A Brief Note on Proofs in Pure Mathematics
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What is pure mathematics?

Pure mathematics is a discipline that enjoys a rich history, dating back to Ancient Greece. The goal is to rigorously establish mathematical truths; to show with absolute certainty that a statement is valid. This is achieved through logical arguments and careful attention to detail, as evidenced by the following anecdote:

A mathematician, a physicist, and an engineer are riding a train through Scotland.

The engineer looks out the window, sees a brown sheep, and exclaims, “Hey! The sheep in Scotland are brown!”

The physicist looks out the window and corrects the engineer, “Strictly speaking, all we know is that there is at least one brown sheep in Scotland.”

The mathematician looks out the window and corrects the physicist, “Strictly speaking, all we know is that at least one side of one sheep in Scotland is brown.”

All jokes aside, this class will be quite different from the lower division math courses you may have taken. Rather than carrying out calculations to find an answer, you will have to write proofs to establish a result. In doing so, you will prove foundational results in calculus - for example, the Fundamental Theorem of Calculus. While you will have seen many of these results before, the proofs may be new to you. This document is intended to help you understand what proofs are and how they work, to smooth the transition from applied to pure mathematics.

Anatomy of a Proof

Unlike in the experimental sciences, where scientists may interpret data in different ways, mathematical results must be universally agreed upon. Any field must start from somewhere, and axioms are basic assumptions that we accept to be true without question. One then uses the rules of logic to obtain new results, called theorems, from the axioms. These theorems can then be used to construct new theorems, and so the field grows.

In order to establish a new theorem, one must provide a proof, an impregnable logical argument to convince others that the statement is true. One starts with something that is known to be true - either an axiom or an earlier theorem. One then proceeds through a series of steps, each following logically from the previous ones, until the desired theorem is reached at the end.

1Unfortunately, a joke can be funny or mathematical but not both. This one was adapted from http://www.cs.northwestern.edu/~riesbeck/mathphyseng.html
2Sometimes, we do question axioms: a famous example comes from geometry. By changing an axiom of classical geometry, 19th century mathematicians developed different geometries, which have since proved very useful in theoretical physics.
When writing a proof, one must decide what level of detail to provide. This is a skill akin to providing driving directions. You should give enough detail so that the reader can follow the proof to the theorem without getting lost. However, it is unnecessary, and indeed unpleasant, to provide every minute instruction - you would not tell someone when to brake or accelerate. When in doubt, though, err on the side of caution - do not leave a logical gap, and be wary of claiming facts are ‘obvious’. There are many cases of “proofs” having stood for decades before being found to be incorrect.

Finally, do not be afraid of using words in your proof. It is often a good idea, especially in a long proof, to give an outline of the proof, and explain what you are doing at the key steps, as opposed to just giving line after line of algebraic manipulation. However, do be concise and precise - you cannot replace the mathematical work with ‘hand-waving’.

The Language of Mathematics

Just as with literature, you cannot read a proof without knowing the language. There are many symbols used in mathematics to help shorten statements, and the table below shows some of the most frequently used. However, it is by no means exhaustive, and sometimes different symbols are used. If you see something you do not recognise, do not hesitate to ask!

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>∀</td>
<td>For all</td>
<td>∃</td>
<td>There exists</td>
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<td>∃ (or :)</td>
<td>Such that</td>
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<td>∧</td>
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<td>⇒</td>
<td>Implies</td>
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<td>If and only if</td>
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<td>□</td>
<td>End of proof</td>
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Sets

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<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>∪</td>
<td>Union</td>
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<td>∩</td>
<td>Intersection</td>
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<tr>
<td>ℝ</td>
<td>Real numbers</td>
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<td>ℚ</td>
<td>Rational numbers</td>
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<tr>
<td>ℤ</td>
<td>Integers</td>
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<tr>
<td>ℕ</td>
<td>Natural numbers</td>
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Greek Letters

<table>
<thead>
<tr>
<th>Symbol</th>
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<td>β</td>
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<td>ε</td>
<td>Epsilon</td>
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<tr>
<td>µ</td>
<td>Mu</td>
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It takes a bit of practice getting used to quantifiers. The first thing to note is that the order they appear in is very important. For instance, consider the following:

\[ ∀x ∈ ℝ \; ∃y ∈ ℝ \; ∀x < y \]  \hspace{1cm} (1)

Translating, this reads “for all real numbers \( x \), there exists a real number \( y \) such that \( x \) is less than \( y \).” This is clearly a true statement, since given any number we can find a larger number. Now consider:

\[ ∃y ∈ ℝ \; ∀x ∈ ℝ \; x < y \]  \hspace{1cm} (2)
This time it reads “there exists a real number $y$ such that for all real numbers $x$, $x$ is less than $y$.” This is now false, since this would imply $y$ is the largest real number, which does not exist.

So be sure to get your quantifiers in the right order!

Another source of difficulty arises from having to negate quantifiers in a statement - i.e. to find the opposite. The trick is to swap every $\forall$ with a $\exists$, and vice versa, and then reverse the remainder of the statement. Suppose we wanted to negate statement (1). We must change the leading “for all” to a “there exists”, and the middle “there exists” to a “for all”. Finally, we need to change the ‘$<$’ to its opposite, ‘$\geq$’. This gives

$$\exists x \in \mathbb{R} \ni \forall y \in \mathbb{R} \quad x \geq y \quad (3)$$

This reads “there exists a real $x$ such that for every real $y$, $x$ is at least as large as $y$.” Note that the original statement says precisely the opposite - for every $x$, there is a larger $y$. For the original statement, for all values of $x$, we needed just one value of $y$ to work, while in the negation, we only need one ‘bad’ value of $x$, but we need all values of $y$ to fail.

