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Discrete Geometry III

Homework # 3 — due November 25th

Please mark **four** of your solutions. Only these will be graded. The problems marked with a are **mandatory**. You can earn **20 points** on every (1 week) homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday.

Exercise 1. Let $P_1, P_2, Q \in \mathcal{P}_d$ be polytopes.

i) Show that

$$(P_1 \cup P_2) + Q \;=\; (P_1 + Q) \cup (P_2 + Q)$$

where '+' is the Minkowski sum.

ii) Assume that $P_1 \cup P_2$ is convex. Show that

$$(P_1 \cap P_2) + Q \;=\; (P_1 + Q) \cap (P_2 + Q).$$

iii) For fixed $Q \in \mathcal{P}_d$, show that the map $\phi_{+Q} : \mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$ given by $\phi_{+Q}(P) := [P + Q]$ is a valuation and yields a linear map $\hat{\phi}_{+Q} : \mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$.

iv) Let $\mathrm{End}(\mathbf{S}\mathcal{P}_d)$ be the abelian group of linear maps $\mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$. Show that the map $\Phi : \mathcal{P} \to \mathrm{End}(\mathbf{S}\mathcal{P}_d)$ given by $\Phi(Q) := \hat{\phi}_{+Q}$ is a valuation and yields $\hat{\Phi} : \mathbf{S}\mathcal{P}_d \to \mathrm{End}(\mathbf{S}\mathcal{P}_d)$.

v) Infer that for $f, g \in \mathbf{S}\mathcal{P}_d$

$$f \star g \;=\; \hat{\Phi}(f)(g)$$

is the product that we constructed in class.

(10 points)

Exercise 2. Let $Q \subset \mathbb{R}^d$ be a fixed polytope and consider the map $\Upsilon_Q : \mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$ defined by $\Upsilon_Q(P) := [\mathrm{conv}(P \cup Q)]$ for $P \in \mathcal{P}_d \setminus \{\emptyset\}$ and $\Upsilon_Q(\emptyset) := 0$.

i) Show that Υ_Q is a valuation.

In particular Υ_Q extends to a homomorphism $\Upsilon_Q : \mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$.
The association $Q \mapsto \Upsilon_Q$ is a map from polytopes $Q \in \mathcal{P}_d$ to homomorphisms $\Upsilon_Q : \mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$. As before, let $\mathrm{End}(\mathbf{S}\mathcal{P}_d)$ be the abelian group of linear maps $\mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$.

ii) Show that the map $Q \mapsto \Upsilon_Q$ is a valuation.

iii) Infer that there is a product $\vee : \mathbf{S}\mathcal{P}_d \times \mathbf{S}\mathcal{P}_d \to \mathbf{S}\mathcal{P}_d$ such that

$$[P] \vee [Q] \;=\; [\mathrm{conv}(P \cup Q)].$$

(10 points)

Exercise 3. Let $e_1, \ldots, e_d \in \mathbb{R}^d$ be the standard basis and $\mathbf{1} = \sum_{i=1}^{d} e_i$.

i) Show that for $d = 2$

$$\big[[0, e_1)\big] \star \big[[0, e_2)\big] \star \big[[0, -\mathbf{1})\big] \;=\; 0.$$

[Hint: The product is associative!]

ii) Show that this holds for all d, that is,

$$\big[[0, e_1)\big] \star \cdots \star \big[[0, e_d)\big] \star \big[[0, -\mathbf{1})\big] \;=\; 0.$$

iii) (Bonus) A collection of vectors $u_1, \dots, u_k$ is positively dependent if there are $\lambda_1, \dots, \lambda_k \geq 0$ not all zero such that $\lambda_1 u_1 + \cdots + \lambda_k u_k = 0$. Show that for positively dependent $u_1, \dots, u_k$

$$\big[[0, u_1)\big] \star \cdots \star \big[[0, u_{k-1})\big] \star \big[[0, u_k)\big] \;=\; 0.$$

(10+3 points)

Exercise 4. An element $f \in \mathbf{S}\mathcal{P}_d$ is an **idempotent** if $f \star f = f$.

i) Let $\mathbf{S}\mathcal{P}_d^0 \subseteq \mathbf{S}\mathcal{P}_d$ be those polytopal simple functions such that $f^{-1}(k)$ is a finite for all $k \neq 0$. Show that if $f \in \mathbf{S}\mathcal{P}_1^0$ is idempotent, then $f = [0]$ or $f = 0$.

ii) Show that $f = [0]$ are the the only non-zero idempotents in $\mathbf{S}\mathcal{P}_d$.
[Hint: Remember that the ring maps $h(f, \cdot)$ distinguish.]

iii) An element $f \in \mathbf{S}\mathcal{P}_d$ is **nilpotent** if $f^k = 0$ for some $k \geq 1$. What are the nilpotent elements of $\mathbf{S}\mathcal{P}_d$?

(10 points)