabla \cdot \left( h \mathbf{v} \circ \mathbf{v} + \frac{1}{2 \text{Fr}^2} h^2 \mathbf{I} \right) &= \frac{1}{\text{Fr}^2} h \nabla h_B
\end{align*}
\]

- \text{Fr} = \frac{\nu'_\text{ref}}{\sqrt{g' h'_\text{ref}}}
- hyperbolic system of conservation laws
- similar to Euler equations, no energy equation
The “Incompressible” Limit
(as Fr → 0)

Zero Froude number shallow water equations:

\[
\begin{align*}
    h_t & + \nabla \cdot (hv) = 0 \\
    (hv)_t & + \nabla \cdot (hv \circ v) + h \nabla h^{(2)} = 0
\end{align*}
\]

- \( h = h_0(t) \) is given through boundary conditions.
- mass conservation becomes a divergence constraint for the velocity field:

\[
\int_{\partial V} h(v \cdot n) \, d\sigma = -|V| \frac{dh_0}{dt} \quad \text{for } V \subset \Omega
\]
Formulation of the Numerical Scheme

Consider a FV method in conservation form:

\[ U^{n+1}_V = U^n_V - \frac{\delta t}{|V|} \sum_{I \in \mathcal{I}_V} |I| F_I \]

\[ F_I(U_I, n_I) := \left( \begin{array}{c} h(v \cdot n) \\ h v (v \cdot n) + h_0 h^{(2)} n \end{array} \right)_I \]

Construction of numerical fluxes:

- advection fluxes from standard explicit FV scheme (applied to an auxiliary system)
- (MAC)-projection corrects advection velocity divergence
- second (exact) projection adjusts new time level divergence of cell-centered velocities
Correction of the Fluxes

1. Projection:

\[(h \nu)_I = (h \nu)^* - \frac{\delta t}{2} h_0 (\nabla h^{(2)})_I\]

corrects convective fluxes on boundary of control volume

2. Projection:

\[(h \nu)^{n+1} = (h \nu)^{**} - \delta t (h_0 \nabla h^{(2)})^{n+1/2}\]

adjusts momentum to obtain correct divergence for new velocity field
Consider a Petrov-Galerkin FE discretization [Süli, 1991]:

- bilinear trial functions for the unknown $h^{(2)}$
- piecewise constant test functions on the dual discretization

Integration over $\Omega$ and divergence theorem leads to:

$$\delta t h_0 \int_{\partial \tilde{V}} \nabla h^{(2)} \cdot n \, d\sigma = \int_{\partial \tilde{V}} \left[ (h \nu)^{**} + (h \nu)^n \right] \cdot n \, d\sigma$$
The (Second) Projection
Discrete Velocity Space

- velocity components at boundary of the dual cells are piecewise linear!
- discrete divergence can be exactly calculated

\[
\begin{align*}
\text{discrete divergence, Laplacian and gradient satisfy } & \quad L = D(G) \\
\text{discrete Laplacian has compact stencil}
\end{align*}
\]
Approximate vs. Exact Projection

- discrete divergence also affected by partial derivatives $u_y$ and $v_x$
- using just the mean values to correct momentum:

\[(h v)^{n+1} = (h v)^{**} - \delta t h_0 \overline{G(h^{(2)})}\]

we obtain $D(v^{n+1}) = O(\delta t \delta x^2)$; approximate projection method

- additional correction of derivatives and their employment in the reconstruction of the predictor step: exact projection method
Generalized Saddle-Point Problems
Nicolaïdes [1982] and Bernardi el al. [1988]

Find \((u, p) \in (\mathcal{X}_2 \times M_1)\), such that

\[
\begin{align*}
  a(u, v) + b_1(v, p) &= \langle f, v \rangle \quad \forall \ v \in \mathcal{X}_1 \\
  b_2(u, q) &= \langle g, q \rangle \quad \forall \ q \in M_2
\end{align*}
\]  

(1)

Theorem

If \(b_i(\cdot, \cdot) \ (i = 1, 2)\) and similarly \(a(\cdot, \cdot)\) satisfy:

\[
\inf_{q \in M_i} \sup_{v \in \mathcal{X}_i} \frac{b_i(v, q)}{\|v\|_{\mathcal{X}_i} \|q\|_{M_i}} \geq \beta_i > 0
\]

Then, (1) has a unique solution for all \(f\) and \(g\).

