Scale-selective Time Integration for Long-Wave Linear Acoustics

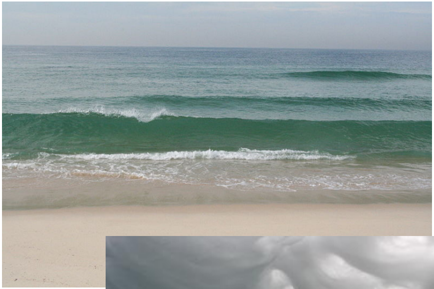
Stefan Vater[1], Rupert Klein[1] & Omar M. Knio[2]

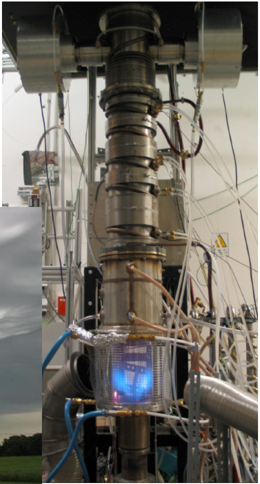
[1]Department of Mathematics and Computer Science
Freie Universität Berlin, Germany

[2]Department of Mechanical Engineering
Johns Hopkins University, Baltimore, USA

Finite Volumes for Complex Applications VI – Problems & Perspectives
Prague, June, 7th 2011

Motivation





Freie Universität Berlin

Outline

Freie Universität Berlin

Three Dimensional Compressible Flow Equations

Non-dimensional form:

$$\begin{aligned}
\rho_t + \nabla\cdot(\rho\boldsymbol{v}) &= 0\\
(\rho\boldsymbol{v})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{v}) + \frac{1}{\mathsf{M}^2}\nabla p + \frac{1}{\mathsf{Ro}}\boldsymbol{\Omega}\times\rho\boldsymbol{v} &= -\frac{1}{\mathsf{Fr}^2}\rho\boldsymbol{k}\\
(\rho e)_t + \nabla\cdot([\rho e + p]\boldsymbol{v}) &= 0
\end{aligned}$$

$$\rho e = \frac{p}{\gamma-1} + \mathsf{M}^2\,\frac{\rho\boldsymbol{v}^2}{2}$$

- slow advection, Rossby waves
- fast acoustic and gravity waves

$\rightsquigarrow$ (semi) implicit discretizations

Model Equations

1D Linear Acoustics:

$$
\begin{aligned}
m_t + p_x \quad &= 0 \\
p_t + c^2 m_x &= q(t, \tfrac{x}{\varepsilon}) \ , \quad \varepsilon \ll 1
\end{aligned}
$$

Desire:

- remove under-resolved modes
- minimize dispersion relation for marginally resolved modes
- correctly compute "slaved" dynamics induced by source term
- no resonance for high wavenumber low frequency source terms

Model Equations – Classical Discretizations

Implicit trapezoidal rule:

$$\frac{m^{n+1} - m^n}{\Delta t} = -\frac{1}{2}\left(\frac{\partial p^n}{\partial x} + \frac{\partial p^{n+1}}{\partial x}\right)$$

$$\frac{p^{n+1} - p^n}{\Delta t} = -\frac{c^2}{2}\left(\frac{\partial m^n}{\partial x} + \frac{\partial m^{n+1}}{\partial x}\right) + q^{n+1/2}$$

- A-stable, symplectic

Backward differencing (BDF(2)):

$$\frac{\frac{3}{2}m^{n+1} - 2m^n + \frac{1}{2}m^{n-1}}{\Delta t} = -\frac{\partial}{\partial x}p^{n+1}$$

$$\frac{\frac{3}{2}p^{n+1} - 2p^n + \frac{1}{2}p^{n-1}}{\Delta t} = -c^2\frac{\partial}{\partial x}m^{n+1} + q^{n+1}$$

