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Discrete Geometry III

Homework # 4 — due December 16th

Please do all the problems. **20 points** on every (1 week) homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday.

Exercise 1. For $c \in \mathbb{R}^d \setminus \{0\}$, consider the map $F_c : \mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$ defined by $F_c(P) := [P^c]$, where P^c is the face in direction c. Show that F_c is a valuation.

(10 points)

Exercise 2. Recall that function $f : \mathbb{R}^d \to \mathbb{R}$ is **piecewise linear** (**PL**, for short) if there is a complete fan $\mathcal{F}$ with maximal cells $F_1, \dots, F_m$ and linear functions $\ell_1, \dots, \ell_m$ such that $f(x) = \ell_i(x)$ for all $x \in F_i$.

i) Let $f_1, \dots, f_r$ be linear function and define

$$f(x) \ := \ \max\{f_i(x) : i = 1, \dots, r\}.$$

Show that f is a piecewise linear function.

ii) Show that a PL function f is convex if and only if it is of the form i).

iii) Show that a function $h : \mathbb{R}^d \to \mathbb{R}$ is the support function of a polytope if and only if h is subadditive and piecewise linear.

(10 points)

Exercise 3. i) For $A = \{a_1, \dots, a_m\} \subset \mathbb{R}^d$ define

$$h_A(x) \ = \ \sum_{i=1}^{m} |a_i^t x|$$

Show that $h_A(x)$ is a piecewise linear function. In fact, h is the support function of a polytope. [Hint: Start with $m = 1$ and use Minkowski sums.]

ii) Show that for a piecewise linear function $h : \mathbb{R}^d \to \mathbb{R}$ there are polytopes $P, Q \subset \mathbb{R}^d$ such that $h = h_P - h_Q$. [Hints: You can assume that the underlying fan $\mathcal{F}$ is induced by a hyperplane arrangement. Argue that there is a PL function h_A that $h + M \cdot h_A$ is convex for some $M \gg 0$. Remember that convexity of a function is a 2-dimensional notion.]

(10 points)

Exercise 4. Let $P \in \mathcal{P}_d$ and let $u \in \mathbb{R}^d \setminus \{0\}$. The **envelope** of P with respect to u is

$$E_u P \ := \ \{x \in P : x + \delta u \notin P \text{ for all } \delta > 0\}$$

Prove that the homomorphism $\phi_u : \mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$ given on generators by $\phi_u([P]) := [E_u P]$ is well-defined.
[Hint: For $f \in \mathbf{S}\mathcal{P}_d$, consider $\lim_{\delta \to 0} f \star [(-\delta u, 0]]$.]

(10 points)