**Methods of Proof**

We have covered what proofs are for, and how to read them, but not how to come up with them! Here I briefly outline three main methods of proof:

**Direct Proof**

This is the most straightforward form of proof, where we start with an axiom or theorem, and use logical rules to proceed from step-to-step until we arrive at our result. For example, consider the following:

**Theorem:** If $0 < a < b$, then $a^2 < b^2$.

**Proof:**

Since $a$ is a positive number, and $a < b$, we have $a^2 = a \cdot a < a \cdot b$. 
Since $b$ is a positive number, and $a < b$, we have $a \cdot b < b \cdot b = b^2$.

Hence $a^2 < a \cdot b < b^2$, and so $a^2 < b^2$. 

Note that the above proof used the fact that multiplying both sides of an inequality by a positive number does not change the inequality; this is an example of leaving out unnecessary details. However, that result would have to have been proven earlier.

It can sometimes be tempting to start from the result you want to prove, and try to reduce it down to something that is known, but this tends to be harder to read. I would recommend the more direct approach.
Proof by Contradiction

This is a powerful method of proof, but can be a little difficult to use at first. The idea is to assume your statement is false, and then show that this leads to a contradiction - something that is obviously not true. Hence the only possibility is for the original statement to have been true.

**Theorem:** There is no smallest positive real number.

**Proof:**

Suppose that there was such a number, call it \( x \). Then by assumption \( x > 0 \). But since \( 0 < \frac{1}{2} < 1 \), multiplying by \( x \) gives \( 0 < \frac{x}{2} < x \), so \( \frac{x}{2} \) is a smaller real positive number, which is a contradiction.

Hence there cannot be a smallest positive real number.

\[ \square \]

A common source of confusion is whether this is equivalent to finding a counterexample - it is **not**! One counterexample is all you need to show a statement is false, but an example is never enough to show that a statement is true. In the above argument we did not just show that one possible value did not work; instead, we had to prove that all possible values failed.

Proof by Induction

The idea of induction is to prove infinitely many statements at once. Suppose we have a sequence of statements, indexed by the natural numbers. For example, consider the statement \( P(n) : 2n \leq 2^n \).

This says that for every natural number \( n \), \( 2n \) is at most \( 2^n \). Note that this is really an infinite sequence of inequalities:

\[
\begin{align*}
P(1) & : 2 \cdot 1 = 2 \leq 2 = 2^1 \\
P(2) & : 2 \cdot 2 = 4 \leq 4 = 2^2 \\
P(3) & : 2 \cdot 3 = 6 \leq 8 = 2^3 \\
& \vdots
\end{align*}
\]

How can we prove all these statements in one fell swoop? There are two steps. The first is the **base step:** we prove that the first statement, \( P(1) \), is true. Next comes the **induction step:** we show that every statement implies the next, i.e. \( P(n) \Rightarrow P(n+1) \). What happens next is like a chain of dominos. We know \( P(1) \) is true. Because \( P(1) \) is true, \( P(2) \) must be true. Because \( P(2) \) is true, \( P(3) \) is true. And so on - we see that \( P(n) \) is true for every \( n \).
Theorem: \( \forall n \in \mathbb{N} \quad 2n \leq 2^n \)

Proof:
Let \( P(n) \) be the statement \( 2n \leq 2^n \). We use mathematical induction.

\( P(1) \): When \( n = 1 \), we have \( 2n = 2 \cdot 1 = 2 \) and \( 2^n = 2^1 = 2 \), and so \( 2n \leq 2^n \) for \( n = 1 \).

\( P(n) \Rightarrow P(n+1) \): Suppose we know \( P(n) \) is true, i.e. \( 2n \leq 2^n \). Then

\[
2(n + 1) = 2n + 2 \\
\leq 2^n + 2 \quad \text{by} \quad P(n) \\
\leq 2^n + 2^n \quad \text{since} \quad n \geq 1 \\
= 2 \cdot 2^n = 2^{n+1}
\]

Hence \( 2(n + 1) \leq 2^{n+1} \), and so \( P(n + 1) \) is true if \( P(n) \) is true.

Hence, by induction, \( P(n) \) is true for every \( n \geq 1 \).

\[ \square \]

You should note that induction can only be applied when the statements are indexed by the natural numbers. There has to be a first case (the base case), and you have to have be able to move to the ‘next’ statement (the induction step). If the statements are indexed by real numbers, then there is no ‘next’ statement. For example, you would not be able to prove the inequality \( 2x \leq 2^x \quad \forall \quad x \in \mathbb{R} \) by induction on \( x \).

Also notice that the power of induction is that in the induction step, you can assume \( P(n) \) to prove \( P(n + 1) \). Thus induction is useful when you can find an instance of \( P(n) \) in \( P(n + 1) \). In the above proof, we saw that \( P(n + 1) \) contained the expression \( 2(n + 1) \), which we could write as \( 2n + 2 \), and then use \( P(n) \) on the \( 2n \) term. Usually some further algebraic manipulation is required, but you know what the result should look like (you are trying to establish \( P(n + 1) \)), so that should guide your calculations. Again referring to the above proof, we knew that our final result should be \( 2(n + 1) \leq 2^{n+1} \), so we did what we had to to get the correct right-hand side.

Parting Thoughts
I have stated above that proofs are used to show that theorems are true. However, good proofs do more than this - they show why theorems are true, and thus suggest further theorems. One of the many wonderful things about mathematics is that there are many ways to prove a theorem, and indeed it is not unheard of for an undergraduate to discover a new (and sometimes better) proof of a classical result. However, it is true that analysis is a very old, well-studied subject, and so the proofs presented are the beautiful results of the work of generations of the world’s greatest mathematicians, so do be sure to enjoy the ride!