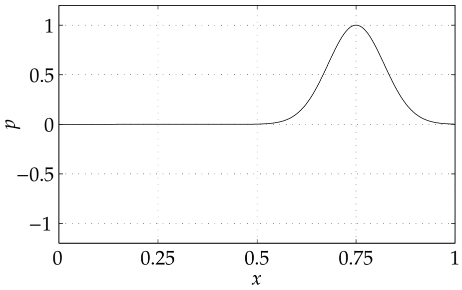
- L-stable, multistep method

Freie Universität Berlin

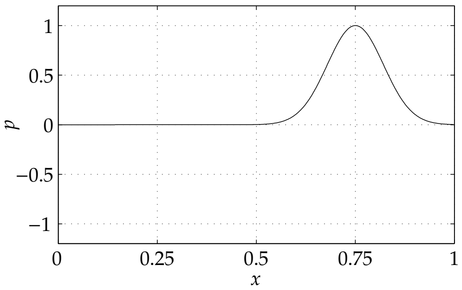
Model Equations – Under-resolved Scales

- staggered grid, 512 grid points, periodic b.c., CFL $= 10$
- simple wave, $c = 1$, single scale data

Implicit trapezoidal rule



BDF(2) scheme



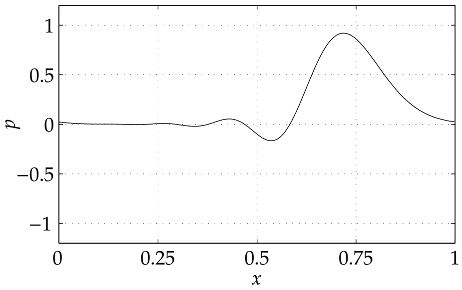
$t = 0$ (Initial conditions)

Freie Universität Berlin

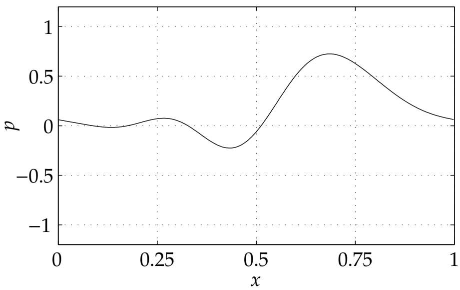
Model Equations – Under-resolved Scales

- staggered grid, 512 grid points, periodic b.c., CFL = 10
- simple wave, $c = 1$, single scale data

Implicit trapezoidal rule



BDF(2) scheme

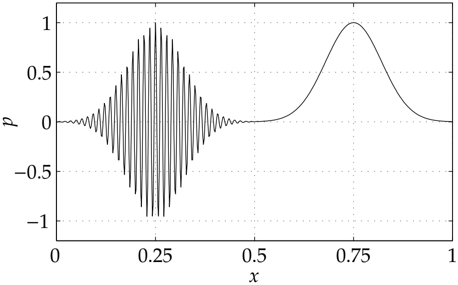


$t = 3$ (154 time steps)

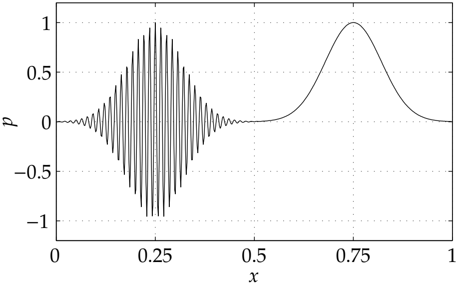
Model Equations – Under-resolved Scales

- staggered grid, 512 grid points, periodic b.c., CFL $= 10$
- simple wave, $c = 1$, multi scale data

Implicit trapezoidal rule



BDF(2) scheme

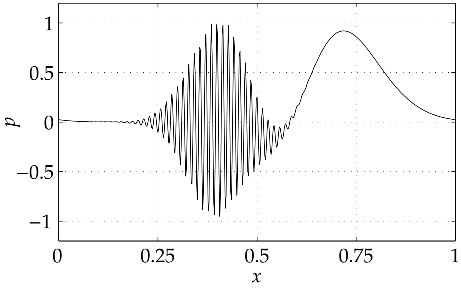


$t = 0$ (Initial conditions)

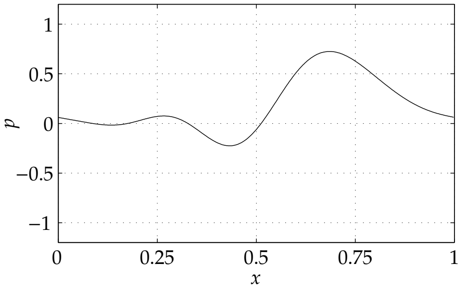
Model Equations – Under-resolved Scales

- staggered grid, 512 grid points, periodic b.c., CFL = 10
- simple wave, $c = 1$, multi scale data

