FREIE UNIVERSITÄT BERLIN
Institut für Mathematik
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Seminar: Submodular functions and convexity

A *submodular function* is a function $f : 2^E \to \mathbb{R}$ that takes values on all subsets of a finite set E such that

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

for all $A, B \subseteq E$. Despite this simple definition, submodular functions give rise to a rich theory and, in particular, make appearances in various parts of mathematics. The goal of the seminar is to get an overview of the main concepts of submodular functions and an impression of the many guises in which submodular functions occur.

In that spirit, the topics fall into "fundamental" and "applications". The "fundamentals" are given priority as they are, well, fundamental. Team-work across topics is indicated and encouraged. If you have suggestions for additional topics, please contact me!

Regarding **participation** you are expected to attend regularly, give a talk (~ 45min.), and submit a term paper (~ 8 pages). The seminar will take place **Wednesdays, 10-12, Arnimallee 2**.

Please subscribe to the **mailinglist** at `https://lists.fu-berlin.de/listinfo/SubModFct`

Foundations of submodular functions

(1) **SubModFcts in Combinatorial Optimization**
Several problems in combinatorial optimization such as spanning trees, matchings, cuts, etc. can be unified under the heading of (poly)matroids which form a special class of submodular functions. Optimization problems over (poly)matroidal structures can be solved efficiently using the greedy algorithm.
If you know how to find a minimal spanning tree using Kruskal's algorithm, all this should be rather easy for you.

(2) **SubModFcts in (discrete) geometry**
Arrangements of hyperplanes or, more generally, arrangements of linear subspaces play an important role in several fields such as algebraic geometry and topology. Their combinatorial features are completely specified by their submodular rank function. In particular, every hyperplane arrangement restricts to a subspace arrangement and every subspace arrangement lifts to a hyperplane arrangement. This raises the question if every (non-negative) submodular function or matroid can be realized as the rank function of an arrangement.
Linear algebra with a view towards geometry should be sufficient background. Connections: (3)

(3) **Operations on SubModFcts** (with Louis Theran)
The richness of the theory of submodular functions is also due to the many operations that preserve submodularity such as Dilworth truncation, (special) convolutions, parallel and principal extensions; see [9, Sect. 2] and [15, Ch. 48]. For example, they enable the lifting of general polymatroids to matroids. The operations are very natural for arrangements and graph structures but also work in complete generality.

(4) **Polyhedra associated to SubModFcts**
To every (non-negative) submodular function f there is an associated convex polytope P_f, the *base polytope*, which encodes f geometrically. In particular, optimization over a submodular function in the sense of (1) corresponds to linear optimization over P_f; see [3]. This geometric point of view essentially shows that the greedy algorithm works for submodular functions. Even stronger, it can be shown [2] that if P is a compact set on which all linear functions can be optimized by a greedy-type algorithm, then P is essentially the polytope associated to a submodular function.

This gives another (geometric) characterization of submodular functions (see also [5, Thm 4.1] and [15, Ch. 44]).
Basic knowledge of linear programming including LP duality should be sufficient. A basic geometric understanding of linear programming is surely a plus. Connections: (5)

(5) **SubModFcts and convex functions**
In some sense, submodularity is a discrete version of convexity. A very elegant and interesting construction of Lovász [9] associates to every set function $f : 2^{[n]} \to \mathbb{R}$ a continuous function $\hat{f} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ such that the following holds: f is submodular if and only if $\hat{f}$ is convex. This is called the *Lovász extension*. Yet another highlight in this area is a close connection between submodular, supermodular, and modular functions and separation theorems a la Farkas or Hahn-Banach. An even broader relation can be build via Murota's *discrete convex analysis* [12]. Connections: (4)
Basic knowledge of convex geometry is advantageous.

(6) **Minimizing SubModFcts**
In relation to convex geometry, it is of interest how to minimize a submodular function, i.e. to find $U \subseteq E$ with $f(U) \leq f(V)$ for all $V \subseteq E$. For practical purposes, the question is also how fast. The first (in a precise sense) *fast* algorithm was given by Schrjiver (see [8, Sect. 14.3]) which makes use of combinatorial concepts as well as the base polyhedra (4).
This, again, requires a little understanding of the greedy algorithm and the geometry of (base) polyhedra.

Applications of submodular functions

- **SubModFcts in convex (real) algebraic geometry** (with Karim Adiprasito)
Hyperbolic polynomials are special homogeneous polynomials with real coefficients that originated in areas such as differential equations (hyperbolic PDEs) and control theory (stable polynomials). In a simple manner, they give rise to a broad class of convex bodies which makes them interesting in convex geometry and optimization. In an equally simple way, hyperbolic polynomials give rise to submodular set functions (see [6]) that are closely related to the geometry of the associated convex bodies. In particular, realizable (over the reals) (poly)matroids give rise to hyperbolic polynomials. This might be the most challenging topic but with beautiful mathematics. You need some background in convex geometry (e.g. Brunn-Minkowski inequality) to understand why the set functions are submodular.

- **SubModFcts in information theory**
Given a finite collection E of random variables, the function that maps a subset to its *entropy* ("= expected surprise" according to Louis) is a submodular function [4]. One might hope that all submodular functions arise this way (in the limit) but this is not the case: the cone of entropy functions is strictly smaller than the submodular cone [16]. Interestingly, if a submodular function is realizable in the sense of (2), then it is also entropic [7] (see [1] for a simple proof).

- **Rigidity of bar-joint-structures** (with Louis Theran)
A bar-and-joint structure is a collection of solid rods connected in a graph like fashion by joints. Such a structure is called *rigid* if there is essentially no way to move the joints except for a motion of the whole structure. The basic question in rigidity theory is to what extend the underlying graph and the lengths of the rods determine the rigidity of the structure. It turns out that the number of "degrees of freedom" of a graph is intricately related to submodular functions [9, Example 1.7] and, in particular, to matroids in the plane [10].
Linear algebra is basically all you need. Enlightenment beyond that can be obtained in the lecture "Rigidity Theory" by Louis Theran this summer term!

- **Cooperative Game theory** (with Britta Peis at TU Berlin)
The objective of cooperative game theory is to study ways to enforce and sustain cooperation among players that are willing to cooperate. For example, suppose there are three neighboring farmers (the "players"). Then it makes certainly sense to share some of the resources in order to achieve a better total payoff. The question is how this total payoff should be shared among the

players so that all of them are satisfied and none of them has an incentive to break the agreement. It turns out that submodularity plays an essential role when it comes to develop "fair" and "stable" cost sharing methods for such cooperative games; see [13, Ch. 15].
A little knowledge of linear optimization could help.

- **SubModFcts in combinatorics** (with César Ceballos)
  Base polyhedra, for certain submodular functions, make appearance in geometric and algebraic combinatorics. For example, the volume of the base polyhedron coming from a graph is the number of spanning trees. The volume of the base polyhedron coming from certain matroids is the number of permutations with given number of descents. The polytopes all belong to the class of *generalized permutahedra* [14]. The generalized permutahedra are not really defined as base polyhedra but it can be shown [11] that submodularity is crucial.
  For this topic it is of advantage to have a background in discrete geometry.

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