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# Discrete Geometry III

## Homework # 5 — due January 20th

Please do all the problems. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday.

**Exercise 1.** For $d \geq 0$ define

$$\mathsf{D}_d \ := \ \{x \in \mathbb{R}^d : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1\}.$$

i) Show that $\mathsf{D}_d$ is a $d$-simplex and determine the vertices of $n\mathsf{D}_d$ for $n \in \mathbb{N}$.

There is a canonical *finite* subdivision $\mathcal{K}_d$ of $\mathbb{R}^d$ whose maximal cells are the cubes $\mathbf{p} + [0,1]^d$ with $\mathbf{p} \in \mathbb{Z}^d$.

ii) Argue that for any $n > 0$, $\mathcal{K}_d \cap n\mathsf{D}_d := \{F \cap n\mathsf{D}_d : F \in \mathcal{K}_d\}$ induces a subdivision of $n\mathsf{D}_d$.

iii) Show that every cell of $\mathcal{K}_d \cap n\mathsf{D}_d$ is affinely isomorphic to

$$\mathsf{D}_{s_1} \times \mathsf{D}_{s_2} \times \cdots \times \mathsf{D}_{s_n},$$

for some $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$ with $s_1 + \cdots + s_n \leq d$.
[Hint: You only have to do this for $s_1 + \cdots + s_n = d$ (why?).]

iv) For $s = (0 \leq s_1 \leq \cdots \leq s_n)$ with $s_1 + \cdots + s_n = d$ count the number of cells of this type. In particular infer that there are exactly $n$ cells isomorphic to $\Delta_d$.

**(10 points)**

**Exercise 2.** For $d = 1$, show that any $f \in \Pi^1$ can be written as

$$f \ = \ \alpha[\![0]\!] + \sigma[\![[0,r)]\!],$$

where $\alpha \in \mathbb{Z}$, $\sigma = \pm 1$, and $r \in \mathbb{R}_{>0}$.

**(10 points)**

**Exercise 3.** i) Show that $f \in \Pi$ is nilpotent if and only if $f \in \Pi_+$.
(This means that $\Pi_+$ is the *nilradical* of $\Pi$.)

ii) Show that $[\![P]\!]$ is invertible in $\Pi$ and give a formula for $[\![P]\!]^{-1}$.

iii) Show that $f \in \Pi$ is invertible if and only if $\chi(f) = \pm 1$.

**(10 points)**

**Exercise 4.** Let $m > 0$ and $P, P_1, \ldots, P_r$ polytopes such that $P = P_1 + \cdots + P_r$. Assume that for every $i$ there is an element $h_i \in \Pi$ such that $mh_i = [\![P_i]\!] - 1$. Show that there is $h \in \Pi$ such that $mh = [\![P]\!] - 1$. Can you give an explicit formula?

**(10 points)**