Implicit trapezoidal rule



BDF(2) scheme

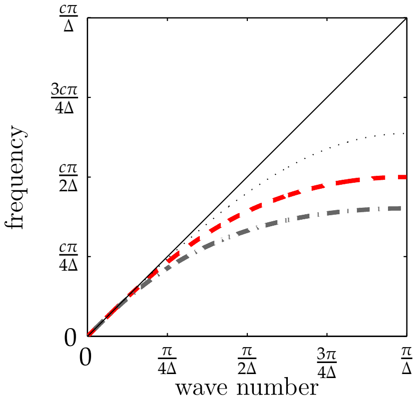


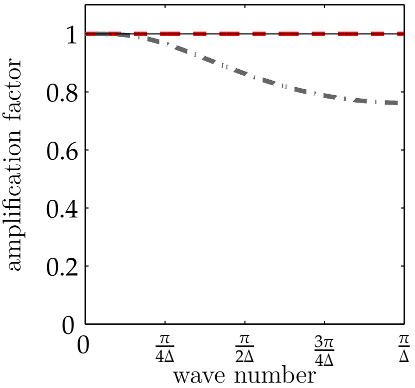
$t = 3$ (154 time steps)

Freie Universität Berlin

Model Equations – Dispersion Relation and Amplitude

$\mathsf{CFL} = 1$

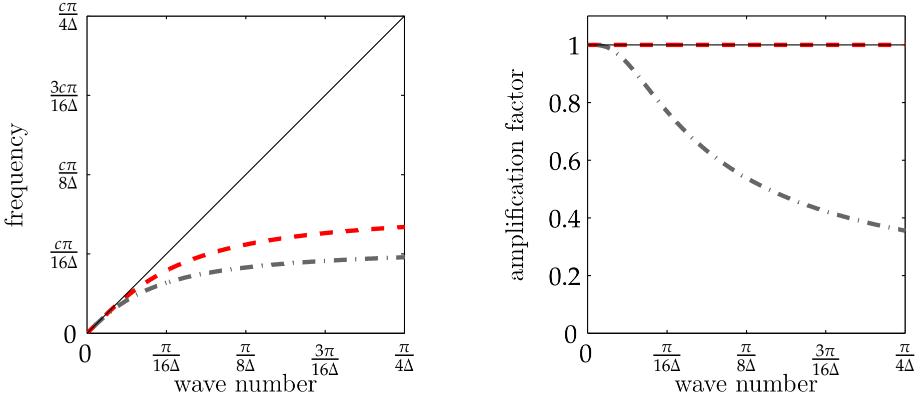




dashed: trapezoidal rule, dashed-dotted: BDF(2), black line: cont. system

Model Equations – Dispersion Relation and Amplitude

$\mathsf{CFL} = 10$



dashed: trapezoidal rule, dashed-dotted: BDF(2), black line: cont. system

Model Equations – Balanced Modes

1D Linear Acoustics:

$$\begin{aligned} m_t + p_x \quad &= 0 \\ p_t + c^2 m_x &= q(t, \tfrac{x}{\varepsilon}) \end{aligned}$$

$$q(t, \tfrac{x}{\varepsilon}) = \sin(\omega t) \cos(\lambda \tfrac{x}{\varepsilon}) \exp(-(\tfrac{x - x_0}{\sigma \varepsilon})^2)$$

Asymptotics:

$$m = \mathcal{O}(\varepsilon) \ , \ p - p_0(t) = \mathcal{O}\left(\varepsilon^2\right) \text{ as } \varepsilon \to 0$$

Scaling should be reproduced by the numerical scheme, especially, when $\Delta t \gg \frac{\Delta x}{c} = \frac{\varepsilon \Delta \xi}{c}$.

Model Equations – Balanced Modes

Up to small perturbations introduced by variation in time:

$$c^2 m_x \approx q\left(t, \tfrac{x}{\varepsilon}\right) \quad \text{and} \quad p \approx 0$$

Balance should be reproduced, at least for $\Delta t \to \infty$!