S. Vater & R. Klein (FU Berlin)  
Projection Method for SWE  
GAMM 2006  
10 / 20
Reformulation of the Poisson-Type Problem

Derive saddle point problem by employing momentum update and divergence constraint:

\[
(h\nu)^{n+1} = (h\nu)^{**} - \delta t (h_0 \nabla h^{(2)})
\]

\[
\frac{1}{2} \nabla \cdot [(h\nu)^{n+1} + (h\nu)^n] = -\frac{dh_0}{dt}
\]

- variational formulation: multiply with test functions \( \varphi \) and \( \psi \) and integrate over \( \Omega \)
- discrete problem with piecewise linear vector and piecewise constant scalar test functions
Existence & Uniqueness
Continuous Problem

- find solution with \((h\mathbf{v})^{n+1} \in H_0(\text{div}; \Omega)\) and \(h^{(2)} \in H^1(\Omega)/\mathbb{R}\)
- test functions in the spaces \([L^2(\Omega)]^2\) and \(L^2(\Omega)\)
- bilinear forms given by:

\[
\begin{align*}
   a(u, v) &:= (u, v)_0 \\
   b_1(v, q) &:= \delta t h_0 (v, \nabla q)_0 \\
   b_2(v, q) &:= (q, \nabla \cdot v)_0
\end{align*}
\]

**Theorem (V. 2005)**

*The continuous generalized saddle point problem has a unique solution* \(((h\mathbf{v})^{n+1}, h^{(2)})\).
Stability of the Discrete Problem?

- \( a(\cdot, \cdot) \) and \( b_1(\cdot, \cdot) \) satisfy discrete inf-sup conditions, open question for \( b_2(\cdot, \cdot) \)

- problem: piecewise linear vector functions not in \( H(\text{div}; \Omega) \) in general (nonconforming finite elements)

- conforming (e.g. Raviart-Thomas) elements do not match with the piecewise linear, discontinuous ansatz functions from the Godunov-Type method

- former version [Schneider et al. 1999] can also be formulated as saddle point problem; but unstable!
Discrete Inf-Sup Condition for $a(\cdot, \cdot)$

To show ("coercivity"):

\[
\inf_{u \in \mathcal{K}_2^h} \sup_{v \in \mathcal{K}_1^h} \frac{a(u, v)}{\|u\| \|v\|} \geq \alpha \quad \text{and} \quad \sup_{u \in \mathcal{K}_2^h} a(u, v) > 0 \quad \forall v \in \mathcal{K}_1^h \setminus \{0\}
\]

\[
v \in \mathcal{K}_1^h \Leftrightarrow 0 = \frac{1}{\delta x} f(u_{ij}, v_{ij}) + \frac{1}{6} g(u_{y,ij}, v_{x,ij})
\]

\[
v \in \mathcal{K}_2^h \Leftrightarrow 0 = \frac{1}{\delta x} f(u_{ij}, v_{ij}) + \frac{1}{4} g(u_{y,ij}, v_{x,ij})
\]

\[\rightsquigarrow \quad \text{one-to-one mapping from } \mathcal{K}_1^h \text{ to } \mathcal{K}_2^h \text{ by multiplying}
\]

\[
\text{partial derivatives of each element with } \frac{4}{6}
\]
Discrete Inf-Sup Condition for $a(\cdot, \cdot)$ (cont.)

- the following estimates can be given for corresponding elements $\mathbf{v} \in \mathcal{K}_1^h$ and $\mathbf{u} \in \mathcal{K}_2^h$ (with $\bar{\mathbf{u}} = \bar{\mathbf{v}}$ and $\nabla \bar{\mathbf{u}} = 2/3 \nabla \bar{\mathbf{v}}$):

\[
\frac{4}{9} a(\mathbf{v}, \mathbf{v}) \leq a(\mathbf{u}, \mathbf{u}) \leq a(\mathbf{u}, \mathbf{v})
\]

- This gives for each $\mathbf{u} \in \mathcal{K}_2^h$, $\|\mathbf{u}\|_{\text{div}, \mathcal{V}} = \|\mathbf{u}\|_0 \neq 0$

\[
\sup_{\mathbf{v} \in \mathcal{K}_1^h} \frac{a(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|_0} \geq \frac{3}{2} \|\mathbf{u}\|_0 = \frac{2}{3} \|\mathbf{u}\|_{\text{div}, \mathcal{V}}
\]

and for $\mathbf{v} \in \mathcal{K}_1^h \setminus \{0\}$ we obtain

\[
\sup_{\mathbf{u} \in \mathcal{K}_2^h} a(\mathbf{u}, \mathbf{v}) \geq \frac{4}{9} a(\mathbf{v}, \mathbf{v}) > 0
\]
Discrete Inf-Sup Condition for $b_1(\cdot, \cdot)$

