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Discrete Geometry III

Homework # 2 — due November 11th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **four** of your solutions. Only these will be graded. The problems marked with a are **mandatory**. You can earn **20 points** on every (1 week) homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday.

Exercise 1. A total order on $\mathbb{N}^n$ is a partial order relation $\preceq$ such that $a \preceq b$ or $b \preceq a$ for all $a, b \in \mathbb{N}^n$. The order is translation invariant if $a + c \preceq b + c$ implies $a \preceq b$.

i) For $a, b \in \mathbb{N}^n, a \neq b$ set $a \prec b$ if $|a| < |b|$ or, if $|a| = |b|$ and the largest index i for which $a_i \neq b_i$ satisfies $a_i < b_i$. Show that this defines a translation invariant total order on $\mathbb{N}^n$.

Let R be a standard graded $\mathbf{k}$-algebra with generators $y_1, \ldots, y_m \in R_1$. Let $\phi : \mathbf{k}[x_1, \ldots, x_m] \to R$ be the ring map given by $\phi(x_i) = y_i$. Define a (possibly infinite) collection of monomials $\mathcal{O}$ inductively as follows: $u_1 := \mathbf{x}^0 \in \mathcal{O}$. For $k \geq 1$, the monomial $u_k = \mathbf{x}^\alpha$ is the smallest (in the order above) monomial such that $\phi(u_k)$ is linearly independent of $\phi(u_1), \ldots, \phi(u_{k-1})$.

ii) Show if $\mathbf{x}^\alpha \notin \mathcal{O}$, then $x_i \mathbf{x}^\alpha \notin \mathcal{O}$ for all $i = 1, \ldots, m$.
[Hint: Use the fact that ϕ is a ring map.]

iii) Show that $H(R, i) = |\{\mathbf{x}^\alpha \in \mathcal{O} : |\alpha| = i\}|$.
[Hint: If not, then we could have added another monomial to $\mathcal{O}$.]

Remark: i) shows that $\mathcal{O}$ is a multicomplex and ii) shows that $\mathcal{O}$ has $H(R, i)$ as its f-vector. Together this gives a proof of Macaulay's theorem.

(10 points)

Exercise 2. Let R be a graded $\mathbf{k}$-algebra. An element $r \in R_1$ is called *regular* if the map $R \xrightarrow{\cdot r} R$ is injective.

i) Show that if $r \in R_1$ is regular, then

$$F(R/\langle r \rangle, t) \;=\; (1-t)F(R, t).$$

[Hint: Pass from the Hilbert function of $R/\langle r \rangle$, which you know, to the Hilbert series.]

Let $\mathbf{k}[\Delta] = \mathbf{k}[x_1, \ldots, x_d, y_1, \ldots, y_d]/I_\Delta$ be the Stanley–Reisner ring for the d-dimensional crosspolytope with $I_\Delta = \langle x_1 y_1, \ldots, x_d y_d \rangle$.

ii) Show that $r_i := x_i - y_i$ is regular for the ring $\mathbf{k}[\Delta]/\langle r_1, \ldots, r_{i-1} \rangle$.

iii) Show that $F(\mathbf{k}[\Delta]/\langle r_1, \ldots, r_d \rangle, t) = (1+t)^d$ by finding a good interpretation for the elements in $\mathbf{k}[\Delta]/\langle r_1, \ldots, r_d \rangle$.

(10 points)

Exercise 3. Let $(\mathbf{S}\mathcal{Q}_d, +, \cdot)$ be the ring of polyhedral simple functions.

i) Let $f \in \mathbf{S}\mathcal{P}_d \subseteq \mathbf{S}\mathcal{Q}_d$. Show that for every $l \in \mathbb{Z} \setminus \{0\}$, the set $f^{-1}(l)$ is a finite union of relatively open polytopes (i.e. it is a polyconvex set).

ii) Give a characterization of the invertible elements of $\mathbf{S}\mathcal{Q}_d$ in terms of polyconvex sets.

iii) Let $\mathbf{PQ}_d$ be the collection of all sets that can be written as finite unions of relatively open polyhedra in $\mathbb{R}^d$. Show that $\mathbf{PQ}_d$ is generated by unions, intersections, and complementation of halfspaces.

(10 points)

Exercise 4. For $S \subseteq \mathbb{R}^d$ define the following four maps $\phi_* : \mathcal{P}_d \to \{0, 1\}$

(a) $\phi_{\subseteq S}(P) = 1$ if and only if $P \subseteq S$;
(b) $\phi_{\supseteq S}(P) = 1$ if and only if $P \supseteq S$;
(c) $\phi_{\cap S}(P) = 1$ if and only if $P \cap S \neq \emptyset$;
(d) $\phi_{\cup S}(P) = 1$ if and only if $P \cup S$ convex,

for any nonempty polytope P and $\phi_*(\emptyset) := 0$.

i) For a non-empty polytope S, which of the four maps is a valuation?

ii) Let $H^< \subset \mathbb{R}^d$ be *open* halfspace. Show that $\phi_{\subseteq H^<}$ is a valuation. For which polytopes P is $\phi_{\subseteq H^<}(P) = 1$?

iii) For a non-empty polyhedron Q define

$$\chi(Q) \;:=\; \begin{cases} 1 & \text{if } Q \text{ is a polytope,} \\ 0 & \text{if } Q \text{ is unbounded but pointed.} \end{cases}$$

(Remember that 'pointed' means that Q does not contain a line.) Show that χ defines a valuation on $\mathcal{Q}_d$. In particular, what is $\chi(Q)$ if Q has a nontrivial lineality space?

iv) Define $\overline{\chi}(Q) = 1$ for every non-empty closed convex polyhedron. Verify that this is a valuation. What is $\chi(\text{relint}(Q))$?

(10 points)

Exercise 5. Let $(\mathbf{k}[\mathcal{J}], +, \cdot)$ and $(\mathbf{k}[\mathcal{J}], +, \star)$ be the two algebra defined in Exercise 6 on Homework #1. Working over a field $\mathbf{k}$ was not necessary and we can define the same algebras over $\mathbb{Z}$. Show that $(\mathbb{Z}[\mathcal{J}], +, \cdot)$ and $(\mathbb{Z}[\mathcal{J}], +, \star)$ are isomorphic to subalgebras of $\mathbf{S}\mathcal{Q}_n$. That is, find polyhedral simple functions $f_A : \mathbb{R}^n \to \mathbb{Z}$ and $g_A : \mathbb{R}^n \to \mathbb{Z}$ for $A \in \mathcal{J}$ such that $e_A \mapsto f_A$ and $e_A \mapsto g_A$ induce isomorphisms.

[Hint: Any collection of polytopes generate a subalgebra of $\mathbf{S}\mathcal{Q}_d$.]

(10 points)