Implicit trapezoidal rule:

$$c^2 \frac{1}{2}\left(\frac{\partial m^n}{\partial x} + \frac{\partial m^{n+1}}{\partial x}\right) = q^{n+1/2} \quad \text{and} \quad \frac{\partial p^{n+1}}{\partial x} = -\frac{\partial p^n}{\partial x}$$

BDF(2) scheme:

$$c^2 \frac{\partial m^{n+1}}{\partial x} = q^{n+1} \quad \text{and} \quad \frac{\partial p^{n+1}}{\partial x} = 0$$

Freie Universität Berlin

Classical Schemes – In Summary

Desired properties:

- minimize dispersion, preserve amplitude for well resolved modes
  $\rightsquigarrow$ trapezoidal rule
- solution should rapidly relax to balanced mode in case of short wave number forcing
  $\rightsquigarrow$ backward differencing formulas

Look for strategy to combine two aspects into one single, scale-dependent time integrator

Multilevel Method – Idea

Consider:

- a separation of pressure and momentum into quasi-spectral components:

$$p = \sum_{\nu=0}^{\nu_M} p^{(\nu)} \; , \;\; m = \sum_{\nu=0}^{\nu_M} m^{(\nu)}$$

- two time discretizations of linear acoustics:

$$S_1(p^{n+1}, m^{n+1}, p^n, m^n, p^{n-1}, m^{n-1}, \dots) = 0$$
$$S_2(p^{n+1}, m^{n+1}, p^n, m^n, p^{n-1}, m^{n-1}, \dots) = 0$$

- perform a convex combination of the two schemes with scale dependent weights (blending):

$$\sum_{\nu=0}^{\nu_M} \mu(\nu) S_1(p^{(\nu),n+1}, \dots) + (1 - \mu(\nu)) S_2(p^{(\nu),n+1}, \dots) = 0$$

Freie Universität Berlin

Multilevel Method – Implicit Trapezoidal Rule/BDF(2)

Implicit Midpoint Rule:

$$\left(1 - c^2 \Delta t^2 \frac{1}{4} \frac{\partial^2}{\partial x^2}\right) p^{n+1} = \mathrm{RHS}^p_{\mathrm{TR}} \ , \ \ m^{n+1} = \mathrm{F}_{\mathrm{TR}}(m^n, p^n, p^{n+1})$$

BDF(2):

$$\left(1 - c^2 \Delta t^2 \frac{4}{9} \frac{\partial^2}{\partial x^2}\right) p^{n+1} = \mathrm{RHS}^p_{\mathrm{BD}} \ , \ \ m^{n+1} = \mathrm{F}_{\mathrm{BD}}(m^n, m^{n-1}, p^{n+1})$$

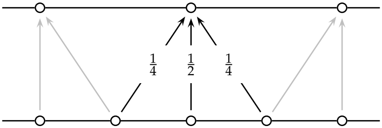
Blended scheme:

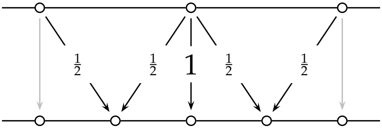
$$\sum_\nu \left(1 - c^2 \Delta t^2 \left(\frac{\mu_\nu}{4} + \frac{4(1-\mu_\nu)}{9}\right) \frac{\partial^2}{\partial x^2}\right) p^{(\nu),n+1} = \sum_\nu \mu_\nu \, \mathrm{RHS}^{p,(\nu)}_{\mathrm{TR}} + (1-\mu_\nu)\, \mathrm{RHS}^{p,(\nu)}_{\mathrm{BD}}$$

Freie Universität Berlin

Scale Splitting – pressure

$P^{(\nu)}$, $R^{(\nu)}$: Multigrid prolongation/restriction:





Scale dependent portions of unknowns $p = \sum_{\nu=0}^{\nu_M} p^{(\nu)}$, where

$$p^{(0)} = \left(R^{(0)} \circ R^{(1)} \circ \cdots \circ R^{(\nu_M - 1)} \right) p$$

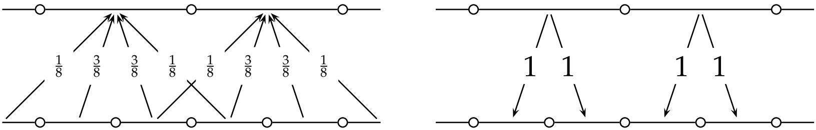
and

$$p^{(\nu)} = \left(I - P^{(\nu-1)} \circ R^{(\nu-1)} \right) \circ \left(R^{(\nu)} \circ R^{(\nu+1)} \circ \cdots \circ R^{(\nu_M - 1)} \right) p$$