- for piecewise bilinear $p \in \mathcal{H}^h \subset H^1(\Omega)/\mathbb{R}$ it follows that $\nabla p \in \mathcal{U}^h$; i.e. piecewise linear
- thus, for arbitrary $p \in \mathcal{H}^h$, we have

$$
\sup_{v \in \mathcal{U}^h} \frac{b_1(v, p)}{\|v\|_0} \geq \frac{b_1(\nabla p, p)}{\|\nabla p\|_0} = \delta t h_0 \frac{(\nabla p, \nabla p)_0}{\|\nabla p\|_0} = \delta t h_0 |p|_1
$$
Convergence Studies
Taylor Vortex

Originally proposed by Minion [1996] and Almgren et al. [1998] for the incompressible flow equations

- smooth velocity field
- nontrivial solution for $h^{(2)}$
- solved on unit square with periodic BC
- $32 \times 32$, $64 \times 64$ and $128 \times 128$ grid cells
- error to exact solution at $t = 3$
## Convergence Studies

### Errors and Convergence Rates

<table>
<thead>
<tr>
<th>Method</th>
<th>Norm</th>
<th>32x32</th>
<th>Rate</th>
<th>64x64</th>
<th>Rate</th>
<th>128x128</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schneider et al.</td>
<td>$L^2$</td>
<td>0.2929</td>
<td>2.16</td>
<td>0.0656</td>
<td>2.16</td>
<td>0.0146</td>
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<tr>
<td></td>
<td>$L^\infty$</td>
<td>0.4207</td>
<td>2.15</td>
<td>0.0945</td>
<td>2.18</td>
<td>0.0209</td>
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</tr>
<tr>
<td>new exact projection</td>
<td>$L^2$</td>
<td>0.0816</td>
<td>2.64</td>
<td>0.0131</td>
<td>2.17</td>
<td>0.0029</td>
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</tr>
<tr>
<td></td>
<td>$L^\infty$</td>
<td>0.1277</td>
<td>2.45</td>
<td>0.0234</td>
<td>2.32</td>
<td>0.0047</td>
<td></td>
</tr>
</tbody>
</table>

- Second order accuracy is obtained in the $L^2$ and the $L^\infty$ norms
- Absolute error obtained with the new exact projection method about four times smaller on fixed grids
Advection of a Vortex
Results for the New Projection Method

Exact projection, central differences (no limiter):

Less deviation from the center line of the channel, loss in vorticity is slightly reduced.
Summary

A new projection method has been presented.

- it is an exact projection method with a projection based on a FE formulation
- numerical results of the new method show considerable accuracy improvements on fixed grids compared to the old formulation
- results supported by theoretical analysis; no local decoupling of the gradient in the 2nd projection

Outlook

- stability of the discrete method; inf-sup for $b_2(\cdot, \cdot)$
- additional degrees of freedom through partial derivatives $u_y$, $v_x$ and $u_x$, $v_y$
- include additional terms (Coriolis etc.)

related talk: M. Oevermann, Wed. 15:10 h (Sect 18, Session 5)
For Further Information/Reading

Th. Schneider, N. Botta, K.J. Geratz and R. Klein.


S. Vater.

A New Projection Method for the Zero Froude Number Shallow Water Equations.

*PIK Report No. 97, Potsdam Institute for Climate Impact Research, 2005.*
The auxiliary system enjoys the following properties:

- It has the same convective fluxes as the zero Froude number shallow water equations.
- The system is hyperbolic.
- Having constant height $h^*$ and a zero velocity divergence at time $t_0$, solutions satisfy at $t_0 + \delta t$:

\[
\nabla \cdot \mathbf{v}^* = O(\delta t) \quad , \quad (h^* \nabla h^*) = O(\delta t^2)
\]
Inf-Sup Condition for $a(\cdot, \cdot)$

- an orthogonal decomposition of $(L^2(\Omega))^2$ is given by
  \[
  \{ v \in H_0(\text{div}; \Omega) \mid \nabla \cdot v = 0 \} \oplus \{ \nabla q \mid q \in H^1(\Omega) \}
  \]

- $\Rightarrow \mathcal{K}_1 = \{ v \in H_0(\text{div}; \Omega) \mid \nabla \cdot v = 0 \} = \mathcal{K}_2$

- for each $u \in \mathcal{K}_2$, $\| u \|_{0,\Omega} \neq 0$, $a(\cdot, \cdot)$ satisfies
  \[
  \sup_{v \in \mathcal{K}_1} \frac{a(u, v)}{\| v \|_{0,\Omega}} \geq \frac{a(u, u)}{\| u \|_{0,\Omega}} = \frac{\| u \|_{2,\Omega}^2}{\| u \|_{0,\Omega}^2} = \| u \|_{\text{div},\Omega}
  \]