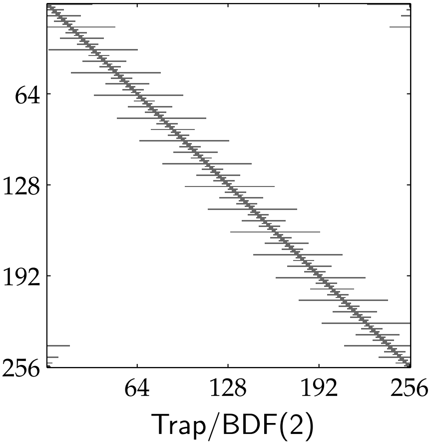
Scale Splitting – momentum

- staggered grid: p node centered, m cell centered
- need different splitting operators to be consistent wrt. level
- momentum corresponds with pressure gradient, i.e.

$$m \leftrightarrow p_x$$



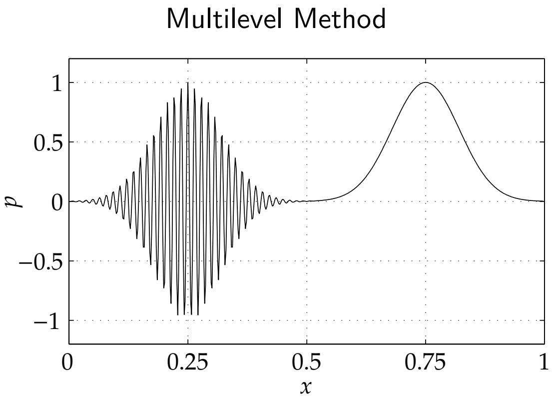
Sparsity Pattern



- pattern depends on weighting function μ_ν
- cannot be solved efficiently with standard iterative methods

Model Equations – Under-resolved Scales

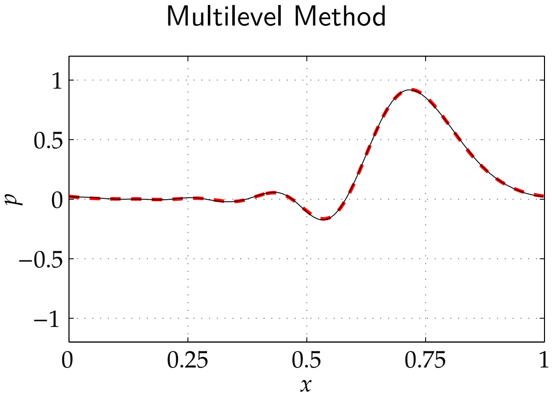
- staggered grid, 512 grid points, periodic b.c., CFL $= 10$
- simple wave, $c = 1$, multi scale data



$t = 0$ (Initial conditions)

Model Equations – Under-resolved Scales

- staggered grid, 512 grid points, periodic b.c., CFL $= 10$
- simple wave, $c = 1$, multi scale data

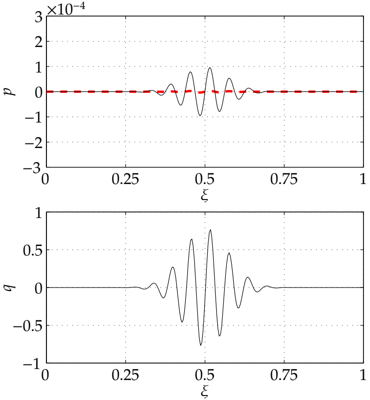


$t = 3$ (154 time steps)

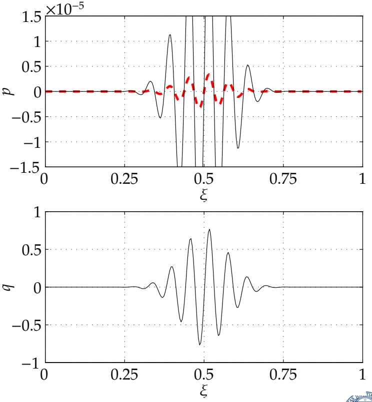
Model Equations – Balanced Modes

- staggered grid, 256 grid points, periodic b.c., CFL $= 8$, $\varepsilon = 0.1$
- $q(t, \frac{x}{\varepsilon}) = \sin(\omega t) \cos(\lambda \frac{x}{\varepsilon}) \exp(-(\frac{x - x_0}{\sigma \varepsilon})^2)$

Implicit trapezoidal rule



BDF(2) scheme



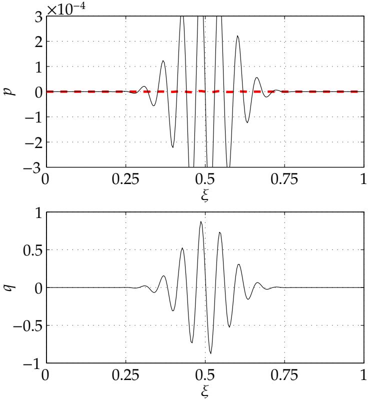
after 1 time step

Freie Universität Berlin

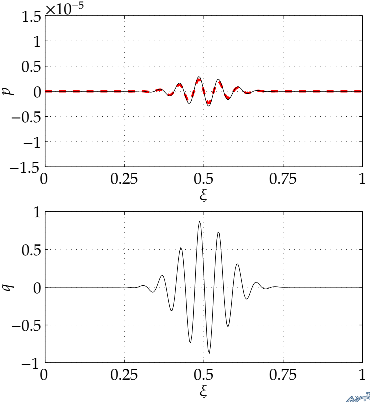
Model Equations – Balanced Modes

- staggered grid, 256 grid points, periodic b.c., CFL $= 8$, $\varepsilon = 0.1$
- $q(t, \frac{x}{\varepsilon}) = \sin(\omega t)\cos(\lambda \frac{x}{\varepsilon})\exp(-(\frac{x-x_0}{\sigma\varepsilon})^2)$

Implicit trapezoidal rule



BDF(2) scheme

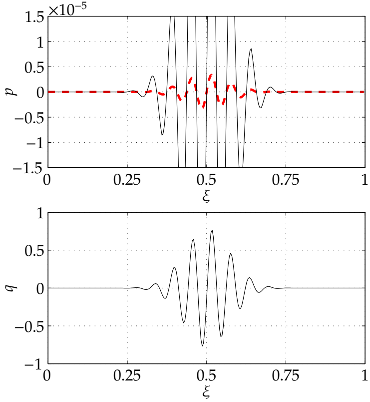


after 18 time steps

Model Equations – Balanced Modes

- staggered grid, 256 grid points, periodic b.c., CFL $= 8$, $\varepsilon = 0.1$
- $q(t, \frac{x}{\varepsilon}) = \sin(\omega t)\cos(\lambda \frac{x}{\varepsilon})\exp(-(\frac{x-x_0}{\sigma\varepsilon})^2)$

Multilevel Method

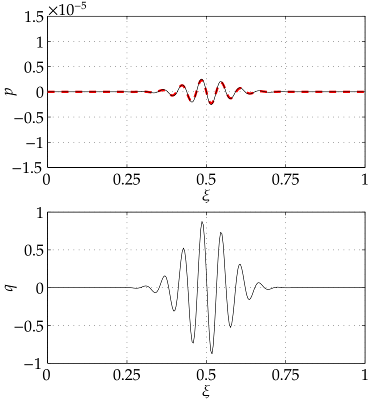


after 1 time step

Model Equations – Balanced Modes

- staggered grid, 256 grid points, periodic b.c., CFL $= 8$, $\varepsilon = 0.1$
- $q(t, \frac{x}{\varepsilon}) = \sin(\omega t)\cos(\lambda \frac{x}{\varepsilon})\exp(-(\frac{x-x_0}{\sigma\varepsilon})^2)$

Multilevel Method



after 18 time steps

Summary

- (semi) implicit solution of geophysical flow equations
- analysis of classical schemes
- multilevel method
- multigrid techniques

- Outlook
  - efficient solution
  - incorporation into semi-implicit method
  - more than one space dimension

Freie Universität Berlin

For Further Information/Reading

S. Vater, R. Klein, O.M. Knio
A Scale-selective Multilevel Method for Long-Wave Linear Acoustics.
submitted to Acta Geophysica, 2011.

Th. Schneider, N. Botta, K.J. Geratz and R. Klein
Extension of Finite Volume Compressible Flow Solvers to Multi-dimensional, Variable Density Zero Mach Number Flows.
Journal of Computational Physics, 155 : 248–286, 1999.

S. Vater & R. Klein
Stability of a Cartesian Grid Projection Method for Zero Froude Number Shallow Water Flows.
Numerische Mathematik, 113(1) : 123–161, 2009.
also available as ZIB-Report No. 07-13, 2007 (`http://www.zib.de`)

R. Klein
Asymptotics, structure, and integration of sound-proof atmospheric flow equation.
Theoretical and Computational Fluid Dynamics, 23 : 161-195, 2009.

Freie Universität Berlin